incomplete 2-out-of-4 two-rail code [2]; and 2) a conventional TSC two-rail checker, which is inside the rectangle in Fig. 1. Note that the stuck-at faults at the input lines A, B, and C are equivalent to the noncodewords at the inputs, i.e. 000, 011, 101, 110, and 111.

Lemma: The proposed 1-out-of-3 code checker is a 3-FT PSFS/PSCD checker.

Proof: The lemma can be proved by considering four cases.

Case 1: The checker is fault-free. Table I shows that the checker maps codeword inputs into codeword outputs, and noncodeword inputs into noncodeword outputs.

Case 2: A single stuck-at fault is present in the checker. From Table II(a), it can be seen there are five stuck-at faults in the checker in the presence of which the checker produces correct codeword at the output, i.e., the faults are undetectable. The five faults are nodes 9 and 12 stuck-at-1, nodes 15, 16, and 20 stuck-at-0. Note that the fault at node 15 or node 16 stuck-at-0 is equivalent to node 20 stuck-at-0. In other words, the checker is TSC for all faults except the three faults: node 9 stuck-at-1, node 12 stuck-at-1, and node 15/16/20 stuck-at-0. Table II(b) shows the checker keeps CD property, and Tables II(a) and II(b) show that the checker satisfies the SFS and SCD properties respectively.

Case 3: The checker has an undetectable fault (shown in Table II(a)) *and another stuck-at-fault.* Since an undetectable fault may prevent a fault from being detected, it is necessary to verify the checker's behavior in the presence of such a fault combination. Table III(a) records the checker's response in the presence of node 9 stuck-at-1 and another single fault. It can be seen that node 2 stuck-at-0 is undetectable in this situation. It can be also shown that node 4 stuck-at-0 becomes undetectable in the presence of node 12 stuck-at-1. Table III(b) shows that the checker keeps its CD property in the presence of a sequence of two faults. It is easy to verify that the checker keeps its SFS/SCD properties under any other combination of two faults.

Case 4: Any combination of two undetectable faults and another single stuck-at fault are present in the checker. Table IV records the checker's response in the presence of the fault sequence (node 9 stuck-at-1, node 2 stuck-at-0, and any other single stuck-at-fault). As can be seen from the table, four more faults (nodes 3, 4, 7, and 14 stuck-at-1) remain undetectable in the presence of the fault sequence. It can be shown that four more faults (nodes 1, 2, 6, and 13 stuck-at-1) remain undetectable in the presence of the fault sequence (node 12 stuck-at-1 and node 4 stuck-at-0). However, the checker still keeps its SCD and SFS properties in the presence of these fault sequences, i.e., the checker is in safe states. On the other hand, the checker is in an unsafe state, as shown in Table IV(b), in the presence of fault sequence of node 9 stuck-at-1, node 2 stuck-at-0, and node 20 stuck-at-0. It can be easily verified that there are only two unsafe states in this case. Another unsafe state occurs for the fault sequence of node 12 stuck-at-1, node 4 stuck-at-0, and node 20 stuck-at-0.

From the preceding, it is clear that the checker is the 3-FT PSFS/PSCD checker. □

IV. Conclusion

A 1-out-of-3 code checker that satisfies the PSFS/PSCD property has been presented. The proposed checker can tolerate three faults ($k = 3$) compared to two faults in [5]. The probability of the checker being in an unsafe state is extremely low. It has been claimed that the checker proposed in [2] is the most efficient in terms of number of gates ($= 20$). However, it uses six levels of gates, and has more input lines than the required three input lines. The checker proposed in this paper uses only eight gates in three levels, and the number of input lines is three.

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A New Family of Bridged and Twisted Hypercubes

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Abstract—We show that by adding eight extra edges, referred to as bridges, to an n-cube ($n > 4$) its diameter can be reduced by 2, and by adding sixteen bridges to an n-cube ($n \geq 6$) its diameter can be reduced by 3. We also show that by adding $\binom{4m}{m-1}+1$ ($m \geq 2$) bridges to an n-cube ($n \geq 4m$ and $n \geq 8$) its diameter can be reduced by $2m$ and by adding $2\binom{4m-3}{m}+1$, ($m > 2$) to an n-cube ($n \geq 4m-2$ and $n \geq 10$) its diameter can be reduced by $2m-1$. We also consider the reduction of diameter of an n-cube by exchanging some independent edges (*twisting*), where two edges are called independent if they are not incident on a common node. We have shown that by exchanging four pairs of independent edges in a d-cube ($d \geq 5$), we can reduce its diameter by 2. By exchanging sixteen pairs of independent edges, the diameter of a d-cube ($d \geq 7$) can be reduced by 3. By exchanging 57 pairs of independent edges, the diameter can be reduced by 4 for $d > 9$. To reduce the diameter by $\lfloor d/2 \rfloor$, ($d \geq 10$) we need to exchange $\binom{d-1}{r-1}$ pairs of independent edges, where $r = \lfloor d/4 \rfloor - 1$.

***Index Terms*—Bridge, diameter, hypercube, routing, twist.**

I. Introduction

The hypercube interconnection scheme is a very popular network topology. An n-dimensional hypercube Q_n consists of $N = 2^n$ nodes interconnected as follows: 1) each node is labeled by an n-bit binary

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number $(a_1 a_2 \cdots a_n)$, 2) two nodes are connected by an edge if and only if their binary labels differ in exactly one bit position. The n-cube has become an interesting topic of research in recent years given its versatile applications in parallel and distributed processing. Many interesting properties of the n-cube have been reported in the literature [1]–[3].

Esfahanian *et al.* [4] have shown that by adding two new edges, the diameter of an n-cube ($n \geq 2$), can be reduced by 1. In [5], it has been shown that by adding $\binom{4m}{m}$ extra edges to a $4m$-dimensional cube ($m \geq 2$), its diameter can be reduced by $2m - 1$. In [6], the effect of adding some extra edges on the performance measures, such as diameter, mean inter-node distance, traffic density, etc. have been discussed.

Two edges of a hypercube are called independent if they are not incident on a common node. The idea of reducing diameter by twisting was first introduced by Hilberts *et al.* [7]. They have shown that by exchanging $(d-1)2^{d-4}$ link pairs in a d-cube ($d = 2m+1$), its diameter can be reduced to $(d+1)/2$. In [4], it has been shown that by exchanging a pair of independent edges of an n-cube ($n \geq 3$), known as twisting, its diameter can be reduced by 1. In [8], performance measures of such twisted cubes have been studied.

In this brief contribution, we have proposed a new family of network topologies, by modifying the original hypercube structure, which will have diameters lesser than that of a hypercube but still retaining the other desirable features of the hypercube, e.g., ease of routing under fault-free and faulty situations, high connectivity, i.e., high degree of fault-tolerance, and so on. Two possible approaches have been considered for this, one involving addition of a few extra edges called bridges, and the other involving exchange of pairs of independent links or twists without the need of extra edges. We first show that by adding $\binom{d}{\lfloor d/4 \rfloor + 1} + 1$ extra edges, termed as *bridges*, to a d-cube ($d > 4$), its diameter can be reduced by $\lfloor d/2 \rfloor$. Then we generalize this scheme to add $\binom{4m}{m+1} + 1$ ($m > 2$) bridges to an n-cube ($n > 4m, n > 8$) to reduce its diameter by $2m$. To reduce the diameter by $2m - 1$, we add $2\binom{4m-3}{m} + 1$ ($m > 2$) bridges to an n-cube ($n > 4m-2, n > 10$). We have also given an algorithm for routing in the bridged hypercube Q_d. Here, the routing ensures path length less than or equal to the diameter in the bridged hypercube Q_d. Here, the routing ensures path length less than or equal to the diameter without much overhead. Routing in other cases can be similarly dealt with.

Next, we have shown that by exchanging four pairs of independent edges in a d-cube ($d \geq 5$), we can reduce its diameter by 2. By exchanging 16 pairs of independent edges, the diameter of a d-cube ($d \geq 7$) can be reduced by 3. By exchanging 57 pairs of independent edges, the diameter can be reduced by four for $d \geq 9$. To reduce the diameter by $\lfloor d/2 \rfloor$, where $d \geq 10$, we need to exchange $\binom{d-1}{r} + 1$ pairs of independent edges where $r = \lfloor d/4 \rfloor + 1$. In [7], one type of twisted hypercube with lower diameter has been developed. But there the number of link pairs exchanged is much more. Starting with a d-cube, where $d = 2m+1, (d-1)2^{d-4}$ link pairs are exchanged to get a graph of diameter $(d+1)/2$. Accordingly, in an 11-cube, one needs to exchange $10.2^7 = 1280$ link pairs to reduce its diameter to 6 by the method given in [7]. In our scheme we need only 121 link pairs to be exchanged in an 11-cube to get a graph of diameter 6.

We introduce a few definitions and notations in Section II. In Section III we show how the extra edges, referred to as bridges, can be connected to a d-dimensional hypercube for reducing the diameter to $\lceil d/2 \rceil$. Section IV deals with a generalization of the idea of adding bridges for reducing the diameter of a hypercube by any given value. We discuss twisted hypercubes in Section V and routing in bridged and twisted hypercubes in Section VI.

II. Notations and Terminologies

A node in a hypercube is represented by a string of binary digits. A subcube is represented by a string over the alphabet $\{0, 1, *\}$. For example, 01*1* represents the subcube formed by the nodes 01010, 01011, 01110, 01111. In order to make the representation more compact we would replace p consecutive occurrences of a given symbol by that symbol raised to the power of p. For example, 00011 will be written as $0^3 1^2$. Let $f(x)$ be the number of ones in the binary representation of a node x. We now define the following sets:

$$W_r = \{x | f(x) = r\}$$
$$A_r = \{x | x \in W_r \text{ and } x \in *^{n-1}0\}$$
$$Br = \{x | x \in W_r \text{ and } x \in *^{n-1}1\}.$$

Also let $\mathrm{HD}(x, y)$ denote the Hamming distance between two nodes x and y.

If a node x is represented by $x = a_1 a_2 \cdots a_n, a_i \in \{0, 1\}$, a node diametrically opposite to x is denoted by $\overline{x} = \overline{a}_1 \overline{a}_2 \cdots \overline{a}_n$.

For a source destination pair (s, t) where $s = a_1 a_2 \cdots a_n$ and $t = b_1 b_2 \cdots b_n$ we define four sets of bit positions S_{00}, S_{01}, S_{10}, and S_{11} as follows:

$S_{00} = \{h | a_h = 0, b_h = 0\}$ $S_{01} = \{h | a_h = 0, b_h = 1\}$ $S_{10} = \{h | a_h = 1, b_h = 0\}$ $S_{11} = \{h | a_h = 1, b_h = 1\}$

Example: Let $s = 01001001$ and $t = 10011011$. Then $S_{00} = \{3, 6\}, S_{01} = \{1, 4, 7\}, S_{10} = \{2\}$ and $S_{11} = \{5, 8\}$.

We denote the cardinalities of the sets S_{00}, S_{01}, S_{10}, and S_{11} by s_0, s_1, s_2, and s_3 respectively. Hence, the number of zeros in $s = (s_0 + s_1)$ and the number of ones $= s_2 + s_3$. Also $\mathrm{HD}(s, t) = s_1 + s_2$.

III. Bridged d-Cube with Diameter $\lceil d/2 \rceil$

The extra edges that we propose to add to a cube are always in the form of $(x, \overline{x})$. We refer to these extra edges as bridges. In a d-dimensional cube ($d > 4$), we take a fixed value of $r, 0 < r < d/2$, and then connect bridges from all the nodes in $W_r \cup W_0$. The cardinality of the set W_r is $\binom{d}{r}$. Hence, the number of such bridges that will be added to the hypercube is $\binom{d}{r} - 1$. To find an expression for the diameter of the bridged hypercube, we first try to unfold the possible paths between two nodes s and t through a bridge.

Let x be a node at which a bridge is connected. To reach t from s, we can reach x by changing some bits in s. Let $S'_{ij} \subseteq S_{ij}, i, j \in \{0, 1\}$, be the sets of bit positions where these changes are made. Let s'_0, s'_1, s'_2, and s'_3 be the cardinalities of the sets $S'_{00}, S'_{01}, S'_{10}$, and S'_{11} respectively. Then we have the following lemma.

Lemma 1: The shortest path between two points s and t, via the bridge $(x, \overline{x})$ is of length $p = 1 + 2(s'_1 + s'_2) + s_0 + s_3$.

Proof: From s, we reach x in $(s'_0 + s'_1 + s'_2 + s'_3)$ steps. Another single step via the bridge leads to the node $\overline{x}$. $\overline{x}$ differs from t in all bits of $S'_{01}, S'_{10}, S_{00} - S'_{00}$ and $S_{11} - S'_{11}$ and nothing else. Hence $p = (s'_0 + s'_1 + s'_2 + s'_3) + 1 + (s'_1 + s'_2 + s_0 - s'_0 + s_3 - s'_3) = 1 + 2(s'_1 + s'_2) + s_0 + s_3$.

Remark: To get a shorter path length between s and t, we would change as few bits in S_{01} and S_{10} as possible.

Lemma 2: There exists a path of length $p_0 = d + 1 + s_2 - s_1$ between s and t, via the bridge connected to the node in W_0.

Proof: Starting from s, we reach the node in W_0 by changing all bits in S_{10} and S_{11}. By lemmma 1, $p_0 = 1 + s_0 + s_3 + 2s_2$. Putting $s_0 + s_3 = d - (s_1 + s_2)$ we get $p_0 = d + 1 + s_2 - s_1$.

Lemma 3: For $s_1 > d - r$ there exists a path of length $p_1 = -d + 2r - 1 + s_1 - s_2$, between s and t, via a bridge connected to some node in B_r.

Proof: $s_1 > d - r$ implies $r > d - s_1$. If we want to reach a node in B_r from s, where $r > d - s_1 = s_0 + s_2 + s_3$, we must change all bits in S_{00} to 1 and also some more bits in S_{01}. The number of

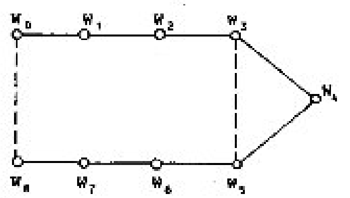

Fig. 1. Eight-cube.

bits in S_{01} to be changed is equal to $s_1' = r-(d-s_1) = s_1-(d-r)$. Note that if the dth bit is in S_{01}, these s_1' bits must include the dth bit to reach a node in B_r. The path length $p_1 = 1+2s_1'-s_0+s_3$ (by Lemma 1). Putting $s_0+s_3 = d-(s_1+s_2)$ and $s_1' = s_1-(d-r)$, we get $p_1 = 1+2r-d+s_1-s_2$.

Lemma 4: For $s_1 \leq d-r$ and $s_2 \leq r$, there exists a path of length $p = 1+s_0+s_3$ between s and t, via a bridge connected to a node in W_r.

Proof:

Case I: $s_2+s_3 \leq r$, i.e., the number of ones in s is less than or equal to r. The number of zeros in a node in W_r is greater than or equal to s_1, since $s_1 \leq d-r$, by the given condition. Hence starting from s, we can reach a node in W_r by changing all (for $s_1 = d-r$) or some (for $s_1 < d-r$) bits in S_{00}. By Lemma 1, $p = 1+s_0+s_3$.

Case II: $s_2+s_3 > r$. By the given condition $s_2 \leq r$, i.e., the number of ones in a node in W_r is greater than or equal to s_2. Hence starting from s, we can reach a node in W_r by changing some bits in S_{11} only. By Lemma 1, path length $p = 1+s_0+s_3$.

Lemma 5: For $s_2 \leq d-r$ and $s_1 \leq r$, there exists a path of length $p = 1+s_0+s_3$, between s and t, via a bridge connected to a node in W_r.

Proof: Similar to Lemma 4, except that we start from t.

Lemma 6: For $r < s_1 < d-r$ and $r < s_2 < d-r$, there exists a path of length $p = d-2r+1+s_2-s_1$ via a bridge connected to a node in W_r.

Proof: Since $s_2 > r$, the number of ones in a node in W_r is less than that in s. Thus to reach a node in W_r from s we have to change some "1" bits to "0". We can change all bits in S_{11} and some $s_2' = s_2-r$ bits in S_{10} to reach a node in W_r. By Lemma 1, $p = 1+s_0+s_3+2s_2'$. Putting $s_0+s_3 = d-(s_1+s_2)$ and $s_2' = s_2-r$ we get $p = 1+d-2r+s_2-s_1$.

Theorem 1: A bridged hypercube of dimension d ($d \geq 8$), where bridges are connected to all the nodes in $W_0 \cup W_r$, has diameter equal to $\lceil d/2 \rceil$, where $r = \lfloor d/4 \rfloor + 1$.

Proof: We can assume w.l.o.g. (without loss of generality) that $s_2 \leq s_1$. Also we need to consider only those s, t for which $\mathrm{HD}(s,t) > \lceil d/2 \rceil$, i.e., $s_0+s_3 \leq \lceil d/2 \rceil - 1$. There can be three possible cases.

Case 1: $s_1 > d-r$.

Case 2: $r < s_1 \leq d-r$.

Case 3: $s_1 \leq r$.

Case 2 can, however, be divided into two subcases:

a) $s_1 \leq d-r, s_2 \leq r$;

b) $r < s_1 < d-r, s_2 > r$.

Note that the case $s_1 = d-r$ and $s_2 > r$ cannot arise, since in that case $s_1+s_2 > d-r+r = d$, but the total number of bits is only d.

Case 1: By Lemma 2, $p_0 = d+1+s_2-s_1$ and by Lemma 3, $p_1 = 1+2r-d+s_1-s_2$. Hence, $\min(p_0, p_1) < (p_0+p_1)/2 = r+1 \leq \lceil d/2 \rceil$.

Case 2:

a) By Lemma 4, the path length $p = 1+s_0+s_3 \leq \lceil d/2 \rceil$. When $s_0+s_3 = \lceil d/2 \rceil - 1$, the path length $p = \lceil d/2 \rceil \leq \mathrm{HD}(s,t)$.

TABLE I
NUMBERS OF BRIDGES ADDED

d	$\binom{d}{r}+1$	$\binom{d}{r}+1 : d \cdot 2^{d-1}$
6	16	.083
8	57	.05
10	121	.02

b) By Lemma 6, $p = d-2r+1+s_2-s_1 < d-2r+1$. For $r = \lfloor d/4 \rfloor + 1, p \leq \lceil d/2 \rceil$.

Case 3: By Lemma 5, the path length $p = 1+s_0+s_3 \leq \lceil d/2 \rceil$. Hence the proof.

Example: An illustrative example is given in Fig. 1 with $d = 8$, where the solid lines represent the already existing lines and the dotted lines represent the bridges.

The number of extra edges that we add is small compared to the total number of edges in the d-cube and as we increase d, the ratio $\binom{d}{r}+1 : d.2^{d-1}$ becomes smaller, where $r = \lfloor d/4 \rfloor + 1$. The values of this ratio for a few values of d are given in Table I.

IV. REDUCTION OF DIAMETER OF A d-CUBE BY ANY VALUE $k \leq \lfloor d/2 \rfloor$

In the previous section, we have considered a bridged hypercube whose diameter is reduced to half of its dimension. Now we will consider how to reduce the diameter of a d-cube by some $k \leq \lfloor d/2 \rfloor$.

A. Reduction by an Even Value

First we consider the case when k is even. Assuming that there are bridges connected to all the nodes in W_r, for a given value of $r, 0 < r < d/2$, we state some more properties (without proof) regarding paths between s and t.

Lemma 7: For $s_2 \geq r$ and the dth bit in S_{01} there is a path of length $p_0 = d-2r-1+s_2-s_1$, between s and t via a bridge connected to a node in A_r.

Lemma 8: For $s_1 \geq r$ and the dth bit in S_{10} there is a path of length $p_0 = d-2r+1+s_1-s_2$, between s and t via a bridge connected to a node in A_r.

Lemma 9: For $s_2 \geq r$ and the dth bit in S_{11} or S_{00} there is a path of length $p_0 = d-2r+1+s_2-s_1$, between s and t via a bridge connected to a node in A_r.

Lemma 10: For $s_1 \geq r$ and the dth bit in S_{01} there is a path of length $p_1 = d-2r+1+s_1-s_2$, between s and t via a bridge connected to a node in B_r.

Lemma 11: For $s_2 \geq r$ and the dth bit in S_{10} there is a path of length $p_1 = d-2r-1+s_2-s_1$, between s and t via a bridge connected to a node in B_r.

Lemma 12: For $s_1 \geq r-1$ and the dth bit in S_{00} or S_{11}, there is a path of length $p_1 = d-2r+3+s_1-s_2$ via a bridge connected to B_r.

Theorem 2: By adding eight extra edges to an n-cube ($n \geq 4$), its diameter can be reduced by 2.

Proof:

Case I: $n = 4$. Let us connect all diametrically opposite pairs by bridges. For this eight extra edges will be added to the cube. Let us take any two points s and t. Let p be the length of the shortest path between s and t. To show that diameter of this bridged 4-cube is 2 we consider only those (s,t) for which $\mathrm{HD}(s,t) > 2$. If $\mathrm{HD}(s,t) = 3$, then $\mathrm{HD}(s,\bar{t}) = 1$ and hence $p = 2$. If $\mathrm{HD}(s,t) = 4$, then $t = \bar{s}$ and hence $p = 1$.

Case II: $n > 4$. We take a four-dimensional subcube $C_3 = 0^{n-4}*^4$. We connect bridges within this subcube as in Case I, i.e., the bridges are of the form $0^{n-4}x$ to $0^{n-4}\bar{x}$, where x is a binary

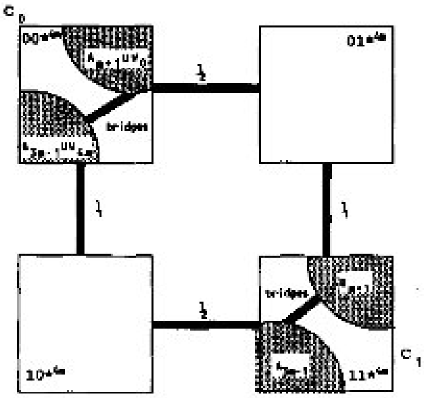

Fig. 2. Connecting bridges in a $(4m+2)$-cube.

string of length 4. Take two nodes $s = ax$ and $t = by$, where x and y are binary strings of length 4, and a and b are binary strings of length $n-4$. Let f be the number of bit positions at which s and t are same. To show that diameter of this bridged n-cube is $n-2$, we consider only those (s,t) for which $\mathrm{HD}(s,t) > n-2$, i.e., $f \le 1$. If $f = 0$, then $x = \overline{y}$. The shortest path between x and y is through C_0 and is of length $n - 4 + 1 = n - 3$. If $f = 1$, there can be two possible cases:

a) If the bit position at which s and t are having the same value is within the last four bits, then shortest path between $0^n x$ and $0^n y$ is of length 2 (by case 1). So the shortest path between s and t is through C_0 and is of length $n - 4 + 2 = n - 2$.
b) If the bit position at which s and t are having the same value is within the first $n-4$ bits, $x = \overline{y}$. The shortest path between s and t is through C_0 and is of length $n-4$ or $n-2$ depending on whether the bit value is 0 or 1.

We generalize this idea to reduce the diameter of a hypercube by some even number $2m (m \ge 2)$ as follows. We take an n'-cube $(n' = n + 4m)$. We take two subcubes $C_0 = 0^n *^{4m}$ and $C_1 = 1^n *^{4m}$. In C_0, we add bridges of the form $(0^n x, 0^n \overline{x})$, where x is a bit string of length $4m$, and in C_1 the bridges are of the form $(1^n x, 1^n \overline{x})$. For C_0, C_1 we define the sets W_r, A_r, B_r as for a $4m$-dimensional cube by ignoring the leftmost n bits of the node representation. In C_0, we connect bridges from all the nodes in A_{m+1} and W_0. In C_1, we connect bridges from all the nodes in B_{m+1}. For example see Fig. 2 when $n = 2$.

Theorem 3: In an n'-cube $(n' = n + 4m, m \ge 2)$, by adding $\binom{4m}{m-1} + 1$ extra edges in the above-mentioned way, we can reduce its diameter by $2m$.

Proof: We can represent any two nodes s and t in this n'-cube as $s = a_1 a_2 \cdots a_n a_{n+1} \cdots a_{n+4m}$ and $t = b_1 b_2 \cdots b_n b_{n+2} \cdots b_{n+4m}$. We define two strings x and y as $x = a_{n+1} a_{n+2} \cdots a_{n+4m}$ and $y = b_{n+1} b_{n+2} \cdots b_{n+4m}$. Let p_0 be the length of a path between $0^n x$ and $0^n y$ in C_0 and p_1 be that between $1^n x$ and $1^n y$ in C_1.

Among the first n bits from left, let p be the number of bits that are 0 in both s and t, and q be the number of bits that are 1 in both s and t. We define s_0, s_1, s_2, and s_3 for the two strings x and y as before. We consider only those (s,t) for which $\mathrm{HD}(s,t) > n + 2m$, i.e., $p + q + s_0 + s_3 \le 2m - 1$. We now consider two paths between s and t, one using a bridge in C_0 and the other using a bridge in C_1. Let the path through C_i be of length P_i, $i \in \{0, 1\}$.

We can easily verify that the total number of bits to be changed to get $0^n x$ from s and t from $0^n y$ is equal to $(n-p-q)+2q = n-p+q$. Hence, $P_0 = n - p + q + p_0$. Similarly $P_1 = n + p - q + p_1$. Now, $\min(P_0, P_1) \le (P_0 + P_1)/2 = n + (p_0 + p_1)/2$.

In finding p_0, p_1 we apply the lemmas where $d = 4m$ and $r = m + 1$. There can be four possible cases as in Theorem 1.

Case 1: $s_1 > d - r = 3m - 1$
Case 2a: $s_1 \le d - r = 3m - 1, s_2 \le r = m + 1$
Case 2b: $m + 1 = r < s_1 < d - r = 3m - 1, s_2 > r = m + 1$
Case 3: $s_1 \le r = m + 1$

Case 1: By Lemma 2, $p_0 = 4m + 1 + s_2 - s_1$ and by Lemma 3, $p_1 = -2m + 3 + s_1 - s_2$. Hence $\min(P_0, P_1) \le n + m + 2 \le n + 2m$ since $m \ge 2$.

Case 2a: By Lemma 4, at least one of p_0, p_1 is equal to $1 + s_0 + s_3$. Hence, at least one of P_0, P_1 is less than or equal to $n + p + q + s_0 + s_3 + 1 < n + 2m$ (as $p + q + s_0 + s_3 \le 2m - 1$).

When $p = q = 0$ and $s_0 + s_3 = 2m - 1$, either P_0 or $P_1 = n + s_0 + s_3 + 1 = n + 2m < \mathrm{HD}(s,t)$.

Case 2b: If the $(n+4m)$th bit is in S_{00} or S_{11}, $p_0 = 2m - 1 + s_2 - s_1$ (by Lemma 9) and $p_1 = 2m + 1 + s_1 - s_2$ (by Lemma 12). Hence, $\min(P_0, P_1) \le n + 2m$.

If the $(n+4m)$th bit is in S_{01}, $p_0 = 2m - 1 + s_2 - s_1$ (by Lemma 7) and $p_1 = 2m - 1 + s_1 - s_2$ (by Lemma 10). Hence, $\min(P_0, P_1) \le n + 2m - 1$.

If the $(n+4m)$th bit is in S_{10}, $p_0 = 2m - 1 + s_1 - s_2$ (by Lemma 8) and $p_1 = 2m - 1 + s_2 - s_1$ (by Lemma 11). Hence, $\min(P_0, P_1) \le n + 2m - 1$.

Case 3: If $s_2 \le 3m - 1$, by Lemma 5, one of p_0, p_1 is equal to $1 + s_0 + s_3$. Hence, at least one of P_0, P_1 is less than or equal to $n + p + q + s_0 + s_3 + 1 \le n + 2m$ (as $p + q + s_0 + s_3 \le 2m - 1$).

If $s_2 > 3m - 1$, the case is similar to case 1 if we interchange s and t, hence the proof.

B. Reduction by an Odd Value

To reduce the diameter by an odd number, say $2m - 1 (m \ge 2)$, we take an $n' = n + 4m - 2$-dimensional cube. We add bridges to the subcubes $C_0 = 0^n *^{4m-2}$ and $C_1 = 1^n *^{4m-2}$ as before. Now we will see that this construction does not give a graph of diameter $n + 2m - 1$ for $m > 2$. But for $m = 2$ this construction gives a graph of diameter $n + 2m - 1$. To do that, in Theorem 3 we substitute d by $4m - 2$ and r by m. We note that path length between two points s and t is at most $n + 2m - 1$ in all cases except in Case 2b when the $(n + 4m - 2)$th bit is in S_{00} or S_{11}.

In Case 2b), when the last bit is in S_{00} or S_{11}

$$P_0 = n - p + q + 2m - 1 + s_2 - s_1$$
$$P_1 = n + p - q + 2m + 1 + s_1 - s_2.$$

If $p - q < s_2 - s_1 - 1$, then $P_1 < n + 2m$ and if $p - q > s_2 - s_1 - 1$, then $P_0 < n + 2m$. $P_1 = P_0 = n + 2m$ only if $p - q = s_2 - s_1 - 1 \cdots (a)$.

For $\mathrm{HD}(x,y) > n + 2m - 1 = n + 3$, $p + q + s_0 + s_3 \le 2$. Also $s_0 + s_3 \ge 1$, since the $(n + 4m - 2)$th bit is in S_{00} or S_{11}. Hence, $p + q \le 1$. Then $p + q$ can be 1 or zero. If $p + q = 1$, $p - q$ is an odd number. But $s_0 + s_3$ is equal to 1. Hence, $s_2 - s_1$ is an odd number. So condition (a) does not hold. If $p + q = 0$, i.e., $p = q = 0$ we can take either of the paths through C_0, which are of lengths $n + 2m - 1 + s_2 - s_1$ and $n + 2m - 1 + s_1 - s_2$.

To reduce the diameter of a hypercube by an odd number $2m - 1 (m > 2)$ we make a slight modification to our previous scheme. We describe this scheme as follows. We take $n' (n' = n + 4m - 2)$ cube. We take two subcubes $C_0 = 0^n *^{4m-2}$ and $C_1 = 1^n *^{4m-2}$. In C_0, we add bridges of the form $(0^n x, 0^n \overline{x})$, where x is a bit string of length $4m-2$, and in C_1 the bridges are of the form $(1^n x, 1^n \overline{x})$. For C_0, C_1 we define the set W_r, A_r, and B_r as for a $4m-2$-dimensional cube by ignoring the leftmost n bits of the node representation. In C_0, we connect bridges from all the nodes in A_m and W_0. In C_1, we connect bridges from all the nodes in B_{m+1}.

Theorem 4: In an n'-cube, where $n' = n + 4m - 2$ $(m > 2)$, by adding $2\binom{4m-3}{m} + 1$ bridges in the above-mentioned way we can reduce its diameter by $2m - 1$.

Proof: The proof is similar to that of Theorem 3.

V. Twisted Hypercubes

We consider reduction of diameter of a d-cube, by exchanging a few pairs of independent links. Consider two independent links (u, v) and (x, y) of the cube. If the links (u, x) and (v, y) are not present in the cube, then to exchange two independent links (u, v) and (x, y), we delete these links and connect (u, x) and (v, y). In this way, the degree of each node remains the same. We call such an exchange of a pair of links a twist. In [4] only 4-cycle twists are considered, where the nodes u, v, x, and y form a four-cycle in the hypercube. There, it is shown that a single four-cycle twist can reduce the diameter of an n-cube $(n > 2)$ by 1. We consider, in general, reduction in diameter by any given value $k, 1 < k < m$, where m depends on the dimension of the cube. First, we show that, by making four twists in a d-cube $(d \geq 5)$, its diameter can be reduced by 2.

Definition: A twist of type r applied on a d-cube consists of the following operation. Let y be a bit string of length $(d-1)$. For all $y0 \in A_r$, we delete the link pair $(y0, y1)$ and $(\overline{y}0, \overline{y}1)$ and connect $(y0, \overline{y}1)$ and $(y1, \overline{y}0)$. We denote the twist of type r by T_r. The total number of link pairs exchanged by such twist is equal to $\binom{d-1}{r}$

Lemma 13: If we delete links of the type $(y0, y1)$ where $y0 \in A_r$, for some given r, then the path length between any two points s and t is $\mathrm{HD}(s,t)$, provided $\mathrm{HD}\ (s,t) > 1$.

Proof: Suppose there is a path in the original cube from s to t as $s \to a_1 \to a_2 \to \cdots \to a_i \to a_j \to \cdots \to a_k \to t$ of length $\mathrm{HD}(s,t)$.

Suppose the link (a_i, a_j) is deleted. For $\mathrm{HD}(s,t) > 1$, $s = a_i$ and $t = a_j$ is not possible. We can assume w.l.o.g. that $s \neq a_i$. We can take an alternative path by changing the rightmost bit in s first, and then changing the required bits.

Remark: If we delete links of type $(y0, y1), y0 \in A_r$ for $r = r_1, r_2, \cdots, r_k$ where

$$r_2 - r_1 > 1$$
$$r_3 - r_2 > 1$$
$$\cdots$$
$$r_k - r_{k-1} > 1,$$

then for two points s and t, where $\mathrm{HD}(s,t) > 1$, we have a path between s and t of length $\mathrm{HD}(s,t)$. When $\mathrm{HD}(s,t) = 1$ and the link (s,t) is deleted, let $s = y0$ and $t = y1$. There exists a node $y_1 0$ such that $\mathrm{HD}(s, y_1 0) = 1$ and the link $(y_1 0, y_1 1)$ is intact. Now we can get a path $s \to y_1 0 \to y_1 1 \to t$, which is of length 3.

Lemma 14: If we apply a type 0 twist on a four-cube its diameter becomes equal to 3.

Proof: Fig 3 shows a four-cube with twist of type 0, where new edges are shown by dotted lines. Let s and t be any two nodes in the above cube. If $\mathrm{HD}(s,t) > 1$, there always exists a path of length $\mathrm{HD}(s,t)$ after the twist. To show that the diameter of this bridged cube in equal to 3, we need to consider only the cases where $\mathrm{HD}(s,t) = 1$ or 4. We can assume w.l.o.g. that s is in $A_k, 0 \leq k \leq 3$.

Case 1: $\mathrm{HD}(s,t) = 1$.

The link (s,t) is deleted only for $s \in A_0$ or $s \in A_3$. If $s \in A_0 = \{0000\}$ then t is equal to 0001 for $\mathrm{HD}(s,t) = 1$ and the link (s,t) is deleted. We take the path $(s \to 0010 \to 0011 \to 0001 = t)$ which is of length 3. The proof is similar for the case $s \in A_3$.

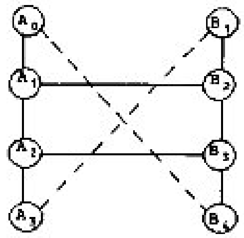

Fig. 3. Four-Cube with a twist of type 0 (T_0).

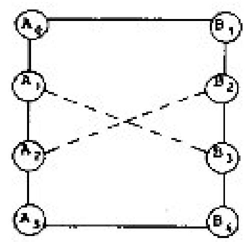

Fig. 4. Four-Cube with a twist of type 1 (T_1).

Case 2: $\mathrm{HD}(s,t) = 4$.

We assume s to be in $A_k, 0 \leq k \leq 3$.

If $s \in A_0$ or $s \in A_3$, s and t are directly connected by a link added as a result of the twist.

If $s \in A_1$, then $t \in B_3$. The path $(s \to 0^4 \to 1^4 \to t)$ is of length 3. The proof is similar for the case when $s \in A_2$.

Lemma 15: If we apply a type 1 twist on a four-cube its diameter becomes equal to 3.

Proof: Referring to Fig. 4, let s and t be any two points in the four-cube. Path length between s and t can be greater than $\mathrm{HD}(s,t)$ only if s and t differ in the last bit. We can assume w.l.o.g. that $s \in A_k, 0 \leq k \leq 3$.

a) $s \in A_1$.

If $t \in B_2$, there can be two cases. If $\mathrm{HD}(s,t) = 1$, we take the path $s \to 0^4 \to 0^3 1 \to t$, which is of length 3. If $\mathrm{HD}(s,t) = 3$, we take the path $s \to \overline{s} \to t$, which is of length 2.

If $t \in B_3$, there can be two cases. If $\mathrm{HD}(s,t) = 4$, path length = 1 owing to a direct link provided by the twist. If $\mathrm{HD}\ (s,t) = 2$, we take the path $s \to \overline{s} \to 1^4 \to t$, which is of length 3.

The proof is similar for the case when $s \in A_2$.

b) $s \in A_0$.

If $t \neq \overline{s}$, there is always a path between s and t of length $\mathrm{HD}(s,t)$. If $t = \overline{s} = 1111$, we take the path $0^4 \to 0^3 1 \to 1^3 0 \to t$, which is of length 3. The proof is similar for the case when $s \in A_3$.

Theorem 5: By exchanging four link pairs in $Q_{n+4}, (n > 0)$ we can reduce its diameter by 2.

Proof: We apply T_0 within a four-dimensional subcube $C_0 = 0^n *^4$. This involves exchange of 1 link pair. We apply T_1 within another one-dimensional subcube $C_1 = 1^n *^4$. This involves exchange of 3 link pairs.

Take any two nodes s and t in Q_{n+4}. Let $s = ax$ and $t = by$ where a and b are strings of length n. Let the path length between s and t be represented by $p(s,t)$.

Case I: If $a = b = 0^n, p(s,t) \leq 3$ (by Lemma 14).

Case II: If $a = b = 1^n, p(s,t) \leq 3$ (by Lemma 15).

Case III: If $\mathrm{HD}(s,t) \leq n+2$, then $p(s,t) \leq n+2$.

Case IV: $\mathrm{HD}(s,t) = n+4$, then $x = \overline{y}$ and there exists a path of length $n+1$ from s to t via one of C_0 and C_1.

Case V: $\mathrm{HD}(s,t) = n+3$. In this case s and t must agree in one bit position.

1) Suppose the bit common to s and t is within the leftmost n bit positions, i.e., $x = \overline{y}$. Either $0^n x$ is connected to $0^n \overline{x}$ or $1^n x$ is connected to $1^n \overline{x}$. Hence the path length is at most $n+1+1 = n+2$.

2) The bit common to s and t is within the rightmost four bit positions, i.e., $a = \overline{b}$.

Subcase a) If x and y do not differ in the rightmost bit, then let $z \in \{0,1\}$ and the path $s = ax \rightarrow z^n x \rightarrow z^n \overline{x} \rightarrow z^n y \rightarrow by = t$ is of length $n+2$.

Subcase b) If $\overline{x}$ and y differ in the rightmost bit, then there may be the following cases.

1) If $b \neq 0^n$ or 1^n, then the path $s \rightarrow z^n x \rightarrow z^n \overline{x} \rightarrow b\overline{x} \rightarrow by$ is of length $n+2$.

2) If $b = 1^n$, then for $x \in A_0 \cup A_3 \cup B_1 \cup B_4$, $y \in A_0 \cup A_3 \cup B_1 \cup B_4$ and the path $s = 0^n x \rightarrow 0^n \overline{x} \rightarrow 10^{n-1}\overline{x} \rightarrow 10^{n-1} y \rightarrow 1^n y$ is of length $n+2$.

On the other hand, for $x \in A_1 \cup A_2 \cup B_2 \cup B_3$, $y \in A_1 \cup A_2 \cup B_2 \cup B_3$ and the path $s = 0^n x \rightarrow 0^n \overline{y} \rightarrow 1^n \overline{y} \rightarrow 1^n y$ is of length $n+2$.

3) If $a = 1^n$, it can be similarly proved that $p(s,t) = n+2$.

Before investigating the effect of a twist of type r on a d-cube we give another lemma regarding the path between two points s and t, when bridges are connected to all the nodes in A_r.

Lemma 16: For $s_1 > d-r-1$ and the dth bit in s is 0, there exists a path of length $p_1 \leq 2r+2$, between s and t, via a bridge connected to some node in A_r.

Proof: For $s_1 > d-r-1, r \geq d - s_1$. If the dth bit is in S_{01}, we can reach a node in A_r from s, by changing all bits in S_{00} to 1 and also some $s_1' = s_1 - (d-r)$ bits in S_{01}. The path length $p_1 = 1 + 2s_1' + s_0 + s_3$ (by Lemma 1). Putting $s_0 + s_3 = d - (s_1 + s_2)$ and $s_1' = s_1 - (d-r)$, we get $p_1 = 1 + 2r - d + s_1 - s_2$. Since $s_1 - s_2 \leq d, p_1 \leq 2r+1$. If the dth bit is in S_{00}, we change all but the dth bit in S_{00} and $s_1' + 1$ bits in S_{01} to reach a node in A_r. Thus $p_1 = 2r - d + 3 + s_1 - s_2$. In this case $s_1 - s_2 \leq d-1$. Hence $p_1 \leq 2r+2$.

Lemma 17: In a d-cube (d being an even number), if we apply twists of type r, where r is an integer between 1 and $d/2-2$, path length between any two points s and t is less than or equal to $\max(2r+2, d-2r+1, d/2, 4)$.

Proof: First consider that only bridges are added, but no links are deleted. We can assume w.l.o.g. that $s_1 \leq s_2$. Because of symmetry of the twisted cube we assume s to be in some $A_k, 0 \leq k \leq d-1$ (proof for the other case is similar).

We define s_0, s_1, s_2, and s_3 as before. There can be four possible cases as in Theorem 1:

Case 1: $s_1 > d-r-1$.
Case 2a: $s_1 \leq d-r-1, s_2 \leq r$.
Case 2b: $r < s_1 < d-r-1, s_2 > r$.
Case 3: $s_1 \leq r$.

Case 1: By Lemma 16, the path length $p \leq 2r+2$.

Case 2a: By Lemma 13, the path length $p = 1 + s_0 + s_3 \leq d/2$, if $\mathrm{HD}(s,t) > d/2$.

Case 2b: If the rightmost bit in s is in S_{01}, by Lemmas 7 and 10, we can show that the path length $p \leq d-2r$. If the rightmost bit in s is in S_{00}, by using Lemma 9 and 12, we can show that the path length $p < d-2r+1$.

Case 3: If $s_2 > d-r-1$, we interchange s and t so that Case 1 becomes applicable. If $s_2 \leq d-r-1$, we interchange s and t so that Case 2a becomes applicable.

Now it follows that the path length is less than or equal to $\max(2r+2, d-2r+1, d/2)$.

Regarding the deletion of links we make the following observations:

Observation 1: For $\mathrm{HD}(s,t) > 1$, there is always a path of length $\mathrm{HD}(s,t)$ between s and t (by Lemma 13).

Observation 2: For $\mathrm{HD}(s,t) = 1$, and the dth bit of s and t are same, the link (s,t) is not deleted.

Observation 3: For $\mathrm{HD}(s,t) = 1$, and the link (s,t) is deleted, there exists a path of length 3, between s and t. In this case s is connected to $\overline{s}$ and t is connected to $\overline{t}$.

Now, consider a path between s and t via a bridge $(x, \overline{x})$ which is of length $\mathrm{HD}(s,x) + 1 + \mathrm{HD}(\overline{x},t)$ before deletion. If $\mathrm{HD}(s,x) > 1$ and $\mathrm{HD}(\overline{x},t) > 1$, the path length does change after deletion (by obs. 1). If $\mathrm{HD}(s,x) = 1$ and the link (s,x) is deleted then s is connected to $\overline{s}$. If $\mathrm{HD}(\overline{s},t) > 1$, then there is a path of length $1 + \mathrm{HD}(\overline{s},t)$ between s and t (by obs. 1), which is less than or equal to $d/2$ for $\mathrm{HD}(s,t) > d/2$. If $\mathrm{HD}(\overline{s},t) = 1$, the path between s and t is of length at most 4 (by obs. 3).

Lemma 18: In a d-cube (d being an even number), if we apply twists of type 0 and type r, where r is an integer between 1 and $d/2-1$, path length between any two points s and t is less than or equal to $\max(r+2, d-2r+1, d/2, 5)$.

Proof: First consider that only new links are added but no link is deleted. We can assume w.l.o.g. that $s_1 \leq s_2$. Because of symmetry of the twisted cube we assume s to be in some $A_k, 0 \leq k \leq d-1$ (proof for the other case is similar).

We define s_0, s_1, s_2, and s_3 as before. There can be four possible cases as in Theorem 1.

Case 1: $s_1 > d-r-1$.
Case 2a: $s_1 \leq d-r-1, s_2 \leq r$.
Case 2b: $r < s_1 < d-r-1, s_2 > r$.
Case 3: $s_1 \leq r$.

Case 1: By using Lemmas 2 and 3 we can show that the path length $p \leq r+2$.

Other cases are treated as in Lemma 17.

Now it follows that the path length is less than or equal to $\max(r+2, d-2r+1, d/2)$.

Regarding deletion of links we have the following observations.

Observation 1: For $\mathrm{HD}(s,t) > 2$, there is always a path of length $\mathrm{HD}(s,t)$ between s and t.

Observation 2: For $\mathrm{HD}(s,t) \leq 2$, and if one of s and t is not connected to a bridge, there exists a path, of length $\mathrm{HD}(s,t)$, between s and t.

Observation 3: For $\mathrm{HD}(s,t) = 1$, and if both of s and t have bridges connected to them, there exists a path, of length at most 3, between s and t.

Observation 4: For $\mathrm{HD}(s,t) = 2$, and if both of s and t have bridges connected to them, there exists a path, of length at most 4, between s and t.

Now, consider a path between two points s and t via a bridge $(x, \overline{x})$ which is of length $\mathrm{HD}(s,x) + 1 + \mathrm{HD}(\overline{x},t)$ before deletion. If $\mathrm{HD}(s,x) > 2$ and $\mathrm{HD}(\overline{x},t) > 2$, the path length does not change after deletion (by obs. 1). Consider $\mathrm{HD}(s,x) \leq 2$ and s is connected to $\overline{s}$. If $\mathrm{HD}(\overline{s},t) > 2$, then there is a path of length $1 + \mathrm{HD}(\overline{s},t)$ between s and t (by obs. 1), which is less than or equal to $d/2$ for $\mathrm{HD}(s,t) > d/2$. If $\mathrm{HD}(\overline{s},t) \leq 2$, the path between s and t is of length at most 5 (by obs. 4).

Lemma 19: If we apply a twist of type 1, on a d-cube (d being an even number ≥ 6), its diameter is reduced by 2.

Proof: By Lemma 17, the path length is less than or equal to $\max(d-2r+1, d/2, 2r+2, 4)$. Putting $r=1$ we get maximum path length of $d-1$ corresponding to $d-2r+1$. But the path length is less than $d-2r+1$ if the dth bit of s and t are different. If the dth bit is in S_{00} or S_{11} the path length is $p_0 = d-2r+1+s_2-s_1$ (by Lemma 9) and $p_1 = d-2r+1+s_1-s_2$ (by Lemma 12). In this case the path length becomes equal to $d - 2r + 1$ only if $s_1 = s_2$, i.e., when HD $(s, t) \leq d - 2$.

Lemma 20: In a six-cube, if we apply twists of type 0 and type 2, maximum path length between any two points is less than or equal to 4.

Proof: According to Lemma 18, the maximum path length is 5 for $d = 6$ and $r = 2$. But the path length can be 5 only if $\text{HD}(\bar{s}, t) = 2$. In that case $\text{HD}(s, t) = 4$. Hence, path length is at most 4.

We take a $d = n + 6$ cube $(n \geq 1)$. In this d-cube we take two subcubes, $C_0 = 0^n *^6$ and $C_1 = 1^n *^6$. We apply twist of type 0 and 2 in C_0 and twist of type 1 in C_1. The total number of link pairs exchanged in the process is 16. Now we have the following theorem.

Theorem 6: By applying twists on a d-cube as above $(d \geq 7)$, we can reduce its diameter by 3.

Proof: Take two points (s, t) in the above cube. When both s and t are in C_0 or C_1, there is a path of length at most 4 between them. We can assume w.l.o.g. that t is outside C_0, C_1. From the discussion in Section III-B, it follows that if no links were deleted in C_0, C_1, the d-cube would have had diameter $d - 3$. Now also this result holds, as whatever links are deleted are of the type which change the d-th bit. If required, we can change the dth bit outside C_0, C_1.

Lemma 21: In an eight-cube, if we apply twists of type 0 and 3 its diameter becomes equal to 5.

Proof: Follows from Lemma 18.

Lemma 22: If we apply a twist of type 2, in an eight-cube its diameter becomes equal to 6.

Proof: Follows from Lemma 17.

Lemma 23: If we apply twists of type 0 and 3 in a nine-cube, its diameter becomes equal to 5.

Proof: Follows from Lemma 18 if we replace $d/2$ by $\lceil d/2 \rceil$.

In a $d = n + 8$ cube $(n \geq 2)$ we exchange 57 link pairs in the following way. We apply twists of type 0 and 3 within one eight-dimensional subcube $0^n *^8$. We apply twist of type 2 within another subcube $1^n *^8$. Now we have the following theorem.

Theorem 7: The above d-cube has diameter equal to $d - 4$.

Proof: The proof is similar to that of Theorem 6.

Theorem 8: In a d-cube (d being an even number ≥ 10) if we apply twists of type 0 and r, where $r = \lfloor d/4 \rfloor + 1$, its diameter becomes equal to $d/2$.

Proof: Follows from Lemma 18.

A. Reduction in diameter by $d/2$, d an Even Number and $d \geq 10$

If we want to reduce the diameter by $d/2$ in an $(n + d)$-cube $(n \geq 1$ and $d \geq 10)$ we apply some twists in a d-dimensional subcube $C_0 = 0^n *^d$ so that any two points within this subcube have a path between them of length at most $d/2$. In another d-dimensional subcube $C_1 = 1^n *^d$ we apply some twists so that any two nodes within this subcube have a path between them of length at most $d/2 + 1$. Now we state the following theorem.

Theorem 9: The $(n+d)$-cube twisted as above will have diameter equal to $n + d/2$.

Proof: We define x, y, p_0, p_1, p, and q as in Theorem 3. Hence, $p_0 \leq d/2$ and $p_1 \leq d/2 + 1$. If $p \geq q$, the path between s and t via C_0 is of length $n - p + q + p_0 \leq n + d/2$. If $p < q$, the path between s and t via C_1 is of length $n + p - q + p_1 < n + d/2 + 1$.

B. Reduction in Diameter by $2m (m \geq 4)$ in $Q_{n+4m} (n > 0)$

We take two $4m$-dimensional subcubes $C_0 = 0^n *^{4m}$ and $C_1 = 1^n *^{4m}$. We apply T_0 and T_{m-1} within C_0 so that any two nodes in C_0 are apart by a distance of at most $2m$. We apply some twists within C_1 so that any two nodes in C_1 are apart by a distance of at most $2m + 2$. We call the resultant graph Q'_{n+4m} and claim that the diameter of Q'_{n+4m} is $n + 2m$.

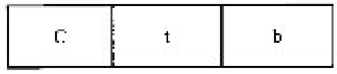

Fig. 5. Tag used for routing

Theorem 10: The $(n + 4m)$-cube twisted as above will have diameter equal to $n + 2m$.

Proof: We represent two nodes s and t as $s = ax$ and $t = by$ where x and y are strings of length $4m$, and a and b are strings of length n. We define $p, q, p_0, p_1, s_0, s_1, s_2, s_3, P_0$, and P_1 as in Theorem 3. Hence $p_0 \leq 2m$ and $p_1 \leq 2m + 2$. We consider only those (s, t) for which $\text{HD}(s, t) > n + 2m$, i.e., $p + q + s_0 + s_3 \leq 2m - 1$.

When $p_0 \neq 1 + s_0 + s_3, p_0 \leq \max(m + 2, 2m - 1) \leq 2m - 1$. Hence, if $q - p \leq 1$, then $P_0 = n + (q - p) + p_0 \leq n + 2m$. If $q - p > 1, P_1 = n - (q - p) + p_1 < n + 2m + 1$, i.e., $P_1 \leq n + 2m$. When $p_0 = 1 + s_0 + s_3, P_0 = n - p + q + 1 + s_0 + s_3 \leq n + p - q + 1 + s_0 + s_3 \leq n + 2m$.

Let $X(k)$ denote the number of link pairs to be exchanged to reduce the diameter by k in $Q_{n+2k} (n > 0)$

Then, we can write

for $k = 2m$, and $m > 2$,

$$X(k) = \sum_{i=3}^{m} \binom{4m - 1}{m + 1} + 57 \{X(4) = 57\}$$

for $k = 2m + 1$, and $m > 2$,

$$X(k) = X(2m) + \binom{4m + 1}{m + 1} + 1.$$

VI. Routing Algorithm

A. Routing in Bridged Q_d of Diameter $\lceil d/2 \rceil$

In this section, we describe the routing strategy in a bridged hypercube of dimension $d (d > 4)$. The routing algorithm merely consists of 1) deciding whether to use a bridge, and 2) if a bridge $(x, \overline{x})$ is to be used, then finding that node x from which a bridge will be used.

A packet to be routed from source node s to the destination node t is transmitted along with a tag as shown in Fig. 5. b is a flag bit, C is a d bit vector and t is the destination node. $b = 1$ indicates that a bridge is to be used and $b = 0$ indicates that we can route the packet without using any bridge. C is called the routing vector. Each bit of C is associated with a specific dimension of the hypercube so that if a bit of C is "1", then we transmit the packet along an edge of the hypercube in that dimension. Thus, given a routing vector C, we scan the bits of C from one end, say, the left end, and whenever we encounter a "1", the packet is transmitted along the corresponding dimension.

When $s_1 + s_2 \leq \lceil d/2 \rceil$ we set $b = 0$ and C is set to $s \oplus t$. When $s_1 + s_2 > \lceil d/2 \rceil$ we set $b = 1$. In this case we have to find the node x from which a bridge will be used. For this, we first enumerate s_0, s_1, s_2, and s_3. For different values of s_1 and s_2 we enumerate p_0, p_1 and p as done in Theorem 1. Then we find the minimum path length. Now, to find the node x to achieve this minimum path length, we have to suitably change certain bits in s in a way as described in the lemma that corresponds to this minimum path length. The routing vector C is then set to $s \oplus x$. Routing from s to x is done by scanning the bits of C from the left. The bridge $(x, \overline{x})$ is then used. Then we set $b = 0$ and set $C = \overline{x} \oplus t$ and route according to this C.

B. Routing in a Twisted Hypercube

We now describe the routing strategy for twisted hypercube Q_d^T on which we apply T_0 and $T_r (r = \lfloor d/4 \rfloor + 1)$. We discuss here the routing for hypercubes of even dimension ($d \geq 10$). The cases for twisted hypercubes of smaller dimension can be dealt with in a similar way. The scheme for routing a message from the source node s to the destination node t is described below.

Case 1): $\mathrm{HD}(s,t) \leq d/2$.

In this case, we define the routing vector $C = s \oplus t$. Let $C = c_1 c_2 \cdots c_n$. For two nodes $a = a_1 a_2 \cdots a_n$ and $b = b_1 b_2 \cdots b_n$, if a and b differ only in the ith bit then transmitting from a to b is equivalent to transmitting along the dimension i.

If $\mathrm{HD}(s,t) = 1$ and link (s,t) is present, then transmit to t.

If $\mathrm{HD}(s,t) = 1$ and link (s,t) is deleted, routing will be via a node s' adjacent to s. From s' send along the dth dimension to t' (t' will be adjacent to t) and then from t' to t.

If $\mathrm{HD}(s,t) > 1$ then let $c_i, c_j, \cdots, c_k, i < j < \cdots < k$ be the bits in C that are 1. There can be the following cases.

Subcase a): s and t do not differ in the rightmost bit.

In this case the deleted links do not affect the path between s and t. The routing is as in the original (untwisted) cube Q_d.

Subcase b): s and t differ in some bits, including the rightmost bit.

If s is not connected to $\bar{s}$, then transmit the packet from s along the dimensions $k, \cdots, j, i$ in that order.

If s is connected to $\bar{s}$, then first transmit the packet along the dimension i and then along the dimension d. Afterwards, transmit to t.

Case 2): $\mathrm{HD}(s,t) > d/2$. In this case, we use the edge $(x, \overline{x})$ and route from s to x, then from x to $\overline{x}$ and finally from $\overline{x}$ to t. Our aim is to find x from s.

If t is such a node that $(t, \bar{t})$ is connected then $x = \bar{t}$.

Otherwise, x is found as follows, depending on the values of s_1 and s_2.

$A) s \in A_k, 0 \leq k \leq d-1$.

Subcase i) $s_1 > d - r - 1$.

Subcase ii) $s_1 \leq d - r - 1, s_2 \leq r$.

Subcase iii) $r < s_1 < d - r - 1, s_2 > r$.

Subcase iv) $s_1 \leq r$.

In each case we find the minimum path length p as done in the proof of Lemma 17. To find the node x to achieve this minimum path length, we have to suitably change certain bits in s in a way as described in the lemma that corresponds to this minimum path length.

$B) s \in B_k, 1 \leq k \leq d$.

Because of the symmetry of the structure, routing can be done as follows:

For a source destination pair (s,t) define a new pair (s', t') where s' differs from s in the dth bit and t' differs from t in the dth bit. Then find a bridge $(y, \overline{y})$ to be used for this new pair (s', t') as in case A. Now for s the corresponding bridge to be used is $(x, \overline{x})$ where x differs from y in the dth bit.

Routing from s to x and from $\overline{x}$ to t, in both the cases A and B above, will be done following the method described above for $\mathrm{HD}(s,t) \leq d/2$.

VII. Conclusion

Hypercubes have various applications in parallel processing. This brief contribution describes different methods for reducing the diameter of a hypercube by adding some extra edges, called bridges, and also by exchanging some pairs of independent edges (without adding extra edges). The addition of bridges will not only reduce the diameter but will also reduce the average internode distance. In this brief contribution, we have aimed at reducing the diameter by adding as few edges as possible. Also, we have given an algorithm for routing in such bridged hypercubes. We add $\binom{d}{r} - 1$ bridges (where $r = \lfloor d/4 \rfloor + 1$) to a d-cube ($d > 4$) to reduce its diameter by $\lfloor d/2 \rfloor$. The extra edges constitute a small fraction of the total number of edges and this fraction is $\binom{d}{r} + 1 : d.2^{d-1}$, which decreases with increase in d. One important result is that the number of bridges to be added to reduce the diameter by k remains constant for all hypercubes of dimension greater than $2k$. Twisting of hypercubes has an extra advantage of reducing the diameter without changing the degree of nodes. Though the routing in this case becomes a little more complicated we have developed a suitable algorithm for routing in a twisted hypercube. Also, the number of link pairs exchanged to reduce the diameter by a given value is much smaller compared to that given in [7]. For example, we apply twists of type 0 and type 3 in an eleven-cube to get a graph of diameter 6. The number of link pairs exchanged in the process is 121, which is much smaller than that required by the method given in [7], which is equal to $10.2^7 = 1280$. It may also be noted that the twists that we have proposed in this brief contribution are different in nature from those given in [7]. It will be interesting to study the improvement of some other performance measures such as average routing distance, traffic density, and so on of the architecture proposed in this brief contribution over the original hypercube.

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