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Effect of toxic substances on a two-species competitive system

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Abstract

In this paper we have considered a two-species competitive system which is also affected by toxic substances. It has been observed that the ratio of the toxic substances of the two species plays a crucial role in shaping the dynamics of the system. Lastly, by using a suitable Liapunov function we have observed that the toxic substances have some stabilizing effect on the system.

Keywords: Competition, interspecific; Toxicology

1. Introduction

The effects of toxic substances on ecological communities is an important problem from an environmental point of view. By using mathematical models Hallam and Clark (1982), Hallam et al. (1983a,b), Hallam and De Luna (1984), De Luna and Hallam (1987), Freedman and Shukla (1990) and others studied the effects of toxicants on various ecosystems.

In this paper we have considered a two-species competitive system which is affected by toxic substances. In the formulation of the model it is assumed that the environment in which the species were present contained pollution which is almost constant, resulting in the growth of the species being affected due to that pollutant.

We are interested in determining the effects of toxic substances on equilibrium levels and stability of biological system in local as well as global sense.

2. The mathematical model

In addition to competition between two species, each species produces a substance toxic to the other, but only when the other is present. The growth equations for such a system can be written as

$$\begin{aligned} \frac{dN_1}{dt} &= N_1(K_1 - \alpha_1 N_1 - \beta_{12} N_2 - \gamma_1 N_1 N_2) \\ \frac{dN_2}{dt} &= N_2(K_2 - \alpha_2 N_2 - \beta_{21} N_1 - \gamma_2 N_1 N_2) \end{aligned} \tag{1}$$

where $N_1(t)$ and $N_2(t)$ denote the population density of two competing species at time t for a common pool of resources and K_1, K_2, α_1, α_2, β_{12}, β_{21}, γ_1 and γ_2 are positive constants. The first three terms represent competitive growth and the last term denotes the effect of toxic substances. Such a system where a toxic sub-

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stance is produced at a constant rate is studied by Maynard-Smith (1974).

3. Equilibria and local stability analysis

The model system (1) has four equilibria, namely $E_0(0,0)$, $E_1((K_1/\alpha_1),0)$, $E_2(0,(K_2/\alpha_2))$ and $E^*(N_1^*,N_2^*)$.

The local and global stability properties of E_0, E_1 and E_2 are well known. We are interested in determining the effect of the toxic substance on the model system, so we shall put emphasis on E^*.

To determine the local stability character, we compute the variational matrix about E^*, which we denote by M.

$$M=\begin{bmatrix} -\alpha_1 N_1^* - \gamma_1 N_1^* N_2^* & -\beta_{12} N_1^* - \gamma_1 N_1^{*2} \\ -\beta_{21} N_2^* - \gamma_2 N_2^{*2} & -\alpha_2 N_2^* - \gamma_2 N_1^* N_2^* \end{bmatrix}. \quad (2)$$

The characteristic equation of M is given by

$$\sigma^2 + b\sigma + c = 0 \quad (3)$$

where

$$b = \alpha_1 N_1^* + \alpha_2 N_2^* + (\gamma_1 + \gamma_2) N_1^* N_2^*$$

$$c = (\alpha_1\alpha_2 - \beta_{12}\beta_{21}) N_1^* N_2^* + (\alpha_1\gamma_2 - \gamma_1\beta_{21}) N_1^{*2} N_2^* + (\alpha_2\gamma_1 - \gamma_2\beta_{12}) N_1^* N_2^{*2}$$

From the above, it is clear that in the absence of toxic substance (i.e., when $\gamma_i = 0$, $i = 1,2$) the interior equilibrium point is locally stable or unstable according as

$$\frac{\alpha_1}{\beta_{21}} > \frac{\beta_{12}}{\alpha_2} \quad \text{or} \quad \frac{\alpha_1}{\beta_{21}} < \frac{\beta_{12}}{\alpha_2} \quad (4)$$

respectively. If the first term of c is less than zero but the second and third term is greater than zero i.e., if the ratio of the toxic substance of the two species lies between

$$\frac{\beta_{12}}{\alpha_2} < \frac{\gamma_1}{\gamma_2} < \frac{\alpha_1}{\beta_{21}}, \quad (5)$$

then the locally unstable system (in the absence of toxic substance) is locally stable (in the presence of toxic substance).

Thus we may conclude that toxic substances have some stabilizing effect on the two-species competitive system.

4. Global stability

Theorem 1. *The interior equilibrium $E^*(N_1^*,N_2^*)$ is globally asymptotically stable if*

$$4(\alpha_1 + \gamma_1 N_2)(\alpha_2 + \gamma_2 N_1) \geq (\beta_{12} + \beta_{21} + \gamma_1 N_1^* + \gamma_2 N_2^*)^2. \quad (6)$$

Proof. We define the following positive definite Liapunov function as

$$V(N_1,N_2) = N_1 - N_1^* - N_1^* \log\left(\frac{N_1}{N_1^*}\right) + N_2 - N_2^* - N_2^* \log\left(\frac{N_{2*}}{N_2}\right) \quad (7)$$

where $E^*(N_1^*,N_2^*)$ is the equilibrium point. Note that $V(N_1,N_2)$ is a positive definite function for all $(N_1,N_2) \neq (N_1^*,N_2^*)$. The time derivative along the solution of Eqs. 1 is

$$\begin{aligned}\dot V(N_1,N_2) &= \frac{N_1 - N_1^*}{N_1}\frac{dN_1}{dt} + \frac{N_2 - N_2^*}{N_2}\frac{dN_2}{dt} \\ &= -\big[(N_1 - N_1^*)^2(\alpha_1 + \gamma_1 N_2) \\ &\quad + (N_1 - N_1^*)(N_2 - N_2^*) \\ &\quad \times (\beta_{12} + \beta_{21} + \gamma_1 N_1^* + \gamma_2 N_2^*) \\ &\quad + (N_2 - N_2^*)^2(\alpha_2 + \gamma_2 N_1)\big]. \end{aligned} \quad (8)$$

The above equation should be considered as a quadratic form in the variables $(N_1 - N_1^*)$ and $(N_2 - N_2^*)$ which is negative definite if the matrix

$$B = \begin{bmatrix} \alpha_1 + \gamma_1 N_2 & (\beta_{12} + \beta_{21} + \gamma_1 N_1^* + \gamma_2 N_2^*)/2 \\ (\beta_{12} + \beta_{21} + \gamma_1 N_1^* + \gamma_2 N^*)/2 & \alpha_2 + \gamma_2 N_1 \end{bmatrix} \quad (9)$$

is positive definite.

The matrix B is positive definite if

$$4(\alpha_1+\gamma_1 N_2)(\alpha_2+\gamma_2 N_1) \geq (\beta_{12}+\beta_{21}+\gamma_1 N_1^*+\gamma_2 N_2^*)^2. \quad (10)$$

Hence the theorem.

From Eq. 10 it is clear that toxic substances play an important role in stabilizing the system, which also reflects the local stability character.

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