Improvement, Deterioration, and Optimal Replacement Under Age-Replacement With Minimal Repair

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Key Words — **Repairable system, Improvement, Deterioration, Partial ordering between life distributions, Non-homogeneous Poisson process, Minimal repair, Optimal replacement time**

Reader Aids —
General purpose: Report of derivations
Special math needed for derivations: Probability theory and methodology
Special math needed to use results: Same
Results useful to: Reliability analysts

Abstract — **Improvement & deterioration for a repairable system are studied, in particular in terms of the effect of ageing on the distribution of the time to first failure under a non-homogeneous Poisson process. For a repairable system undergoing minimal repair, the optimal replacement time under the age replacement policy is discussed.**

1. INTRODUCTION

The performance of a repairable system can be better or worse with the passage of time. A repairable system improves (deteriorates) with time if the times between two successive repairs tend to get larger (smaller) in some sense. Ascher & Feingold [1] defined system improvement (deterioration) in terms of orderings between interarrival times (times between successive failures). However, their definition is valid only under the assumption of *s*-independence of these interarrival times. Ebrahimi [8] and Deshpande & Singh [7] defined system improvement (deterioration) by considering the entire history of the system. They compared the conditional inter-arrival times through several known partial and complete orderings [6,7] between probability distributions.

We consider a system with minimal repair, *ie*, the failed system is restored to a condition which is the statistically the same as its condition just prior to failure. Most repairs involve the replacement of only a very small fraction of a system's parts. A system subject to minimal repair can be modeled by an NHPP [1, 11].

Section 2 defines improvement (deterioration) of the system by comparing inter-arrival times through a few partial and complete orderings between probability distributions; then it derives some results and connections thereof. Section 3 determines the optimal replacement time under age-replacement policy (wherein items are replaced at failure or at prefixed time) for a system subject to minimal repair.

Notation (Statistics)

- $S(n)$ arrival time of failure n; $n = 1,2,\ldots$
- $X(n)$ interarrival time between failures $n-1$ & n; $n = 1,2,\ldots$
- $N(t)$ number of failures in $[0,t]$
- $f_X(x), \bar{F}_X(x), r_X(x)$ [pdf, Sf, failure rate] of X
- $g_Y(x), \bar{G}_Y(x)$ [pdf, Sf] of Y
- $e_X(x)$ mean residual life of X: $[\int_x^\infty \bar{F}_X(u)\, du]/\bar{F}_X(x)$
- $\sigma_X^2(x)$ $\mathrm{Var}\{X-x|X>x\}$, the variance of residual life of X.
- s_k ordered real numbers; if there is only 1 number, then $s_1 = X(1)$
- $X(n;s_{n-1},\ldots,s_1)$ $X(n)$, given $S(k) = s_k$, for $k = n-1,\ldots,1$.

Acronyms[1] *& Nomenclature*

- MRL mean residual-life
- NHPP non-homogeneous Poisson process
- MRLF mean residual-life function
- DLR, ILR [decreasing, increasing] likelihood ratio: $f_X(x+t)/f_X(t)$ is [non-increasing, non-decreasing] in t for all $x \ge 0$
- DFR, IFR [decreasing, increasing] failure rate: $r(x)$ is [non-increasing, non-decreasing] in x for all $x \ge 0$
- DMRL, IMRL [decreasing, increasing] mean residual life: $e(x)$ is [non-increasing, non-decreasing] in x for all $x \ge 0$
- DVRL, IVRL [decreasing, increasing] variance of residual life: $\sigma^2(x)$ is [non-increasing, non-decreasing] in x for all $x \ge 0$
- NDVRL, NIVRL net [decreasing, increasing] variance of residual life: $\sigma^2(x)$ [$\ge$, $\le$] $\sigma^2(0)$ for all $x \ge 0$
- NWU, NBU new [worse, better] than used: $\bar{F}(x+y)$ [$\le$, $\ge$] $\bar{F}(x)\cdot\bar{F}(y)$ for all $x,y \ge 0$
- NWUE, NBUE new [worse, better] than used in *s*-expectation: $e_X(x)$ [$\ge$, $\le$] $e_X(0)$ for all $x \ge 0$
- NWUFR, NBUFR new [worse, better] than used in failure rate: $r(x)$ [$\le$, $\ge$] $r(0)$ for all $x \ge 0$
- SS, XSS $g(x)$ is [star-shaped, anti-star-shaped] if $g(x)/x$ is [increasing, decreasing] for all $x \ge 0$.

Notation (Logic)

- $X \overset{LR}{\ge} Y$ X is larger than Y in likelihood ratio ordering: $f_X(x)/g_Y(x)$ is non-decreasing in x for all $x \ge 0$

[1]The singular & plural of an acronym are always spelled the same.

$X \overset{FR}{\geq} Y$ X is larger than Y in failure rate ordering: $r_X(x) \leq r_Y(x)$ for all $x \geq 0$

$X \overset{ST}{\geq} Y$ X is larger than Y in stochastic ordering: $\bar{F}_X(x) \geq \bar{G}_Y(x)$ for all $x \geq 0$

$X \overset{MR}{\geq} Y$ X is larger than Y in mean residual life ordering: $e_X(x) \geq e_Y(x)$ for all $x \geq 0$

$X \overset{E}{\geq} Y$ X is larger than Y in s-expectation ordering: $E\{X\} \geq E\{Y\}$

$X \overset{VR}{\geq} Y$ X is larger than Y in variance of residual life ordering: $\sigma_X^2(x) \geq \sigma_Y^2(x)$ for all $x \geq 0$

$X \overset{V}{\geq} Y$ X is larger than Y in variance ordering: $\sigma_X^2(0) \geq \sigma_Y^2(0)$

$X \overset{r(0)}{\geq} Y$ X is larger than Y in initial failure rate ordering: $r_X(0) \leq r_Y(0)$.

Other, standard notation is given in "Information for Readers & Authors" at the rear of each issue.

Assumptions

1. Repair times are negligible.
2. $S(0) = X(0) = 0$.
3. The repair function does exactly what it is assumed to do — neither better nor worse. In particular, repair never damages anything. ◄

Definitions

1. *Minimal repair:* The failed system is restored to a condition which is the statistically the same as its condition just prior to failure; *ie*, if the system fails at time t and undergoes *minimal repair*, then the Sf of the repaired system is $\bar{F}_{X(1)}(t+x)/\bar{F}_{X(1)}(t)$, and the system is modelled by an NHPP.[2]

2. *Age Replacement Policy:* A unit is replaced (with a like-new unit) upon failure or at a specified age, whichever comes first.

3. [Ξ = LR, FR, ST, MR, E, VR, V, $r(0)$] *Ξ-Improving (Deteriorating):* A point process $\{N(t): t \geq 0\}$ consisting of interarrival times $X(1)$, $X(2)$, ... is *Ξ-improving (Ξ-deteriorating)* if $X(j; s_{j-1}, \ldots, s_1) \geq^{\Xi} (\leq^{\Xi})\ X(i; s_{i-1}, \ldots, s_1)$ for every $j \geq i \geq 1$ and every $0 < s_1 < \ldots < s_{j-1}$, with strict inequality for at least one pairing of interarrival times.[3]

2. IMPROVEMENT & DETERIORATION

Ebrahimi [8] and Deshpande & Singh [5] defined system improvement (deterioration) by comparing the conditional interarrival times through ST, LR, FR, MR, E orderings between probability distributions. Other partial orderings of distributions and their corresponding relationships are available in the literature [6,7]. Definition 3 covers improvement (deterioration) of a repairable system through other partial orderings.

[2]Most repairs involve the replacement of only a very small fraction of a system's constituent parts.

[3]The equality will hold in each case for a homogeneous Poisson process (HPP), because it is neither improving nor deteriorating.

2.1 General Case

The effect of ageing of $X(1)$ on improvement (deterioration) of the system undergoing minimal repairs at failure is discussed in theorem 1.

Theorem 1. The stochastic process $\{N(t): t \geq 0\}$ generated by a minimal repair policy, *ie*, by an NHPP, is improving (deteriorating) —

a. in the VR sense iff $\bar{F}_{X(1)}(x)$ is IVRL (DVRL).
b. in the V sense iff $\bar{F}_{X(1)}(x)$ is IVRL (DVRL).
c. in the $r(0)$ sense iff $\bar{F}_{X(1)}(x)$ is DFR (IFR). ◄

Thus for a system undergoing only minimal repair —

- improvement (deterioration) in the VR or V sense is the same.
- improvement (deterioration) in the $r(0)$ or FR or ST sense is the same [5].

In some situations, a system undergoing minimal repair might not be improving (deteriorating) monotonically, but the 'time after every repair up to the next failure' can be compared with the 'time up to the first failure'. We show that improvement (deterioration) defined by such a comparison is equivalent to certain properties of the distribution of $X(1)$.

Theorem 2. The stochastic process $\{N(t): t \geq 0\}$ generated by a minimal repair policy is improving (deteriorating):

$X(n+1; s_n, \ldots, s_1) \geq (\leq)\ X(1)$, for all $n \geq 1$ and for all $0 < s_1 < \ldots < s_n$ —

a. in the LR sense iff $\bar{F}_{X(1)}(x)$ is DLR (ILR).
b. in the FR sense iff $\bar{F}_{X(1)}(x)$ is DFR (IFR).
c. in the ST sense iff $\bar{F}_{X(1)}(x)$ is NWU (NBU) [5].
d. in the MR sense iff $\bar{F}_{X(1)}(x)$ is IMRL (DMRL).
e. in the E sense iff $\bar{F}_{X(1)}(x)$ is NWUE (NBUE) [5].
f. in the VR sense iff $\bar{F}_{X(1)}(x)$ is IVRL (DVRL).
g. in the V sense iff $\bar{F}_{X(1)}(x)$ is NIVRL (NDVRL).
h. in the $r(0)$ sense iff $\bar{F}_{X(1)}(x)$ is NWUFR (NBUFR). ◄

In the comparisons of theorem 2, different orderings lead to different properties of the distribution of $X(1)$. Comparisons based on weaker orderings lead to successively weaker ageing properties of the distribution of $X(1)$.

2.2 Special Case

Notation

$\lambda(t)$ intensity of NHPP
$\Lambda(t)$ mean of NHPP: $\int_0^t \lambda(y)\,dy$
$\Delta\Lambda(x; t)$ $\Lambda(t+x) - \Lambda(t)$.

$\{N(t): t \geq 0\}$ is a NHPP with intensity $\lambda(t)$ and mean $\Lambda(t)$. The improvement (deterioration) of the corresponding repairable system is discussed in terms of $\lambda(t)$.

Theorem 3. The NHPP $\{N(t): t \ge 0\}$ with intensity function $\lambda(t)$ is improving (deteriorating) —

a. in the LR sense iff $\lambda(x+t)\cdot\exp[-\Delta\Lambda(x;t)]$ is non-decreasing (non-increasing) in t for all $x,t \ge 0$.

b. in the FR sense iff $\lambda(t)$ is non-increasing (non-decreasing) in t for all $t \ge 0$.

c. in the $r(0)$ sense iff $\lambda(t)$ is non-increasing (non-decreasing) in t for all $t \ge 0$.

d. in the stochastic sense iff $\Delta\Lambda(t_2-t_1;\ t_1) \ge (\le)\ \Delta\Lambda(t_2-t_1;\ t_1+x)$ for $0 \le t_1 \le t_2$, and $x \ge 0$ [8].

e. in the MR sense iff $\exp[\Lambda(x+t)]\cdot\int_{t+x}^{\infty}\exp[-\Lambda(y)]\,dy$ is non-decreasing (non-increasing) in t for all $x,t \ge 0$.

f. in the E sense iff $\exp[\Lambda(t)]\cdot\int_t^{\infty}\exp[-\Lambda(y)]\,dy$ is non-decreasing (non-increasing) in t for all $t \ge 0$. ◄

Theorem 4. Let $\{N(t): t \ge 0\}$ be a NHPP with intensity function $\lambda(t)$. Then,

$X(n+1;\ s_n,\ldots,s_1) \ge (\le)\ X(1)$, for $0 < s_1 < \ldots < s_n$, —

a. in the LR sense iff $\lambda(x+t)\cdot\exp[-\Delta\Lambda(x;\ t)] \ge (\le)\ \lambda(x)\cdot\exp[-\Delta\Lambda(x;\ 0)]$, for all $t,x \ge 0$.

b. in the FR sense iff $\lambda(x+t) \le (\ge)\ \lambda(x)$, for all $t,x \ge 0$, *ie*, $\Lambda(t)$ is concave (convex) for all $t \ge 0$.

c. in the $r(0)$ sense iff $\lambda(t) \le (\ge)\ \lambda(0)$, for all $t \ge 0$.

d. in the stochastic sense iff $\Lambda(t)$ is sub-additive (super-additive) for all $t \ge 0$.[4]

e. in the MR sense iff $\exp[\Lambda(x+t)]\cdot\int_{t+x}^{\infty}\exp[-\Lambda(y)]\,dy \ge (\le)\ \exp[\Lambda(x)]\cdot\int_x^{\infty}\exp[-\Lambda(y)]\,dy$, for all $x,t \ge 0$.

f. in the E sense iff $\exp[\Lambda(t)]\cdot\int_t^{\infty}\exp[-\Lambda(y)]\,dy \ge (\le)\ \int_0^{\infty}\exp[-\Lambda(y)]\,dy$, for all $t \ge 0$.

Proposition 1. Let $\Lambda(t)$ be XSS (SS), then,

$$X(n+1;\ s_n,\ldots,s_1) \overset{E}{\ge} (\overset{E}{\le})\ X(1), \text{ for every } n \ge 1 \text{ and } 0 < s_1 < \ldots < s_n.$$

3. OPTIMAL REPLACEMENT TIME UNDER AGE REPLACEMENT POLICY WITH MINIMAL REPAIR

Notation

- $R(t)$ operational cost of a unit operating during $[0,t)$, $R(0) = 0$
- ζ age at which there is: a) unplanned replacement, or b) a major unrepairable breakdown, a r.v., $\zeta > 0$.
- c_1 cost of the repair/replacement at ζ
- T age at which there is a planned replacement
- c_2 cost of the planned replacement at T
- η $\min(\zeta,T)$
- $c_0^i(x)$ cost of minimal repair i at age x
- $\bar{F}(x)$ Sf of a new unit
- $M(t)$ an NHPP with mean function $\int q\cdot\bar{F}^{-1}(y)\,dF(y)$
- S_j arrival-time j of $M(t)$
- $'$ implies ordinary derivative
- $\phi(t)$ & $\psi(t)$ continuously differentiable functions on $[0,\infty)$; $\phi(0) = \psi(0) = 0$; $\phi'(t) > 0$, $\psi'(t) > 0$ on $[0,\infty)$
- $D(t)$ $\psi'(t)/\phi'(t)$
- $\Gamma(t)$ $D(t)\cdot\phi(t) - \psi(t)$.

Savits [9] considered a model with an underlying stochastic process $\{R(t),\ 0 \le t \le \zeta\}$. Under the age replacement policy, a planned replacement occurs whenever a functioning unit reaches age T.

Assumptions (Additional)

4. If an operating unit fails at age x, it is replaced by a new unit with probability p, or minimally repaired with probability $q = 1-p$.
5. $M(t)$ and ζ are s-independent. ◄

Then [4,10],

$$\mathrm{Sf}\{\zeta\} = \bar{G}(t) = \exp\left[\int_0^t p\cdot\bar{F}^{-1}(y)\ dF(y)\right] = \bar{F}_p(t),$$

$$R(t) = \sum_{i=1}^{M(t-)} c_0^i(s_i),\ 0 \le t \le \zeta.$$

The long-run average cost-rate for such a system under age replacement policy is:

$$J(T) = \lim_{t\to\infty}\{C(t)/t\} = A(T)/\mathrm{E}\{\eta\},$$

$$\mathrm{E}\{\eta\} = \int_0^T \bar{G}(y)\ dy,$$

$$A(T) = c_1\cdot F_p(T) + c_2\cdot\bar{F}_p(T) + \int_0^T q\cdot h(y)\cdot\bar{F}^{-1}(y)\cdot\bar{F}_p(y)\ dF(y),$$

$$h(y) \equiv \mathrm{E}\{c_0^{M(y)+1}(y)\}.$$

We determine the T that minimizes $J(T)$ by using an optimization lemma [9].

Lemma 1. $C(t) = [R + \psi(t)]/\phi(t)$, $t \ge 0$, $R > 0$, $D(t)$ is non-decreasing (non-increasing) over $[0,\infty)$. Then,

i. $\Gamma(t)$ is non-decreasing (non-increasing) over $[0,\infty)$.

ii. If $D(t)$ is non-decreasing and $\lim_{t\to\infty}\{\Gamma(t)\} > R$, (4-1)

there exists at least one point $t_0 < \infty$ which is a global minimum of $C(t)$. Such points are the only solution of:

$$\Gamma(t) = R. \quad (4\text{-}2)$$

[4]A function $\Lambda(t)$ is sub-additive (super-additive) iff $\Lambda(t+x) \le (\ge)\ \Lambda(t) + \Lambda(x)$ *for all* $t,x \ge 0$.

The minimum of $C(t)$ subject to (4-1) is $C(t_0) = D(t_0)$, t_0 is any solution of (4-2). The global minimum and hence the solution of (4-2) is unique if $D(t)$ is strictly increasing.

iii. If $D(t)$ is non-decreasing and $\lim_{t\to\infty}\{\Gamma(t)\} = R$, then $C(t)$ is a non-increasing function of t, so that a minimum of $C(t)$ occurs at $t = \infty$. However, $C(t)$ can assume the minimum value at all points $t \in [u, \infty)$ for some finite u.

iv. For the remaining cases, such as: a) the limit in (4-1) is less than R, or b) $D(t)$ is non-increasing over $[0, \infty)$, then $C(t)$ is a non-increasing function of t and the minimum of $C(t)$ occurs only at $t = \infty$. ◄

3.1 Constant Probability for Replacement

Assumption: $c_0^i(y) = c_0$.

$$A(T) = c_a \cdot F_p(T) + c_2 \cdot \bar{F}_p(T)$$

$$E(\eta) = \int_0^T \bar{F}_p(y)\, dy,$$

$$J(T) = A(T)/\mathrm{E}\{\eta\}$$

$$c_a = (q/p)\cdot c_0 + c_1$$

Compare $J(T)$ & $C(T)$ using lemma 1.

$$R = c_2,\ \psi(T) = c_b \cdot F_p(T)$$

$$\phi(T) = \int_0^T \bar{F}_p(y)\, dy,$$

$$c_b = (q/p)\cdot c_0 + c_1 - c_2$$

Hence the conditions of lemma 1 are satisfied.

$$D(T) = r_F(T)\cdot[c_0 + p\cdot(c_1 - c_0 - c_2)].$$

If $r_F(T)$ is a non-decreasing (non-increasing) function of T and $c_1 > c_0 + c_2$, then $D(T)$ is increasing (decreasing) over $[0, \infty)$.

$\lim_{T\to\infty}\{\Gamma(T)\} = c_b \cdot [p \cdot \lim_{T\to\infty}\{r_F(T)\}\cdot \mathrm{E}\{\eta\} - 1] > c_2$, when $r_F(T) \to \infty$. Therefore from lemma 1, the optimal replacement time T_0 is finite and is the solution of:

$$r_{F_p}(T)\cdot \int_0^T \bar{F}_p(z)\, dz + \bar{F}_p(T) = c_a/c_b$$

$$r_{F_p}(t) = p \cdot r_F(T).$$

Example

Let F be Weibull with:

$$r_F(t) = k\cdot\lambda\cdot(\lambda\cdot t)^{k-1},\ \lambda>0,\ k>1.$$

The optimal replacement time T_0 is finite and is the solution of:

$$p\cdot k\cdot\lambda^k\cdot T^{k-1}\cdot \int_0^T \exp(-p(\lambda\cdot y)^k)\ dy\ +\ \exp(-p(\lambda\cdot T)^k)$$

$$= c_a/c_b.$$

The l.h.s. can be evaluated as an incomplete Gamma integral whose values are readily available in the tables.

Remark: If F is DFR and $c_0 > c_1 + c_2$, then the optimal replacement time $T_0 = \infty$; *ie*, there should be no replacement.

3.2 No Unplanned Replacements ($p=0$)

The model, considered by Block *et al* [4], is a generalization of the minimal repair model [2,3]. Then [10],

$$A(T) = c_2 + \int_0^T h(y)\cdot \bar{F}^{-1}(y)\, dF(y),$$

$M(t)$ is a NHPP with mean function $\Lambda(t) = \int_0^T r(y)\, dy$.

$$h(y) = \mathrm{E}\{c_0^{M(y)+1}(y)\} = \Sigma_x\, c_0^{x+1}(y)\cdot \mathrm{poim}(x;\Lambda(y))$$

$$= c_0(y)\cdot \exp[-\Lambda(y)\cdot \bar{c}_0(y)].$$

$$A(T) = c_2 + \int_0^T c_0(y)\cdot \exp[-\Lambda(y)\cdot\bar{c}_0(y)]\cdot r_F(y)\, dy,$$

$$\mathrm{E}\{\eta\} = T.$$

The long run average cost-rate is:

$$J(T) = A(T)/T.$$

Compare $J(T)$ with $C(T)$ in lemma 1.

$$R = c_2,$$

$$\psi(T) = \int_0^T c_0(y)\cdot\exp[-\Lambda(y)\cdot\bar{c}_0(y)]\cdot r_F(y)\, dy,$$

$$\phi(T) = T.$$

Hence the conditions of lemma 1 are satisfied.

$$D(T) = c_0(T)\cdot\exp[-\Lambda(T)\cdot\bar{c}_0(T)]\cdot r_F(T)$$

$$= \exp[-\Lambda(T)\cdot\bar{c}_0(T)]\cdot[c_0'(T)\cdot r_F(T)\ +\ c_0(T)\cdot r_F'(T)$$

$$+\ c_0(T)\cdot r_F(T)\cdot[\Lambda(T)\cdot c_0'(T)\ -\ \Lambda'(T)\cdot\bar{c}_0(T)]].$$

Example

Let the operational cost be constant: $c_0(t) = c_0$.

$$D(T) = c_0[r_F'(T) + (c_0 - 1)\cdot r_F^2(T)]\cdot\exp[-\Lambda(T)\cdot\bar{c}_0]$$

$$\Gamma(T) = \exp[-\Lambda(T)\cdot\bar{c}_0]\cdot[c_0\cdot T\cdot r_F(T) + c_0/\bar{c}_0] - c_0/\bar{c}_0$$

$$\lim_{T\to\infty}\{\Gamma(T)\} > c_2, \text{ for } c_0 > 1$$

Thus if F is IFR, then $D(T)$ is non-decreasing over $[0,\infty)$ and hence, from lemma 1, the optimal replacement time T_0 is finite and is given by a solution of:

$$\exp[-\Lambda(T)\cdot\bar{c}_0]\cdot[c_0\cdot T\cdot r_F(T) + c_0/\bar{c}_0]$$

$$= [c_2 - c_0\cdot(c_2-1)]/\bar{c}_0.$$

If D(T) is non-increasing over $[0,\infty)$, then $T_0 = \infty$.

For $0<c_0<1$, $\lim_{T\to\infty}\{\Gamma(T)\} = -c_0/\bar{c}_0 < 0 < c_2$

Hence for any F, the optimal choice is $T_0 = \infty$. Thus in all cases, the best policy is never to replace a unit but to minimally-repair it on failure. ◄

APPENDIX

A.1 Proof of Theorem 1

a. $x_{s_{j-1},\ldots,s_1}(j) \overset{\text{VR}}{\geq} (\overset{\text{VR}}{\leq})\ x_{s_{i-1},\ldots,s_1}(i)$

$$\Leftrightarrow\ \sigma^2_{\bar{F}_{X_{s_n,\ldots,s_1}(n+1)}}(x) \geq (\leq)\ \sigma^2_{\bar{F}_{X_{s_{n-1},\ldots,s_1}(n)}}(x)$$

for all $x \geq 0$ and for all $0 < s_1 < \ldots < s_n$;

$$\Leftrightarrow\ \sigma^2_{\bar{F}_{X(1)}}(x+s_n) \geq (\leq)\ \sigma^2_{\bar{F}_{X(1)}}(x+s_{n-1})$$

for all x $\geq$ 0 and for all $0 < s_{n-1} < s_n$;

$\Leftrightarrow\ \sigma^2_{\bar{F}_{X(1)}}(x)$ is a non-decreasing (non-increasing) function of x;

$\Leftrightarrow\ \bar{F}_{X(1)}$ is IVRL (DVRL).

b. $X_{s_{j-1},\ldots,s_1}(j) \overset{\text{V}}{\geq} (\overset{\text{V}}{\leq})\ X_{s_{i-1},\ldots,s_1}(i)$

$$\Leftrightarrow\ \sigma^2_{\bar{F}_{X_{s_n,\ldots,s_1}(n+1)}}(0) \geq (\leq)\ \sigma^2_{\bar{F}_{X_{s_{n-1},\ldots,s_1}(n)}}(0)$$

for all $0 < s_1 < \ldots < s_n$;

$\Leftrightarrow\ \sigma^2_{\bar{F}_{X(1)}}(s_n) \geq (\leq)\ \sigma^2_{\bar{F}_{X(1)}}(s_{n-1})$ for all $0 < s_{n-1} < s_n$;

$\Leftrightarrow\ \bar{F}_{X(1)}$ is IVRL (DVRL).

c. $X_{s_{j-1},\ldots,s_1}(j) \overset{r(0)}{\geq} (\overset{r(0)}{\leq})\ X_{s_{i-1},\ldots,s_1}(i)$

$$\Leftrightarrow\ r_{\bar{F}_{X_{s_n,\ldots,s_1}(n+1)}}(0) \geq (\leq)\ r_{\bar{F}_{X_{s_{n-1},\ldots,s_1}(n)}}(0)$$

for all $0 < s_1 < \ldots < s_n$;

$\Leftrightarrow\ r_{\bar{F}_{X(1)}}(s_n) \geq (\leq)\ r_{\bar{F}_{X(1)}}(s_{n-1})$ for all $0 < s_{n-1} < s_n$;

$\Leftrightarrow\ \bar{F}_{X(1)}$ is DFR (IFR).

A.2 Proof of Theorem 2

Observe that

$$\Pr\{X(n+1) > x \mid S(n)=s_n,\ldots,S(1)=s_1\}$$

$$= \Pr\{S(n+1) > x+s_n \mid S(n)=s_n,\ldots,S(1)=s_1\}$$

$$= \Pr\{S(n+1) > x+s_n \mid S(n)=s_n\}$$

$$= \Pr\{X(1) > x+s_n \mid X(1) > s_n\} \text{ for all } s_n > s_{n-1} > \ldots$$

The proof of the theorem follows by further observing that

$f_{X_{s_n,\ldots,s_1}(n+1)}$ is given by $\dfrac{f_{X(1)}(x+s_n)}{\bar{F}_{X(1)}(s_n)}$;

$$r_{\bar{F}_{X_{s_n,\ldots,s_1}(n+1)}}(x) = r_{\bar{F}_{X(1)}}(x+s_n);$$

$$e_{\bar{F}_{X_{s_n,\ldots,s_1}(n+1)}}(x) = e_{\bar{F}_{X(1)}}(x+s_n);$$

$$\sigma^2_{\bar{F}_{X_{s_n,\ldots,s_1}(n+1)}}(x) = \sigma^2_{\bar{F}_{X(1)}}(x+s_n).$$

A.3 Proof of Theorem 3

Observe that for $0 < s_1 < \ldots < s_{n-2} < t$

$$\Pr\{X(n) > x \mid S(n-1) = t,\ S(n-2) = s_{n-2},\ldots,S(1) = s_1\}$$

$$= \Pr\{X(n) > x \mid S(n-1) = t\}$$

$$= \exp\left\{\int_t^{x+t} \lambda(y)\ dy\right\}$$

$$= \exp\{-\Lambda(x+t) + \Lambda(t)\}.$$

Hence

$$f_{X_{t,s_{n-2},\ldots,s_1}(n)}(x) = \lambda(x+t)\ \exp\left\{-\int_t^{x+t}\lambda(y)\ dy\right\}.$$

a. By definition, NHPP is improving (deteriorating) in the LR sense iff for $t_1 > t_2$, the ratio of

$$\lambda(x+t_1)\exp\left\{-\int_{t_1}^{x+t_1}\lambda(y)\,dy\right\} \text{and } \lambda(x+t_2)$$

$$\exp\left\{-\int_{t_2}^{x+t_2}\lambda(y)\,dy\right\} \text{ is}$$

non-decreasing (non-increasing) in $x \geq 0$,

$$\Leftrightarrow \frac{\lambda(x+t_1)\exp\{-\Lambda(x+t_1)\}}{\lambda(x+t_2)\exp\{-\Lambda(x+t_2)\}}$$

is non-decreasing (non-increasing)

in $x \geq 0$,

$$\Leftrightarrow \frac{\lambda'(x+t_1)}{\lambda(x+t_1)} - \frac{\lambda'(x+t_2)}{\lambda(x+t_2)} \geq (\leq)\ \lambda(x+t_1)$$

$$-\ \lambda(x+t_2) \text{ for all } x \geq 0$$

assuming that $\lambda'(x)$ exists,

$$\Leftrightarrow \frac{\partial}{\partial x}\left\{\log\frac{\lambda(x+t_1)}{\lambda(x+t_2)}\right\} \geq (\leq)\ \lambda(x+t_1)-\lambda(x+t_2)$$

for all $x \geq 0$,

$$\Leftrightarrow \frac{\lambda(x+t_1)}{\lambda(x+t_2)} \geq (\leq) \exp\left\{\int_{t_1}^{x+t_1}\lambda(y)\,dy - \int_{t_2}^{x+t_2}\lambda(y)\,dy\right\}$$

for all $x \geq 0$

$$\Leftrightarrow \lambda(x+t_1)\exp\left\{-\int_{t_1}^{x+t_1}\lambda(y)\,dy\right\}$$

$$\geq (\leq)\ \lambda(x+t_2)\exp\left\{-\int_{t_2}^{x+t_2}\lambda(y)\,dy\right\}$$

for all $x \geq 0$,

$$\Leftrightarrow \lambda(x+t)\exp\left\{-\int_{t}^{x+t}\lambda(y)\,dy\right\}$$

is non-decreasing (non-increasing)

in t for all $x \geq 0$.

This means that the conditional density function of $X(n)$ is non-decreasing (non-increasing) in t, the time of $(n-1)^{\text{th}}$ failure.

b. NHPP is improving (deteriorating) in the FR sense

$$\Leftrightarrow \frac{\lambda(x+t_1)\exp\left\{-\int_{t_1}^{x+t_1}\lambda(y)\,dy\right\}}{\exp\left\{-\int_{t_1}^{x+t_1}\lambda(y)\,dy\right\}} \leq (\geq)$$

$$\frac{\lambda(x+t_2)\exp\left\{-\int_{t_2}^{x+t_2}\lambda(y)\,dy\right\}}{\exp\left\{-\int_{t_2}^{x+t_2}\lambda(y)\,dy\right\}}$$

for all $x \geq 0$ and $t_1 > t_2$,

$\Leftrightarrow \lambda(x+t_1) \leq (\geq)\ \lambda(x+t_2)$ for all $x \geq 0$ and $t_1 > t_2$,

$\Leftrightarrow \lambda(t)$ is non-increasing (non-decreasing) in t for all $t \geq 0$.

c. NHPP is improving (deteriorating) in the $r(0)$ sense

$$\Leftrightarrow r_{F_{X_{t_1,s_{n-1},\ldots,s_1}(n+1)}}(0) \leq (\geq)\ r_{F_{X_{t_2,s_{n-1},\ldots,s_1}(n+1)}}(0)$$

for all $t_1 > t_2$,

$\Leftrightarrow r_{F_{X(1)}}(t_1) \leq (\geq)\ r_{F_{X(1)}}(t_2)$ for all $t_1 > t_2$,

$\Leftrightarrow \lambda(t_1) \leq (\geq)\ \lambda(t_2)$ for all $t_1 > t_2$,

$\Leftrightarrow \lambda(t)$ is non-increasing (non-decreasing) in t for all $t \geq 0$.

d. By definition, NHPP is improving (deteriorating) in MR sense iff for all $t_1 > t_2$ and $x \geq 0$,

$$\frac{\int_x^\infty \exp\left\{-\int_{t_1}^{u+t_1}\lambda(y)\,dy\right\}du}{\exp\left\{-\int_{t_1}^{x+t_1}\lambda(y)\,dy\right\}}$$

$$\geq (\leq)\ \frac{\int_x^\infty \exp\left\{-\int_{t_2}^{u+t_2}\lambda(y)\,dy\right\}du}{\exp\left\{-\int_{t_2}^{x+t_2}\lambda(y)\,dy\right\}}$$

$$\Leftrightarrow \frac{\int_x^\infty \exp[-\Lambda(u+t_1)]\,du}{\exp[-\Lambda(x+t_1)]}$$

$$\leq (\leq)\ \frac{\int_x^\infty \exp[-\Lambda(u+t_2)]\,du}{\exp[-\Lambda(x+t_2)]}$$

$$\Leftrightarrow \exp\{\Lambda(x+t)\}\int_{x+t}^{\infty}\exp[-\Lambda(u)]\,du \text{ is non-decreasing}$$

(non-increasing) in t for all $x,\ t \geq 0$.

e. $$\mathrm{E}\left[x_{s_n,\ldots,s_1}(n+1)\right] = \int_0^{\infty}\left\{-\int_t^{x+t}\lambda(y)\,dy\right\}dx$$

$$= \int_0^{\infty}\exp\left\{-\int_0^{x+t}\lambda(y)\,dy + \int_0^{t}\lambda(y)\,dy\right\}dx$$

$$= \exp[\Lambda(t)]\int_t^{\infty}\exp[-\Lambda(z)]\,dz$$

Hence from the definition, the result follows.

Proof of Theorem 4

We shall prove *d*. The proofs of the remaining parts follow on similar lines as those for theorem 3.

d. $X_{s_n,\ldots,s_1}(n+1) \overset{\text{ST}}{\geq} (\overset{\text{ST}}{\leq})\ X(1)$ for every $n \geq 1$ and for

$0 < s_1 < \ldots < .s_n$

$$\Leftrightarrow \exp\{-\Lambda(t+s_n) + \Lambda(s_n)\} \geq (\leq)\ \exp\{-\lambda(t)\}$$

for all $t \geq 0$,

$\Leftrightarrow \Lambda(t)$ is sub-additive (super-additive).

Proof of Proposition 1

Consider for $t \geq 0$

$$\exp\{-\Lambda(t)\}\int_0^{\infty}\exp[-\Lambda(y)]\,dy - \int_t^{\infty}\exp[-\Lambda(y)]\,dy$$

$$= \exp\{-\Lambda(t)\}\int_0^{t}\exp[-\Lambda(y)]\,dy$$

$$- [1-\exp\{-\Lambda(t)\}]\int_t^{\infty}\exp[-\Lambda(y)]\,dy$$

$$\leq (\geq)\ \exp\{-\Lambda(t)\}\int_0^{t}\exp[-(y\Lambda(t))/t]\,dy$$

$$- [1 - \exp\{-\Lambda(t))\}]\int_t^{\infty}\exp[-(y\Lambda(t))/t]\,dy$$

since $\dfrac{\Lambda(y)}{y}$ is non-increasing (non-decreasing) in y,

$= 0.$

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