ELSEVIER

12 October 1995

PHYSICS LETTERS B

Physics Letters B 359 (1995) 339–342

Aharonov–Bohm interaction for a deformed non-relativistic spin $\frac{1}{2}$ particle

P. Roy [1], R. Roychoudhury [2]

Physics and Applied Mathematics Unit, Indian Statistical Institute, Calcutta 700035, India

Received 31 May 1995
Editor: M. Dine

Abstract

Starting from the κ deformed Dirac equation we derive the κ deformed Pauli equation. Aharonov–Bohm interaction is studied within the framework of this equation. In particular we show that the deformation parameter helps in creating additional bound states.

During the last couple of years the quantum deformation of the Poincaré algebra has been studied by many authors [1–8]. The classical Poincaré algebra can be deformed by applying the contraction process to the $\mathrm{so}_q(3, 2)$ algebra. Then taking the limit of the de-Sitter curvature $R \to \infty$ with a suitable limit of the real deformation parameter q such that $\varepsilon^{-1} = \lim(R \ln q)$, one can obtain the κ deformed Poincaré algebra. From the κ Poincaré algebra one can obtain various deformed relativistic wave equations, e.g., the κ Klein–Gordon equation [4], the κ Dirac equation [4,7] as well as other ones [9]. Recently, the κ Dirac equation has been used to study different quantum mechanical models [10–12]. Here, starting from the κ Dirac equation we shall derive a κ deformed non-relativistic equation for a spin $\frac{1}{2}$ particle. This equation is the κ deformed analogue of the Pauli equation. We shall study the Aharonov–Bohm effect within the framework of this equation. In particular we shall examine whether or not there exist non-zero energy bound states in this model.

We recall that the algebraic structure of the κ deformed Poincaré algebra is given by [7–10] (we take $\kappa = \varepsilon^{-1}$)

$$\begin{aligned}
&[P_i, P_j] = 0, \quad [P_i, P_0] = 0,\\
&[M_i, P_j] = i\varepsilon_{ijk}P_k, \quad [M_i, P_0] = 0,\\
&[L_i, P_0] = iP_0, \quad [L_i, P_j] = i\delta_{ij}\varepsilon^{-1}\sinh(\varepsilon P_0),\\
&[M_i, M_j] = i\varepsilon_{ijk}M_k, \quad [M_i, L_j] = i\varepsilon_{ijk}L_k,\\
&[L_i, L_j] = -i\varepsilon_{ijk}\ [M_k\cosh(\varepsilon P_0) - \tfrac{1}{4}\varepsilon P_k P_l M_l],
\end{aligned} \tag{1}$$

where $P_\mu = (P_0, P_i)$ are the deformed energy and momenta, the M_i, L_i are spatial rotation and deformed boost generators, respectively. The coalgebra and antipode for this algebra can also be defined.

[1] Email address: pinaki@isical.ernet.in.
[2] Email address: raj isical.ernet.in.

Elsevier Science B.V.
SSDI 0370-2693(95)01079-3

Based on this algebra (1) the deformed Dirac operator can be found and it is given by [7–10]

$$\mathscr{D} = -\exp\left(-\tfrac{1}{2}\varepsilon P_0\right)\gamma_i P_i + \frac{1}{\varepsilon}\gamma_4 \sinh(\varepsilon P_0) - \tfrac{1}{2}\varepsilon\gamma_4 P_i P_i. \tag{2}$$

The corresponding κ deformed Dirac equation reads

$$\mathscr{D}\psi = M\left(1 + \tfrac{1}{4}\varepsilon^2 M^2\right)^{1/2}\psi, \tag{3}$$

where M denotes the mass.

To bring this equation to a more tractable form, we operate from from the left by $\exp(\varepsilon P_0)$, expand and retain terms upto $O(\varepsilon)$. The resulting equation which is the κ Dirac equation to $O(\varepsilon)$ is given by

$$\left[(\gamma_4 P_0 - \gamma_i P_i) + \tfrac{1}{2}\varepsilon\gamma_4\left(P_0^2 - P_i P_i\right) - M P_0\right]\psi = M\psi. \tag{4}$$

Now we introduce the gauge interaction through the minimal coupling prescription

$$P_0 \to P_0 = H = E, \qquad P_i \to \hat{P}_i = P_i - eA_i. \tag{5}$$

Eq. (3) can then be written as

$$H\psi = \left[\left(\gamma_4\gamma_i \hat{P}_i + \gamma_4 M\right) - \tfrac{1}{2}\varepsilon\left(H^2 - \hat{P}_i\hat{P}_i - \gamma_4 M H\right)\right]\psi. \tag{6}$$

Eq. (5) is highly nonlinear and cannot be solved without some approximation. In this respect we follow Ref. [10] and note that the Hamiltonian corresponding to the undeformed part is

$$H_0 = \left(\gamma_4\gamma_i\hat{P}_i + \gamma_4 M\right). \tag{7}$$

Now substituting (7) on the right-hand side of Eq. (6) the eigenvalue equation to $O(\varepsilon)$ is found to be

$$\left[H_0 - \tfrac{1}{2}\varepsilon\left(H_0^2 - \hat{P}_i\hat{P}_i - \gamma_4 M P_0\right)\right]\psi = E\psi, \tag{8}$$

Next we choose the following representation of the γ matrices:

$$\gamma_4 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \gamma_i = \begin{pmatrix} 0 & \sigma_i \\ -\sigma_i & 0 \end{pmatrix}, \tag{9}$$

where σ_i are the Pauli matrices.

Using the representation of the gamma matrices in (9) we can write Eq. (8) in the following form:

$$\left[M + \tfrac{1}{2}(\boldsymbol{\sigma}\cdot\boldsymbol{H})\varepsilon e\right]\phi + \left[1 + \tfrac{1}{2}\varepsilon M\right](\boldsymbol{\sigma}\cdot\hat{\boldsymbol{P}})\chi = E\phi, \tag{10}$$

$$\left[1 - \tfrac{1}{2}\varepsilon M\right](\boldsymbol{\sigma}\cdot\hat{\boldsymbol{P}})\phi + \left[-M + \tfrac{1}{2}\varepsilon e\boldsymbol{\sigma}\cdot\boldsymbol{H}\right]\chi = E\chi, \tag{11}$$

where we have taken $\psi = \binom{\phi}{\chi}$, ϕ and χ being two-component spinors and $\boldsymbol{H} = \mathrm{curl}(\boldsymbol{A})$ is the magnetic field.

We shall now take the non-relativistic limit of Eqs. (10) and (11). Writing $E = E' + M$, $E'\ M$ we get from (10) and (11)

$$\left[1 + \tfrac{1}{2}\varepsilon M\right](\boldsymbol{\sigma}\cdot\hat{\boldsymbol{P}})\chi = \left[E' - \tfrac{1}{2}\varepsilon e(\boldsymbol{\sigma}\cdot\boldsymbol{H})\right]\phi, \tag{12}$$

$$\chi = \frac{1}{2M}\left[1 + \frac{E' - \varepsilon e/2(\sigma\cdot H)}{2M}\right]^{-1} \times\left[1 - \tfrac{1}{2}\varepsilon M\right](\boldsymbol{\sigma}\cdot\hat{\boldsymbol{P}}). \tag{13}$$

Eliminating the lower component χ from the above equations we get to $O(\varepsilon)$

$$\frac{1}{2M}\left[P^2 + e(1 + \varepsilon M)(\boldsymbol{\sigma}\cdot\boldsymbol{M})\right]\phi = E'\phi \tag{14}$$

Eq. (14) is the κ deformed Pauli equation. It can be seen from (14) that the magnetic moment has increased by an amount proportional to the deformation parameter.

We now turn to the Aharonov–Bohm interaction [13]. The AB interaction is important in different contexts [14,15]. In an ideal situation the magnetic field is pointlike and concentrated at the origin. The problem can then be treated by the method of the self-adjoint extension [15–17]. However, a physically more appealing way to treat the problem is to consider a magnetic field on a ring and at the end the radius of the ring is allowed to shrink to zero. This method has been extensively used by Hagen [18,19] and Bordag et al. [20]. In the present work we shall follow Refs. [18–20].

In our case the vector potentials are given by

$$eA_i = -\frac{\varepsilon_{ij}x_j}{r^2}, \quad r > R, \qquad = 0, \quad r < R, \tag{15}$$

where $r^2 = x^2 + y^2$ and $i = 1, 2$. The corresponding magnetic field is given by

$$eH = \frac{\alpha}{R}\delta(r - R). \tag{16}$$

We take $\alpha > 0$ and consider the negative value of the spin projection. In this case the magnetic moment interaction is attractive and may produce bound states. For $\alpha < 0$, the spin projection has to be reversed.

Using (15) and (16) we find from Eq. (14)

$$\left[-\frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}r}r\frac{\mathrm{d}}{\mathrm{d}r} + \frac{[m - \alpha\theta(R - r)]^2}{r^2} - (1 + \varepsilon M)\frac{\alpha}{R}\delta(r - R)\right]\phi_m = \bar{E}\phi_m, \tag{17}$$

where $\bar{E} = 2ME'$ and since the motion is essentially planar we have set $p_z = 0$, for simplicity.

The wave functions ϕ_m satisfy the boundary conditions

$$\phi_m(R + 0) = \phi(R - 0), \tag{18}$$

$$R\frac{\mathrm{d}}{\mathrm{d}r}\phi_m\Big|_{R-0}^{R+0} = -(1 + \varepsilon M)\phi_m(R). \tag{19}$$

The bound state solutions of Eq. (17) are given by

$$\begin{aligned}\phi_m(r) &= A_m I_{|m|}\big(\sqrt{-\bar{E}}r\big), \quad r < R, \\ &= B_m K_{|m-\alpha|}\big(\sqrt{-\bar{E}}r\big), \quad r > R,\end{aligned} \tag{20}$$

where A_m and B_m are arbitrary constants. The bound state energies can now be determined using (18) and (19). Plugging (20) in (19) and using (18) we find

$$z\frac{K'_{|m-\alpha|}(z)}{K_{|m-\alpha|}(z)} = -(1 + \varepsilon M) + z\frac{I'_{|m|}(z)}{I_{|m|}(z)}, \tag{21}$$

where $z = \sqrt{-\bar{E}}R$.

Now expanding the RHS and LHS of (21) in powers of z we get

$$\begin{aligned}&-|m - \alpha| - \frac{z^2}{4(|m - \alpha| - 1)} \\ &\quad = -(1 + \varepsilon M)\alpha + |m| + \frac{z^2}{4(|m| + 1)}, \\ &\quad |m - \alpha| > 1,\end{aligned} \tag{22}$$

$$\begin{aligned}&-|m - \alpha| - |m - \alpha|\frac{(1 - m - \alpha)}{(1 + |m - \alpha|)}(z/2)^{2|m-\alpha|} \\ &\quad = -(1 + \varepsilon M) + |m|, \\ &\quad |m - \alpha| < 1.\end{aligned} \tag{23}$$

From (22) and (23) it follows that bound states with non-vanishing binding energy exist if

$$0 < m < \left(1 + \tfrac{1}{2}\varepsilon M\right) \tag{24}$$

and in the undeformed case (i.e, $\varepsilon = 0$) only zero energy bound states exist.

Now writing $\alpha = N + \bar{\alpha}$, $0 < \bar{\alpha} < 1$ we find

$$\bar{E}_i = -\frac{4}{R^2}\left[\frac{(|i - \alpha|)(|i| + 1)}{|i - \alpha| + |i|}\right]^{1/\alpha}(\alpha\varepsilon M), \quad i = 0, 1, \ldots, N - 1, \tag{25}$$

$$\bar{E}_N = -\frac{4}{R^2}\left[\frac{\Gamma(1 + \bar{\alpha})}{\Gamma(1 - \bar{\alpha})}\frac{\alpha\varepsilon M}{\bar{\alpha}}\right]^{1/\bar{\alpha}}. \tag{26}$$

However, in the case $N = 0$ there is only one bound state with energy given by

$$\bar{E}_0 = -\frac{4}{R^2}\left[\frac{\Gamma(1 + \bar{\alpha})}{\Gamma(1 - \bar{\alpha})}\varepsilon M\right]^{1/\bar{\alpha}}. \tag{27}$$

From (25)–(27) it can be observed that with all other parameters held fixed, the limit $R \to 0$ would make the energy levels infinite and so is not admissible. However, if both $\varepsilon \to 0$ and $R \to 0$ such that $\varepsilon M/R^2 =$ finite non-zero constant, then the energy levels in (25) become finite while $\bar{E}_N$ behaves like $(\varepsilon M)^{1/\bar{\alpha}-1}$, $0 < \bar{\alpha} < 1$. In the $N = 0$ case the correct limiting behaviour is $\varepsilon^{1/\bar{\alpha}}/R^2 =$ finite as $\varepsilon \to 0$, $R \to 0$. Note that this limiting behaviour is not admissible in the case of (25) and (26), since this would make (26) finite and (25) infinite.

Another approach which is perhaps more sensible, is to consider a small non-zero value of R and a suitable small value of ε [20,21]. In this case all the energy levels i.e., (25)–(27) will remain finite.

Conclusion. Here we have studied the Aharonov–Bohm interaction within the framework of κ deformed Pauli equation. It has been shown that the deformation parameter creates non-zero energy bound states. We can hope that some future experiment would detect these states. The existence of non-zero bound states can also be understood in the following way: the Pauli equation with $g = 2$ is supersymmetric [22] and hence admits zero energy bound states. However, in the the case of the deformed Pauli equation $g \neq 2$ and so SUSY is broken, giving rise to non-zero energy bound states.

References

[1] J. Lukiersky, A. Nowicki, H. Ruegg and V.N. Tolstoy, Phys. Lett. B 264 (1991) 331.
[2] J. Lukiersky, A. Nowicki and H. Ruegg, Phys. Lett. B 271 (1991) 321.
[3] S. Giler et al., Phys. Lett. B 286 (1991) 57.
[4] J. Lukiersky, A. Nowicki and H. Ruegg, Phys. Lett. B 293 (1993) 344.
[5] E. Celegnini et al., J. Math. Phys. 31 (1990); 32 (1992) 1155, 1159.
[6] H. Bacry, J. Phys. A 26 (1993) 419.
[7] A. Nowicki, E. Sorace and M. Tarlini, Phys. Lett. B 302 (1993) 419.
[8] J. Lukiersky and H. Ruegg, Phys. Lett. B 329 (1994) 613.
[9] S. Giler et al., Mod. Phys. Lett. A 8 (1993) 3785.
[10] L.C. Biedenharn, B. Muller and M. Tarlini, Phys. Lett. B 318 (1993) 613.
[11] J. Lukiersky, H. Ruegg and W. Ruhl, Phys. Lett. B 313 (1993) 357.
[12] P. Roy and R. Roychoudhury, Phys. Lett B 339 (1994) 87.
[13] Y. Aharonov and D. Bohm, Phys. Rev. 115 (1959) 485.
[14] M.G. Alford and F. Wilczek, Phys. Rev. Lett. 62 (1989) 1071.
[15] Ph.s. de Gerbert, Phys. Rev. D 40 (1989) 1346.
[16] R. Jackiw, in: Beg Meorial Volume (World Scientific, Singapore, 1992).
[17] Ph.s. de Gerbert and R. Jackiw, Commun. Math. Phys. 124 (1989) 229.
[18] C.R. Hagen, Phys. Rev. Lett. 64 (1990) 503.
[19] C.R. Hagen, Intern. J. Mod. Phys. A 6 (1991) 3119.
[20] M. Bordag and S. Voropaev, J. Phys. A 26 (1993) 7637.
[21] J. Dey et al., Phys. Lett. B 331 (1994) 355.
[22] R. Jackiw, Phys. Rev. D 29 (1984) 2375.