

A General composition Theorem and its Applications to Certain  
Partial Orderings of Distributions

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*Key Words and Phrases:* Likelihood ratio ordering,  $TP_2$  ordering, failure rate ordering, stochastic ordering, composition theorem for positively ordered functions, characterization of failure rate ordering.

## Abstract

A composition theorem for functions obeying certain positive ordering is proved. The novelty of the present version is that unlike earlier results which assume both components of the composition to be distributions or survival functions, one of the components is allowed to be negative and unbounded. The theorem is applied to yield very simple proofs of characterizations for failure rate orderings of distributions given recently by Capéraà (1988).

**Key Words.** Likelihood ratio ordering,  $TP_2$  ordering, failure rate ordering, stochastic ordering, composition theorem for positively ordered functions, characterizations of failure rate ordering.

# A General Composition Theorem and its Applications to Certain Partial Orderings of Distributions

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## 1 Introduction and Summary

Stochastic ordering of distributions has been an important tool in the theory of reliability and statistical inference in general. One of the earlier definitions of stochastic ordering was given by Lehmann (1955): distribution function  $F_2$  is said to be stochastically larger than  $F_1$  if  $F_2(z) \leq F_1(z)$  for every  $z$ , or equivalently, if  $\bar{F}_2(z) \geq \bar{F}_1(z)$  for every  $z$ , where  $\bar{F}_1 = 1 - F_1$  and  $\bar{F}_2 = 1 - F_2$  are corresponding *survival functions*. If  $X_1$  and  $X_2$  are random variables with

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distribution functions  $F_1$  and  $F_2$  respectively, then one of the basic properties of this ordering is that for every nondecreasing function  $g$ ,

$$E[g(X_2)] \geq E[g(X_1)].$$

In some cases a pair of distributions may satisfy a stronger condition called *likelihood ratio ordering*. Suppose distributions  $F_1$  and  $F_2$  possess densities  $f_1$  and  $f_2$  respectively. Then the condition required for *likelihood ratio ordering* is given by

$$\frac{f_2(z)}{f_1(z)} \text{ is nondecreasing in } z. \quad (1.1)$$

Condition (1.1) is related to  $TP_2$  functions. A nonnegative measurable function  $h(x, y)$  is said to be  $TP_2$  if

$$\begin{vmatrix} h(x_1, y_1) & h(x_1, y_2) \\ h(x_2, y_1) & h(x_2, y_2) \end{vmatrix} \geq 0 \text{ for every } x_1 \leq x_2 \text{ and } y_1 \leq y_2. \quad (1.2)$$

Let  $h(i, z)$  denote a probability density function  $f_i(z)$  for  $i = 1, 2$ . Then the monotone likelihood ratio condition is clearly seen to be equivalent to the condition  $h$  is  $TP_2$ .

Keilson and Sumita (KS) (1982) studied orderings which lie between likelihood ratio and stochastic ordering. These are defined by the condition

$$\frac{\overline{F}_2(z)}{\overline{F}_1(z)} \text{ is monotone increasing in } z \quad (1.3)$$

and

$$\frac{F_2(z)}{F_1(z)} \text{ is monotone increasing in } z. \quad (1.4)$$

Writing  $F(\theta, z)$  and  $\overline{F}(\theta, z)$  for  $F_\theta(z)$  and  $\overline{F}_\theta(z)$  respectively, for  $\theta = 1, 2$ , it follows that conditions (1.3) and (1.4) are equivalent to requiring  $F$  and  $\overline{F}$  to be  $TP_2$  functions. Later we will assume that  $\theta$  takes values in an open interval contained in  $R$ .

Observe that condition (1.3) implies that for every  $z$  and  $\delta > 0$ ,

$$\frac{\overline{F}_1(z)}{\overline{F}_1(z + \delta)} \geq \frac{\overline{F}_2(z)}{\overline{F}_2(z + \delta)},$$

which in turn implies that for every  $z$ ,

$$\frac{f_1(z)}{\overline{F}_1(z)} \geq \frac{f_2(z)}{\overline{F}_2(z)}. \quad (1.5)$$

Conversely if (1.5) holds then

$$-\frac{d}{dz} \ln \overline{F}_1(z) \geq -\frac{d}{dz} \ln \overline{F}_2(z),$$

so that (1.3) holds. Thus the ordering defined by (1.5) can be termed as *failure rate (fr) ordering*. Condition (1.4) can be given a similar interpretation. It is easy to see that (1.4) is equivalent to

$$\frac{f_1(z)}{F_1(z)} \leq \frac{f_2(z)}{F_2(z)}. \quad (1.6)$$

To interpret the ratios in (1.6), imagine that  $F_1$  is the life distribution of a component. Given that the component has failed by time  $z$ , then the probability that it survives up to time  $z - \delta$  is approximately  $\delta$  times the ratio appearing on the left side of the inequality (1.6). The corresponding ordering can be termed as *survival rate (sr) ordering*. Keilson and Sumita (1982) show that the  $TP_2$  ordering implies *fr* as well as *sr* ordering and it is easy to see that each of these two imply stochastic ordering. In the following we will write  $X_2 >_{fr} X_1$ , or  $F_2 >_{fr} F_1$ , to denote *fr* ordering. Similarly,  $X_2 >_{st} X_1$  will denote stochastic ordering,  $X_2 >_{sr} X_1$  will denote *sr* ordering and  $X_2 >_l X_1$  will denote likelihood ratio ordering.

Two recent articles derive results related to the above partial orderings. Capéraà (1988) derived several characterizations of *fr* and *sr* orderings and gave applications to computing asymptotic efficiency of rank tests. One of the features of the characterizations given by Capéraà is that they involve functions which are allowed to assume negative values. This feature widens the scope of applications of these orderings.

Lynch, Mimmack and Proschan (LMP) (1987) extend the composition result for  $TP_2$  density functions to corresponding results for distribution functions and survival functions. From these generalizations it follows that the *fr* ordering is preserved under the operation of convolution with a common distribution having the *increasing failure rate (IFR)* property. As a result, if  $F_i$  dominates  $G_i$  in *fr* ordering for  $i = 1, 2$  and if  $F_2$  and  $G_1$  are

IFR, then  $F_1 * F_2$  dominates  $G_1 * G_2$  in  $fr$  ordering. A similar preservation holds for  $sr$  ordering.

In this paper we achieve two main goals:

(1) We extend the results of LMP concerning the composition of  $TP_2$  functions. LMP assume both components of the composition are distribution or survival functions. We assume a more general first component; it may be unbounded or even assume negative values.

(2) We obtain the characterizations of Capéraà (1988) from our composition theorem as simple transparent corollaries, much shorter in proof than the lengthy Capéraà proof.

Remark. It is easy to verify that partial orderings discussed above are preserved under a common nondecreasing transformation. For example, if  $g$  is nondecreasing and  $X_2 >_{fr} X_1$  then  $g(X_2) >_{fr} g(X_1)$ .

## 2 A Preservation Theorem.

As noted above we plan to use the  $TP_2$  technique. However, due to the fact that we will be dealing with functions taking on negative values, many of the familiar results have to be extended to accommodate such functions. Fortunately, the tools needed for extensions to negative valued functions do hold.

It is important to take note of the difference between the statements one generally obtains when one is considering a pair of densities possessing the monotone likelihood ratio property where the pair is taken from a family of densities and the pair of functions we will be considering. In the case we deal with, the ratio is formed by functions, the numerator of which may take negative values while the denominator is assumed to be positive valued. Clearly for this case, the pair does not come from a family of density or distribution functions. What we use mainly is the  $TP_2$  like property given in relation (1.2).

Definition. A pair of measurable real functions,  $(g_1, g_2)$ , is said to satisfy the  $DP_2$  condition if

- (i)  $g_1$  is nonnegative while  $g_2$  may take negative values.
- (ii) for every  $x_1 \leq x_2$ ,

$$g_1(x_1)g_2(x_2) \geq g_1(x_2)g_2(x_1).$$

Here  $DP_2$  denotes the positivity of the second order determinant.

**Lemma 2.1** *Suppose that a pair  $(g_1, g_2)$  satisfies the  $DP_2$  condition. Let  $(a, b)$  and  $(c, d)$  be a pair of intervals such that  $a \leq c$  and  $b \leq d$ . Then*

$$\int_a^b g_1(x) dx \int_c^d g_2(x) dx \geq \int_a^b g_2(x) dx \int_c^d g_1(x) dx. \quad (2.1)$$

*Proof.* First assume that the intervals are disjoint. If for some  $x$ , both  $g_1$  and  $g_2$  take the value 0, then the contribution to both sides of (2.1) is 0. As a result we may assume that the set where both functions are simultaneously 0 has been removed. Now the  $DP_2$  condition implies that the ratio  $g_2(x)/g_1(x)$  is nondecreasing in  $x$ . This monotonicity together with the nonnegativity of  $g_1$  implies that either  $g_2(x)$  has the same sign for all  $x$  or there exists an  $x_- \leq x_+$  such that  $g_2$  is negative for  $x < x_-$  and positive for  $x > x_+$ . It also follows that the set  $\{x : g_1(x) > 0\}$  is an interval containing  $x_-$  and  $x_+$ . From these observations the inequality (2.1) is seen to be trivially true if either  $g_1(b)$  or  $g_1(c)$  is 0. Hence it will be assumed in the following that both of these numbers are positive.

Due to the assumption of  $DP_2$ , it follows that

$$g_1(c)g_2(x) \geq g_1(x)g_2(c) \quad \text{for every } x \in (c, d),$$

$$g_1(b)g_2(c) \geq g_1(c)g_2(b)$$

and

$$g_1(x)g_2(b) \geq g_1(b)g_2(x) \quad \text{for every } x \in (a, b).$$

Hence using these inequalities in succession, it follows that

$$\begin{aligned} g_1(b)g_1(c) \int_a^b g_1(x) dx \int_c^d g_2(x) dx &\geq g_1(b)g_2(c) \int_a^b g_1(x) dx \int_c^d g_1(x) dx \\ &\geq g_1(c)g_2(b) \int_a^b g_1(x) dx \int_c^d g_1(x) dx \\ &\geq g_1(c)g_1(b) \int_a^b g_2(x) dx \int_c^d g_1(x) dx. \end{aligned}$$

Since  $g_1(b)g_1(c) > 0$ , the desired inequality is established for the case of disjoint intervals.

To see that it extends to the case of overlapping intervals, suppose  $c < b$  so that the inequality to be proved becomes

$$\begin{aligned} & \left( \int_a^c g_1(x) dx + \int_c^b g_1(x) dx \right) \left( \int_c^b g_2(x) dx + \int_b^d g_2(x) dx \right) \\ & \geq \left( \int_a^c g_2(x) dx + \int_c^b g_2(x) dx \right) \left( \int_c^b g_1(x) dx + \int_b^d g_1(x) dx \right). \end{aligned}$$

Expanding the two sides and using the result for the disjoint intervals  $(a, c)$ ,  $(c, b)$ ,  $(b, d)$  the desired inequality follows.

Remark. Lemma 2.1 points out that the  $DP_2$  notion leads to what is known as a *local property*. If the pair of functions were positive then the *local property* would be equivalent to having the  $TP_2$  property for the measures generated by the pair of functions.

The main use of Lemma 2.1 will be made in the following:

**Corollary 2.1** *Suppose a pair of functions  $g_1, g_2$  satisfies the  $DP_2$  property, as in Lemma 2.1. Let  $h_i$  be the function obtained by convolving  $g_i$  with a uniform distribution; that is,*

$$h_i(x) = \frac{1}{2\theta} \int_{-\theta}^{\theta} g_i(x-u) du.$$

Then

- (i) *the pair  $h_1, h_2$  preserves the  $DP_2$  property.*
- (ii) *if  $g_i$  is increasing (decreasing) so is  $h_i$ .*

**Proof.** Suppose  $x_2 > x_1$ . Then the functions  $h_i$  evaluated at these points are integrals of the functions  $g_i$  over the intervals which satisfy the conditions in Lemma 2.1. The assertion (i) now follows from Lemma 2.1. The assertion (ii) follows from the fact that the intervals over which integrals are taken are of the same length and that one interval lies to the right of the other.

The motivation for considering the above result is that we may encounter functions which are discontinuous. Corollary 2.1 says that the smoothed version of the given functions will persist in having the  $DP_2$  property so that we can establish the results for the smoothed versions and then extend those to the general functions by letting  $\theta$  go to 0. For this purpose the following result is useful.



**Lemma 2.2** Suppose  $g$  is a nonnegative increasing function, integrable with respect to measure  $\mu$ , and let

$$h(x; \theta) = \frac{1}{\theta} \int_0^\theta g(x - u) du.$$

Then the sequence

$$\int h(x; 1/n) d\mu(x) \uparrow \int g(x) d\mu(x),$$

as  $n \rightarrow \infty$  on the positive integers.

In the above if  $g$  is decreasing, then by redefining  $h(x; \theta)$ , where convolution with the uniform distribution on  $(-\theta, 0)$  is used, it follows that monotone convergence of the sequence of integrals holds.

**Proof.** It is clear that  $h(x; \theta)$  is bounded above by the integrable function  $g(x)$  for every  $\theta$ . Further, for  $n > m$ ,  $h(x; 1/n) \geq h(x; 1/m)$  for every  $x$  so that by the monotone convergence theorem the first part of Lemma 2.2 follows. The same argument holds for the case where  $g$  is decreasing.

**Theorem 2.1** Let  $(g_1, g_2)$  be a pair of functions satisfying the  $DP_2$  property and the survival functions  $\overline{F}(\theta, t)$  be  $TP_2$  in  $(\theta, t)$ . Suppose that for  $i = 1, 2$ ,

$$\int g_i(t) dF_\theta(t) \text{ exists and is finite.}$$

Further suppose that  $g_1(t)$  is increasing in  $t$ . Then for  $i = 1, 2$ ,

$$h_i(\theta) = \int g_i(t) dF_\theta(t)$$

is  $DP_2$ , or equivalently,

$$\int g_1(t) dF_1(t) \int g_2(t) dF_2(t) \geq \int g_1(t) dF_2(t) \int g_2(t) dF_1(t) \quad (2.2)$$

**Proof.** Without loss of generality,  $\theta$  may be assumed to take values 1 and 2. Suppose  $F_i$  has density  $f_i$ , and  $g_i$  possesses derivative  $g_i'$  for  $i = 1, 2$ . Due to the assumption of monotonicity of  $g_1$ , the derivative  $g_1'$  will be nonnegative. Note that the assumed  $TP_2$  property for  $\overline{F}(\theta, t)$  implies

that the family of distribution functions  $F(\theta, \cdot)$  is stochastically ordered with respect to  $\theta$ .

To prove the theorem we have to verify that

$$D = \begin{vmatrix} \int g_1(t)f_1(t) dt & \int g_2(t)f_1(t) dt \\ \int g_1(t)f_2(t) dt & \int g_2(t)f_2(t) dt \end{vmatrix} \geq 0,$$

Using the basic composition formula the above relation can be rewritten as

$$D = \iint_{s < t} \begin{vmatrix} g_1(s) & g_1(t) \\ g_2(s) & g_2(t) \end{vmatrix} \begin{vmatrix} f_1(s) & f_2(s) \\ f_1(t) & f_2(t) \end{vmatrix} ds dt.$$

When the above determinants are expanded one gets four terms. When each term is integrated by parts with respect to  $t$ , where  $-\bar{F}_i$  is used as the integral of  $f_i$ , two terms arise. For example,

$$\begin{aligned} g_1(s)f_1(s) \int_s^\infty g_2(t)f_2(t) dt &= g_1(s)f_1(s)[-g_2(t)\bar{F}_2(t)|_s^\infty + \int_s^\infty g_2'(t)\bar{F}_2(t) dt] \\ &= g_1(s)f_1(s)[g_2(s)\bar{F}_2(s) + \int_s^\infty g_2'(t)\bar{F}_2(t) dt]. \end{aligned}$$

The determinant formed by the first terms is easily seen to be 0 while the one formed by the second terms yields

$$D = \iint_{s < t} \begin{vmatrix} g_1(s) & g_1'(t) \\ g_2(s) & g_2'(t) \end{vmatrix} \begin{vmatrix} \frac{f_1(s)}{\bar{F}_1(t)} & \frac{f_2(s)}{\bar{F}_2(t)} \end{vmatrix} ds dt.$$

To show that the first determinant is nonnegative, note that the  $DP_2$  property for the  $g$  functions implies that for every  $\delta > 0$

$$g_1(x)[g_2(x) - g_2(x - \delta)] \geq g_2(x)[g_1(x) - g_1(x - \delta)],$$

and hence,

$$g_1(x)g_2'(x) \geq g_2(x)g_1'(x).$$

Since  $g_1$  is nonnegative, the above inequality implies

$$g_1(t)g_2'(t)g_1(s) \geq g_2(t)g_1'(t)g_1(s) \geq g_2(s)g_1'(t)g_1(t),$$

where the last inequality follows from the  $DP_2$  property and the nonnegativity of  $g_1'$ .

Since the  $\overline{F}$  functions are  $TP_2$ , a property stronger than  $DP_2$ , the non-negativity of the second determinant follows easily by a similar argument. This completes the proof of the result when the  $g$  functions and  $\overline{F}$  are assumed to be differentiable.

Suppose now that the  $g_i$  are not continuous. By Lemma 2.1, it follows that the function  $g_1$  can be replaced by a smoothed version which satisfies the conditions of the theorem and then using Lemma 2.2 the required inequalities can be established by taking limits.

In order to replace  $g_2$  by a smooth version, first notice that the function  $g_1$  is nondecreasing, so that the set where  $g_1$  is 0 is of the form  $(-\infty, l)$ . Further the assumption regarding the monotonicity of the ratio  $g_2(x)/g_1(x)$  implies that  $g_2$  has to be nonpositive on this set. Consequently, on the set where  $g_1$  is 0, the contribution to the left side of (2.2) is 0 while the right side is nonpositive. Thus the right side of (2.2) is increased by making  $g_2$  equal to 0 whenever  $g_1$  is 0. Since the set where both are 0 can be ignored, we may assume that the ratio  $g_2(x)/g_1(x)$  is well defined everywhere.

Let  $g_3(x) = g_2(x)/g_1(x)$ . According to the assumption of the theorem,  $g_3$  is monotone increasing. As seen earlier, there exists an  $x_0$ , such that  $g_3$  is nonpositive for  $x \leq x_0$  and nonnegative for  $x \geq x_0$ . Considering the positive and negative parts of  $g_3$  it follows that one can find smoothed versions which are bounded above by these parts. Since  $g_1$  is nonnegative multiplying these by  $g_1$  we get a smoothed version of  $g_2$ . By an argument similar to the one used in Lemma 2.2, the convergence of the integrals will lead to the required result for discontinuous functions  $g_2$ .

The proof of the preservation theorem is now complete.

Remark. In Theorem 2.1 if  $F$  is assumed to have the  $TP_2$  property rather than  $\overline{F}$ , analogous results may be obtained for nonincreasing functions.

### 3 Characterizations of FR and SR Ordering

Recently Capéraà (1988) derived four characterizations of  $fr$  and  $sr$  orderings. It will be shown below that these characterizations follow easily from Theorem 3.1. From the paper by Capéraà (1988) it seems that condition (d) below is very useful in applications. Since the treatment of  $sr$  ordering is very similar we present the results only for  $fr$ . As before it is assumed that the random variables  $X_i$  have distribution functions  $F_i$  for  $i = 1, 2$ .

Theorem 3.1 (Capéraà) *The following conditions are equivalent.*

- (a)  $F_2 >_{fr} F_1$ .
- (b) For a pair of nondecreasing functions  $a$  and  $b$ , where  $b$  is nonnegative, if  $E[a(X_1)b(X_1)] = 0$ , then  $E[a(X_2)b(X_2)] \geq 0$ .
- (c) For every nonnegative nondecreasing function  $b$ , with finite expectations  $E[b(X_1)]$  and  $E[b(X_2)]$ , the distribution function  $H_2(\cdot, b)$  is stochastically larger than  $H_1(\cdot, b)$ , where

$$H_i(x, b) = \frac{\int_{-\infty}^x b(u) dF_i(u)}{E[b(X_i)]}$$

is a weighted distribution corresponding to  $F_i$  with weight function  $b$  for  $i=1,2$ .

- (d) Suppose  $\alpha$  and  $\beta$  are functions having finite expected values under  $F_1$  and  $F_2$ . Further suppose that  $\beta$  is nonnegative and  $\alpha/\beta$  and  $\beta$  are nondecreasing. Then

$$\frac{\int_{-\infty}^{\infty} \alpha(x) dF_2(x)}{\int_{-\infty}^{\infty} \beta(x) dF_2(x)} \geq \frac{\int_{-\infty}^{\infty} \alpha(x) dF_1(x)}{\int_{-\infty}^{\infty} \beta(x) dF_1(x)}.$$

**Proof of Theorem.** To show that (a) implies (d), suppose that the functions  $\alpha$  and  $\beta$  satisfy the monotonicity conditions stated in (d). Define  $g_1(t) = \beta(t)$  and  $g_2(t) = \alpha(t)$ . Then from the assumptions, the pair  $g_1, g_2$  is  $DP_2$ . Let  $F(2, t) = \overline{F}_2(t)$  and  $F(1, t) = \overline{F}_1(t)$ . Then the conditions of Theorem 2.1 are met and the resulting pair  $h_1, h_2$  is  $DP_2$ . This however is equivalent to the desired inequality.

To show the converse, suppose (d) holds and define

$$\beta(x) = \begin{cases} 1, & x > t_1 \\ 0, & x \leq t_1 \end{cases} \quad \alpha(x) = \begin{cases} 1, & x > t_2 \\ 0, & x \leq t_2, \end{cases}$$

where  $t_1 < t_2$ . Then

$$\frac{\overline{F}_2(t_2)}{\overline{F}_1(t_2)} \geq \frac{\overline{F}_2(t_1)}{\overline{F}_1(t_1)},$$

and hence  $F_2 >_{fr} F_1$ .

To show that (d) implies (c), we choose  $\beta(x) = b(x)$  and  $\alpha(x) = b(x)I_{(t, \infty)}(x)$ , where  $I$  is the indicator function. When the ratio in (d) is formed with this

choice of integrands and the integrations are carried out with respect to the measures defined by  $F_1$  and  $F_2$ , the inequality is equivalent to the one leading to the stochastic ordering.

The assertion that (c) implies (a) is proved easily (as pointed out by Capéraà (1988)) by taking  $b$  to be an indicator function of  $x > p$ , and then observing that the stochastic ordering of the resulting distributions is equivalent to the monotonicity of the ratio of the survival functions.

The proof that (b) implies (d) is quite simple and is given by Capéraà (1988). We give it here for completeness. Define

$$e = \frac{\int_{-\infty}^{\infty} \alpha(x) dF_2(x)}{\int_{-\infty}^{\infty} \beta(x) dF_2(x)}.$$

Then the inequality in (d) is equivalent to

$$\int_{-\infty}^{\infty} (\alpha(x) - e\beta(x)) dF_1(x) \geq 0. \quad (3.1)$$

Choose functions  $a$  and  $b$  in (b) as follows. Let

$$a(x) = \frac{\alpha(x)}{\beta(x)} - e$$

and let  $b(x)$  be  $\beta(x)$ . Then it is clear that  $E[a(X_1)b(X_1)] = 0$ . Hence (3.1) follows from (b) and hence (d) holds.

Finally, to show that (d) implies (b), suppose  $E[a(X_1)b(X_1)] = 0$ , where  $a$  is a nondecreasing function and  $b$  is a nonnegative nondecreasing function. Choose  $\beta(x) = b(x)$  and  $\alpha(x) = a(x)b(x)$ . Then  $\alpha$  and  $\beta$  satisfy the conditions in (d). However, in this case the right side of the inequality in (d) is 0 due to the assumed condition on  $X_1$ . Thus the inequality in (d) implies that

$$\int_{-\infty}^{\infty} \alpha(x) dF_2(x) \geq 0, \quad (3.2)$$

and hence  $E[a(X_2)b(X_2)] \geq 0$ , where  $X_2$  has distribution function  $F_2$ . This completes the proof of Theorem 3.1.

Remark. The inequality in (d) above was proved by Bickel and Lehmann (1975) under the more stringent condition of monotone likelihood ratio.

## 4 Application to MRL Ordering

Let

$$\mu_{F_i}(x) = \frac{\int_x^\infty \bar{F}_i(u) du}{\bar{F}_i(x)}$$

denote the mean residual life (mrl) function of  $F_i$ , the distribution of a non-negative random variable,  $i = 1, 2$ . Let  $\mu_i$  be the expected value corresponding to  $F_i$ . We say that  $F_2$  is larger than  $F_1$  in mrl ordering ( $F_2 >_{mrl} F_1$ ), if

$$\mu_{F_1}(x) \leq \mu_{F_2}(x) \text{ for every } x. \quad (4.1)$$

Let

$$\bar{H}_i(x) = \frac{\int_x^\infty \bar{F}_i(u) du}{\mu_i}. \quad (4.2)$$

Then inequality (4.1) is equivalent to

$$\frac{d}{dx} \log \bar{H}_2(x) \geq \frac{d}{dx} \log \bar{H}_1(x) \text{ for every } x.$$

Consequently, the *mrl* ordering is equivalent to

$$\frac{\bar{H}_2(x)}{\bar{H}_1(x)} \text{ is increasing in } x.$$

These  $H$  functions are called the *equilibrium survival functions*. Apart from a normalizing constant  $\mu_i$ , relation (4.2) reveals that  $\bar{F}_i$  plays the role of density function. From this consideration and from the discussion in Section 1 it follows that

$$F_2 >_{fr} F_1 \implies F_2 >_{mrl} F_1.$$

Since we are dealing with nonnegative random variables, the integrals in (d) of Theorem 3.1 are taken over 0 to  $\infty$ . Using condition (d) with this restriction on the  $H$  functions we see that the *mrl* ordering can be expressed as

$$\frac{\int_0^\infty \alpha(x) \bar{F}_2(x) dx}{\int_0^\infty \beta(x) \bar{F}_2(x) dx} \geq \frac{\int_0^\infty \alpha(x) \bar{F}_1(x) dx}{\int_0^\infty \beta(x) \bar{F}_1(x) dx}, \quad (4.3)$$

where the functions  $\alpha$  and  $\beta$  satisfy the same conditions as in (d) of Theorem 3.1. If we restrict  $\alpha$  to be nonnegative, then the condition  $\alpha/\beta$  nondecreasing would force  $\alpha$  to be nondecreasing. Define

$$\alpha^*(x) = \int_0^x \alpha(u) du$$

and

$$\beta^*(x) = \int_0^x \beta(u) du,$$

with  $\alpha^*(0) = \beta^*(0) = 0$ . Using integration by parts in (4.3), the inequality can be rewritten as

$$\frac{E\alpha^*(X_2)}{E\beta^*(X_2)} \geq \frac{E\alpha^*(X_1)}{E\beta^*(X_1)}, \quad (4.4)$$

where the random variables  $X_i$  are assumed to be distributed according to the distribution functions  $F_i$ . The inequality (4.4) may be taken as a criterion for *mrl* ordering where  $\alpha^*$  and  $\beta^*$  are nonnegative nondecreasing convex functions tending to 0 as  $x \rightarrow 0$  and the ratio of their derivatives is increasing.

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