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A Necessary and Sufficient Condition for Two-Person Nash Implementation

BHASKAR DUTTA

and

ARUNAVA SEN

Indian Statistical Institute, New Delhi

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The main result of this paper is to characterize the class of two-person social choice correspondences which are Nash-implementable. The characterization result is used to formulate domain restrictions which allow the construction of non-dictatorial and Pareto-efficient social choice correspondences which are Nash-implementable.

1. INTRODUCTION

In an important paper, Maskin (1977), it was shown that if an n-person social choice correspondence with $n \geqq 3$ satisfies the conditions of monotonicity and no-veto power, then it can be implemented in Nash equilibrium. These conditions are however, not sufficient for the implementation of two-person social choice correspondences. Moreover the condition of no-veto power, which is not a necessary condition for implementation, is unacceptably strong in this environment. In this paper we address the two-person implementation problem and characterize completely the class of implementable social choice correspondences. The two-person problem is an important one in the theory of incentives. It has a bearing on a wide variety of bilateral contracting and negotiating problems.

Why is the two-person problem different from its many-person counterpart? Consider the case where there are two states of nature, θ and ϕ. Let f be a social choice correspondence which picks outcomes $f(\theta)$ and $f(\phi)$ in states θ and ϕ respectively. It is well-known that if f is implementable, then it must be truthfully implementable.[1] In other words, there must exist a "direct revelation mechanism" with the following properties—(a) each individual can announce either θ or ϕ (b) if all individuals announce θ (resp ϕ), the outcome is $f(\theta)$ (resp. ϕ)) (c) the unanimous announcement of θ (resp. ϕ) is a Nash equilibrium with respect to state θ (resp. ϕ); i.e. truth-telling is a Nash equilibrium. Suppose $n \geqq 3$. Observe then that *any* social choice correspondence f can be truthfully implemented. This can be achieved by constructing a direct revelation mechanism with the following property—if at least $n-1$ individuals announce θ (resp. ϕ), then the outcome is $f(\theta)$ (resp. $f(\phi)$). No individual can change the outcome by deviating from a unanimous announcement, so that truth-telling is clearly a Nash equilibrium. The

1. See Dasgupta, Hammond and Maskin (1979). Of course, the converse is not true, i.e. a social choice correspondence which is truthfully implementable may not be implementable.

restriction $n \geqq 3$ is crucial because it allows the planner to identify a deviant from a truth-telling strategy combination. If instead $n=2$ and individual 1 announces θ and individual 2, ϕ, then there is no way for the planner to ascertain whether state θ has occurred and 2 is lying, or state ϕ has occurred and 1 is lying. Clearly, if truth-telling is to be sustained as an equilibrium, there must exist an outcome which is *simultaneously* no better than $f(\theta)$ for 2 in state θ and no better than $f(\phi)$ for 1 in state ϕ. One way to summarize the discussion above would be to say that for a social choice correspondence to be implementable when $n=2$, it must satisfy a *self-selection constraint*. In the many-person case however, this constraint is always trivially satisfied.

The main result of the paper is that a condition which we call Condition β is both necessary and sufficient for implementation of two-person social choice correspondences. The ideas behind the formulation of Condition β have an applicability beyond that of the present context. In another paper (Dutta and Sen (1988)), we use them to derive a necessary and sufficient condition for implementation in Strong Nash equilibrium.

A classical result due to Hurwicz and Schmeidler (1978) and Maskin (1977) states that if a two-person, Pareto-efficient, social choice correspondence defined on the domain of all possible strong orderings is implementable, then it must be dictatorial. We use our characterization result to formulate domain restrictions which allow us to avoid the Hurwicz-Schmeidler result. These domain restrictions are stated generally and are satisfied in two important specialized environments. The first of these is the "cardinal-utility, lottery" framework where the set of outcomes is the space of lotteries over a finite set of social alternatives. Individual preferences over lotteries are assumed to be representable by a von Neumann–Morgenstern utility function. The second is the class of "economic environments" including, for example, exchange economics with a mild restriction on preferences. We give a sufficient condition for implementation in these environments and provide examples of social choice correspondences which satisfy this sufficient condition.

The paper is organized as follows. Section 2 contains the essential notation and definitions while Section 3 introduces the crucial Condition β. The main result is proved in Section 4 and Section 5 discusses the possibility of implementation in restricted domains.

2. NOTATION

The set of social outcomes is denoted by A and the set of agents by $I=\{1,2\}$. Each agent $i \in I$ has a preference ordering on the set A. An ordered pair of preference orderings is called a preference profile. In the profile $R=(R_1, R_2)$, agent i's preference ordering is R_i. For any $i \in I$, P_i denotes the asymmetric component of R_i. The set of admissible preference profiles is denoted by $\mathscr{R}$. For any $i \in I$, $R \in \mathscr{R}$ and $a \in A$, $L_i(R, a)$ denotes the set $\{c \in A \mid aR_ic\}$. It is the lower contour set for agent i at outcome a according to preference ordering R_i. Similarly, given $i \in I$, $R \in \mathscr{R}$ and $a \in A$, $SL_i(R, a)=\{c \in A \mid aP_ic\}$ and $SU_i(R, a)=\{c \in A \mid cP_ia\}$. For any $i \in I$, $R \in \mathscr{R}$ and $C \subseteq A$, let $M_i(R, C)=\{a \in C \mid aR_ic \forall c \in C\}$. Thus $M_i(R, C)$ is the set of maximal elements in C for agent i according to preference ordering R_i. A social choice correspondence (SCC) f, associates a non-empty set, $f(R) \subseteq A$ with every profile $R \in \mathscr{R}$. A mechanism or game form G is a 3-tuple (S_1, S_2, π) where S_1 and S_2 are sets and π is a function $\pi: S_1 \times S_2 \mapsto A$. The set S_i is the strategy set for agent i and the function π specifies an outcome for every vector of strategies. For every $R \in \mathscr{R}$, the pair (G, R) constitutes a game in normal form. We let $NE(G, R)$ denote the set of Nash equilibrium strategies of the game (G, R). The mechanism implements the SCC f if for all $R \in \mathscr{R}$, $\pi(NE(G, R))=f(R)$.

3. CONDITION β

In this section, we introduce Condition β which is necessary and sufficient for Nash implementation in two-person settings.

Definition 3.1. A SCC f satisfies Condition β if there exists a set A^* which contains the range of f, and for each $i \in I$, $R \in \mathscr{R}$ and $a \in f(R)$ there exists a set $C_i(R, a) \subseteq A^*$, with $a \in C_i(R, a) \subseteq L_i(R, a)$ such that for all $R^1 \in \mathscr{R}$, we have

(i) (a) for all $b \in f(R^1)$, $C_1(R, a) \cap C_2(R^1, b) \neq \phi$. (b) Moreover, there exists $x \in C_1(R, a) \cap C_2(R^1, b)$ such that if for some $R^2 \in \mathscr{R}$, $x \in M_1(R^2, C_1(R, a)) \cap M_2(R^2, C_2(R^1, b))$, then $x \in f(R^2)$.

(ii) if $a \notin f(R^1)$, there exists $j \in I$ and $b \in C_j(R, a)$ such that $b \notin L_j(R^1, a)$.

(iii) $[M_i(R^1, C_i(R, a)) \backslash \{a\}] \cap M_j(R^1, A^*) \subseteq f(R^1) \forall i \in I$ and $j \neq i$.

(iv) $M_1(R^1, A^*) \cap M_2(R^1, A^*) \subseteq f(R^1)$.

Moore and Repullo (1988) have shown that parts (ii), (iii) and (iv) of Condition β are necessary and sufficient for the Nash-implementability of many-person social choice correspondences. Part (ii) is the well-known condition of (Maskin) monotonicity while part (iv) is a unanimity condition. Part (iii) is more subtle—according to it, any alternative which is R_i^1-maximal for individual i in $C_i(R, a)$ and R_j^1-maximal for individual j in A^* must be a value of f at the profile R^1.

Part (i) of Condition β is specific to the two-person problem. Part (i)(a) is the self-selection constraint mentioned in the Introduction. In addition, part (i)(b) requires that it should be possible to make appropriate selections from the $C_i(.)$ sets.

4. THE CHARACTERIZATION RESULT

We are now ready to present our main result.

Theorem 4.1. *The SCC f is implementable if and only if it satisfies Condition β.*

Before proving the theorem formally, we will describe some of the arguments verbally. We begin with necessity. In the two-person model, a game form can be represented as a matrix with elements belonging to the set A. Suppose the SCC f can be implemented by the game form G, described below.

	t_1	t_2 .	.
s_1	a		
s_2		b	
.			
.			

Assume without loss of generality that $(s_1, t_1) \in NE(G, R)$ and $(s_2, t_2) \in NE(G, R^1)$, so that $a \in f(R)$ and $b \in f(R^1)$. Let $C_1(R, a)$ denote the set of outcomes which 1 can obtain if 2 plays t_1 and $C_2(R, a)$, the set of outcomes which 2 can obtain if 1 plays s_1. (The former is the set of outcomes in the first column and the latter, the set of outcomes in the first row.) Since $(s_1, t_1) \in NE(G, R)$, it must be the case that $C_i(R, a) \subseteq L_i(R, a)$, $i \in I$. Defining the sets $C_i(R^1, b) \forall i$, analogously, we conclude that $C_i(R^1, b) \subseteq L_i(R^1, b)$.

Now, the outcome corresponding to the strategy combination (s_2, t_1), say x, must clearly lie in the set $C_1(R, a) \cap C_2(R^1, b)$. Suppose x is maximal for 1 and 2 in the sets $C_1(R, a)$ and $C_2(R^1, b)$ respectively according to some profile R^2. Then, $x \in \pi[NE(G, R^2)]$ which implies that $x \in f(R^2)$, thus establishing (i) of condition β. For a discussion of parts (ii)–(iv) which are common to the many-person problem, see Moore–Repullo (1988).

The game form which we use to prove sufficiency is similar to some of the game forms used in the Nash implementation literature (see Repullo (1987)).[2] One difference is that agents in addition to announcing a profile R, an outcome $a \in f(R)$ and a non-negative integer, also have the option of raising a "flag". If only one agent, say i, raises the flag, then he can choose any outcome from the set $C_i(R, a)$ where R and a are the announcements of agent j. If both agents raise flags, then the agent who has announced the higher integer gets to choose any outcome in the set A^* specified in Condition β.

We now proceed to more formal arguments.

Proof. (Necessity): Let G be the game form that implements f, and let $A^* = \{a \in A \mid a = g(s) \text{ for some } s \in S_1 \times S_2\}$. For each $R \in \mathscr{R}$ and $a \in f(R)$, let $\bar{s}(R, a) \in S_1 \times S_2$ be such that $\bar{s}(R, a) \in NE(G, R)$ and $\pi(\bar{s}(R, a)) = a$. For each $i \in I$, let $C_i(R, a) = \{c \in A \mid c = \pi(s_i, \bar{s}_j(R, a)) \text{ for some } s_i \in S_i\}$. Clearly, $C_i(R, a) \subseteq A^*$ and $a \in C_i(R, a) \subseteq L_i(R, a)$. Observe that $\pi(\bar{s}_1(R, a), \bar{s}_2(R^1, b)) \in C_1(R, a) \cap C_2(R^1, b) \forall R^1 \in \mathscr{R}$ and $b \in f(R^1)$. Let $x = \pi(\bar{s}_1(R, a), \bar{s}_2(R^1, b))$ and suppose that for some $R^2 \in \mathscr{R}$, x is R^2-maximal for 1 in $C_1(R, a)$ and R^2-maximal for 2 in $C_2(R^1, b)$. Since agents 1 and 2 by unilateral deviations from $(\bar{s}_1(R, a), \bar{s}_2(R^1, b))$ can only obtain $C_1(R, a)$ and $C_2(R^1, b)$, it follows that $x \in \pi(NE(G, R^2))$. Hence $x \in f(R^2)$, establishing part (i) of Condition β. The proofs of parts (ii)–(iv) can be found in Moore–Repullo (1988).

(Sufficiency): Suppose f satisfies Condition β. The game form G is constructed as follows. $S_i = \{(R^i, a^i, r^i, k^i) \in \mathscr{R} \times A \times \{F, NF\} \times N \mid a^i \in f(R^i)\}$ where $\{F, NF\}$ is the set comprising the 2 elements "flag" and "no flag" and N is the set of non-negative integers. The outcome function π is defined by:

(i) if $s_1 = (R, a, NF, k^1)$ and $s_2 = (R, a, NF, k^2)$, then $\pi(s) = a$.

(ii) if $s_1 = (R^1, a^1, NF, k^1)$ and $s_2 = (R^2, a^2, NF, k^2)$, then $\pi(s) = x$ where $x \in C_1(R^2, a^2) \cap C_2(R^1, a^1)$ and $x \in M_1(R, C_1(R^2, a^2) \cap M_2(R, C_2(R^1, a^1))$ implies that $x \in f(R)$ for all R. The existence of such an alternative is guaranteed by (i) of Condition β.

(iii) if $s_i = (R^i, a^i, F, k^i)$ and $s_j = (R^j, a^j, NF, k^j)$, then individual i gets to choose any outcome from $C_i(R^j, a^j)$,

(iv) if $s_i = (R^i, a^i, F, k^i)$, $s_j = (R^j, a^j, F, k^j)$ and $k^i > k^j$, then individual i gets to choose any outcome from A^*. Ties are broken in favour of agent 1.[3]

Let the true preference profile be R^* and let $a \in f(R^*)$. We claim first that the strategy combination $\bar{s}_1 = \bar{s}_2 = (R^*, a, NF, 0)$ is a Nash equilibrium of (G, R^*). To see this, observe that any deviation by i will get him an outcome in $C_i(R^*, a)$. Since $C_i(R^*, a) \subseteq L_i(R^*, a)$, such deviations are not profitable. Thus, $f(R^*) \subseteq \pi(NE(G, R^*))$.

We now establish that $\pi[NE(G, R^*)] \subseteq f(R^*)$. To do this, we consider various candidate equilibrium strategy combinations.

2. See Jackson (1988) for a general criticism of these game forms, especially the "integer game" construction.

3. We have chosen to be a little informal in our description of the game form. To be more exact we ought to have enlarged individual strategy spaces to allow individual i to signal the outcome he wishes to choose in (iii) and (iv).

Case 1: $\bar{s}_1 = (R, c, NF, k^1)$ and $\bar{s}_2 = (R, c, NF, k^2)$. The outcome is $\pi(\bar{s}_1, \bar{s}_2) = c$. Suppose $c \notin f(R^*)$. From part (ii) of Condition β, there exists $j \in I$ and $b \in C_j(R, c)$ such that $b \notin L_j(R^*, c)$. Agent j can therefore deviate profitably by raising his flag and picking b. Clearly $(\bar{s}_1 \bar{s}_2) \notin NE(G, R^*)$.

Case 2: $\bar{s}_1 = (R^1, a^1, NF, k^1)$ and $\bar{s}_2 = (R^2, a^2, NF, k^2)$. Let $\pi(\bar{s}_1, \bar{s}_2) = x$. Any deviation by 1 will yield an outcome in $C_1(R^2, a^2)$ and any deviation by 2, an outcome in $C_2(R^1, a^1)$. Therefore, if $(\bar{s}_1, \bar{s}_2) \in NE(G, R^*)$, then $x \in M_1(R^*, C_1(R^2, a^2)) \cap M_2(R^*, C_2(R^1, a^1))$. It follows from the construction of π and part (i) of Condition β that $x \in f(R^*)$.

Case 3: $\bar{s}_1 = (R^1, a^1, F, k^1)$ and $\bar{s}_2 = (R^2, a^2, NF, k^2)$. Let $\pi(\bar{s}_1, \bar{s}_2) = c$. By the construction of π, $c \in C_1(R^2, a^2)$. Suppose $(\bar{s}_1, \bar{s}_2) \in NE(G, R^*)$. Then clearly $c \in M_1(R^*, C_1(R^2, a^2))$. Also since 2 can (by raising his flag and announcing $k^2 > k^1$) obtain any outcome in A^*, it must be the case that $c \in M_2(R^*, A^*)$. If $c \neq a^2$ then (iii) of Condition β ensures that $c \in f(R^*)$. If $c = a^2$, then $a^2 \in f(R^2)$ and $C_i(R^2, a^2) \subseteq L_i(R^*, a^2) \forall i \in I$, and part (ii) of condition β implies that $c \in f(R^*)$.

Case 4: $\bar{s}_1 = (R^1, a^1, NF, k^1)$ and $\bar{s}_2 = (R^2, a^2, F, k^2)$. This case is the symmetric opposite of Case 3. The same arguments can be used to show that if $\pi(\bar{s}_1, \bar{s}_2) = c$, then $c \in f(R^*)$.

Case 5: $\bar{s}_1 = (R^1, a^1, F, k^1)$ and $\bar{s}_2 = (R^2, a^2, F, k^2)$. Let $\pi(\bar{s}_1, \bar{s}_2) = c$. If $(\bar{s}_1, \bar{s}_2) \in NE(G, R^*)$, then clearly $c \in M_1(R^*, A^*) \cap M_2(R^*, A^*)$, so that $c \in f(R^*)$ by part (iv) of Condition β.

Cases 1 to 5 exhaust all possible types of candidate equilibria. Therefore $NE(G, R^*) \subseteq f(R^*)$. ||

5. DOMAIN RESTRICTIONS

In this section we discuss a set of domain restrictions which allow us to escape from the Hurwicz–Schmeidler impossibility result. The particular assumptions we make are the following.

Assumption 5.1. Let A, the space of outcomes be a compact subset of some finite-dimensional Euclidean space. Moreover, for each individual, the set of admissible preferences is the set of continuous orderings over A. We continue to denote the set of admissible profiles by R.

Assumption 5.2. For all $R \in \mathcal{R}$, for all closed balls[4] $B \subset A$, $M_1(R, B) \cap M_2(R, B) = \phi$.

Definition 5.1. A SCC f satisfies Condition β^* if

(i) for all pairs $(R, a), (R', b) \in \mathcal{R} \times A$ with $a \in f(R)$ and $b \in f(R')$, $SL_1(R, a) \cap SL_2(R', b) \neq \emptyset$.

4. A closed ball B is defined as the set $\{x \in A \mid d(x, x_0) \leq \varepsilon\}$ for some $x_0 \in A$ and $\varepsilon > 0$, where $d(\cdot)$ is the Euclidean distance function.

(ii) for all $R, R' \in \mathscr{R}$ and $a \in A$, if $a \in f(R) - f(R')$, then there exists $i \in I$ such that $SL_i(R, a) \cap SU_i(R', a) \neq \varnothing$.
(iii) for all $R \in \mathscr{R}$, $M_1(A, R) \cap M_2(A, R) \subseteq f(R)$.

Notice that the self-selection and monotonicity conditions of Condition β have been strengthened in Condition β^*. On the other hand some of the conditions in Condition β^* have now been dropped and the $C_i(.)$ sets dispensed with altogether.

Proposition 5.1. *Suppose Assumptions 5.1 and 5.2 hold. Then a SCC which satisfies Condition β^* is implementable.*

Proof. Let f be a SCC satisfying Condition β^*. Part (i) of β^* and the continuity of preferences imply that for all pairs $(R, a), (R', b) \in \mathscr{R} \times A$ with $a \in f(R)$ and $b \in f(R')$, there exists a closed ball $\bar{W}(R, R', a, b) \subset SL_i(R, a) \cap SL_j(R', b)$. Let $W(i, R, a) = \bigcup_{\{(b, R'): b \in f(R')\}} \bar{W}(R, R', a, b)$. For all $i \in I$, let $Q(i, R, a) = \{R' \in \mathscr{R} \mid SL_i(R, a) \cap SU_i(R_i, a) \neq \phi\}$. Fix $i \in I$ and let $R' \in Q(i, R, a)$. By the continuity of preferences, there exists a closed ball $\bar{Z}(i, R, R', a) \subset SL_i(R, a) \cap SU_i(R', a)$. Let $Z(i, R, a) = \bigcup_{\{R' \in Q(i, R, a)\}} \bar{Z}(i, R, R', a)$.

For all $i \in I$, $R \in \mathscr{R}$ and $a \in f(R)$, let $C_i(R, a) = \{a\} \cup W(i, R, a) \cup Z(i, R, a)$ and $A^* = A$. We now show that f satisfies condition β with respect to the $C_i(R, a)$ and A^* sets.

Observe that by construction, $W(i, R, a)$ and $Z(i, R, a)$ are both subsets of $L_i(R, a)$, so that $C_i(R, a) \subset L_i(R, a)$. Let $R' \in \mathscr{R}$ and $b \in f(R')$. Since $\bar{W}(R, R', a, b) \in C_i(R, a) \cap C_j(R', b)$, self-selection is satisfied. Let $x \in \bar{W}(R, R', a, b)$. By Assumption 5.2, $x \notin M_1(\hat{R}, \bar{W}(R, R', a, b)) \cap M_2(\hat{R}, \bar{W}(R, R'a, b))$ for any $\hat{R} \in \mathscr{R}$. Hence, part (1)(b) of β is also satisfied. Suppose $a \notin f(R')$. Then by part (ii) of β^*, $R' \in Q(1, R, a) \cup Q(Z, R', a)$. Assume w.l.o.g. that $R' \in Q(1, R, a)$. Let $x \in \bar{Z}(1, R, R', a)$. Observe that $x \in C_1(R, a)$ but $x \notin L_1(R, a)$. To check part (iii) of β, suppose that $x \in M_i[R', C_i(R, a)] \backslash \{a\} \cap M_j(R', A)$. Clearly, either $x \in W(i, R, a)$ or $x \in Z(i, R, a)$. Suppose the former is true, and let $x \in \bar{W}(R, R'', a, c)$ for some pair $(R'', c) \in \mathscr{R} \times A$. Since $x \in M_i(R', C_i(R, a))$, we must have $x \in M_i(R, \bar{W}(R, R'', a, c))$. Also, $x \in M_j(R', A)$ implies that $x \in M_j(R', \bar{W}(R, R'', a, c))$. However this contradicts Assumption 5.2. A similar argument shows that $x \in Z(i, R, a)$ also contradicts Assumption 5.2. Finally observe that parts (iv) and (iii) of β and β^* respectively, are identical. Therefore, f satisfies Condition β. The result follows by applying Theorem 4.1. ||

Remark. Moore and Repullo (1990) prove a related though not identical result (Corollary 4). One feature of their proof is that the $C_i(\cdot)$ sets are chosen to be open sets. This construction eliminates several "undesirable" equilibria simply because individuals are forced to maximize on open sets. Our proofs avoids this unattractive feature but at the cost of a new assumption (Assumption 5.2).

We now provide examples of two environments where it is possible to find SCCs which satisfy Condition β^*.

Example 1. Let $X = \{a, \ldots, a_k\}$ represent a finite set of social alternatives and let $\mathscr{L}$ denote the set of lotteries over the elements of X. Both individuals have von Neuman-Morgenstern utility functions defined over X. Individual i's utility function u_i induces an ordering R_i over $\mathscr{L}$ in the following manner: for all $p, q \in \mathscr{L}$, pR_iq iff $u_i \cdot p \geq u_i \cdot q$ (where $\cdot$ signifies the inner product). Let U be the set of 2-tuples of utility functions over X satisfying

Assumption 5.3. $\forall u \in U, \forall i \in I$ and $\forall$ distinct $a_j, a_k \in X$, $u_i(a_j) \neq u_i(a_k)$.

Assumption 5.4. For all $u \in U$, $u_i \neq u_j$ for $i \neq j$.

Assumption 5.5. For all $u \in U$, either [there is a pair $(a_j, a_k) \in X$ such that $u_i(a_j) > u_i(a_k) \Rightarrow u_j(a_j) > u_j(a_k)$ or [there exist pairs $(a_j, a_k), (a_m, a_n)$ such that

$$\frac{u_i(a_j) - u_i(a_k)}{u_i(a_m) - u_i(a_n)} \neq \frac{u_j(a_j) - u_j(a_k)}{u_j(a_m) - u_j(a_n)}.$$

Assumption 5.3 expresses the restriction that individuals' preferences over X are strong orderings, while Assumption 5.4 rules out complete unanimity. We will comment on Assumption 5.5 shortly.

In this setting, we identify the set $\mathscr{L}$ as the set of alternatives A. A social choice correspondence, then, associates a non-empty subset $f(R) \subset \mathscr{L}$ where $R = (R_1, R_2) \in \mathscr{R}$ if it is induced by some $u \in U$. It is immediate that Assumption 5.1 is satisfied. Note also that Assumption 5.3 and 5.4 together imply Assumption 5.2.

Let $\bar{q} \in \mathscr{L}$ denote the *uniform* lottery over X. An immediate consequence of Assumption 5.3 is that for any $i \in I$, $R \in \mathscr{R}$, the sets $SL_i(R, \bar{q})$ and $SU_i(R, \bar{q})$ are nonempty. We now show that Assumption 5.5 also implies that $SL_i(R, \bar{q}) \cap SL_j(R, \bar{q}) \neq \phi$.

Lemma 5.1. *Suppose Assumption* 5.3 *and* 5.5 *hold. Then, for all* $R \in \mathscr{R}$, $SL_i(R, \bar{q}) \cap SL_j(R, \bar{q}) \neq \phi$.

Proof. Choose any u satisfying Assumption 5.3 and 5.5. Suppose there is a pair $(a_j, a_k) \in X$ such that $u_1(a_j) > u_1(a_k)$ and $u_2(a_j) > u_2(a_k)$. Construct $q \in \mathscr{L}$ such that $q_i = \bar{q}_i$ for $i \neq j, k$, $q_j = \bar{q}_j + \varepsilon$ and $q_k = \bar{q}_j - \varepsilon$, where $0 < \varepsilon < 1/K$. It is obvious that $q \in SL_i(R, \bar{q}) \cap SL_j(R, \bar{q})$, where R is induced by u.

Suppose now that for all $a_j, a_k \in X$, $u_i(a_j) > u_i(a_k) \Rightarrow u_j(a_k) > u_j(a_j)$. From Assumption 5.5, there exist pairs $(a_j, a_k), (a_m, a_n)$ such that

$$\frac{u_1(a_j) - u_1(a_k)}{u_1(a_m) - u_1(a_n)} \neq \frac{u_j(a_j) - u_j(a_k)}{u_j(a_m) - u_j(a_n)}.$$

Assume w.l.o.g. that $[u_1(a_j) > u_1(a_k) > u_1(a_m) > u_1(a_n)]$ and $[u_2(a_n) > u_2(a_m) > u_2(a_k) > u_2(a_j)]$ and

$$\frac{u_1(a_j) - u_1(a_k)}{u_1(a_m) - u_1(a_n)} > \frac{u_2(a_j) - u_2(a_k)}{u_2(a_m) - u_2(a_n)}.$$

Choose $\varepsilon_1, \varepsilon_2$ such that (i) $0 < \varepsilon_i < 1/K$ for $i = 1, 2$, and

$$\text{(ii)} \quad \frac{u_1(a_j) - u_1(a_k)}{u_1(a_m) - u_1(a_n)} > \frac{\varepsilon_1}{\varepsilon_2} > \frac{u_2(a_j) - u_2(a_k)}{u_2(a_m) - u_2(a_n)}.$$

Now, construct $q \in \mathscr{L}$ such that (i) $q_i = \bar{q}_i$ for $i \notin \{j, k, m, n\}$ (ii) $q_j = \bar{q}_j - \varepsilon_2$, (iii) $q_k = \bar{q}_k + \varepsilon_2$ (iv) $q_m = \bar{q}_m + \varepsilon_1$, (v) $q_n = \bar{q}_n - \varepsilon_1$. It can be checked that $q \in SL_1(R, \bar{q}) \cap SL_2(R, \bar{q})$. ‖

We turn now to a discussion of Assumption 5.5. This assumption rules out utility profiles of the following kind: individual 1 has a utility function u with $u(a_1) > u(a_2) \cdots > u(a_k)$, individual 2 has a utility function v with $v(a_k) > v(a_{k-1}) \cdots v(a_1)$ and $u(a_j) = \lambda v(a_{k+1-j}) + \mu$, for some $\lambda > 0$ and $j = 1, \ldots, K$. In other words, individual 2's utility function is the inverse (up to an affine transformation) of 1's utility function. Observe that a "slight" perturbation in the utility function of one individual (i.e. changing the utility number associated with some outcome) is sufficient to ensure that the assumption holds. This suggests that the assumption is likely to be satisfied "almost" always.

For any $R \in \mathscr{R}$ let $B(R) = \{q \in \alpha \mid qR_i\bar{q} \forall i \in I\}$, and $PO(R) = \{q \in \mathscr{L} \mid$ there is no $\hat{q} \in \mathscr{L}$ such that $\hat{q}P_iq$ and $\hat{q}R_jq\}$, so that $PO(R)$ is the set of *Pareto efficient* lotteries in $\mathscr{L}$ given R.

Consider the following SCC f: for all $R \in \mathscr{R}, f^*(R) = B(R) \cap PO(R)$. First, note that f^* is well-defined since $B(R) \cap PO(R) \neq \phi$. Moreover, f^* satisfies Condition β^*. To check this, observe that Lemma 5.1 guarantees that f^* satisfies part (i) of Condition β^*. It is also obvious that f^* satisfies part (iii) of Condition β^*. Finally, both $B(R)$ and $PO(R)$ satisfy part (ii) of Condition β^*, and hence so does $B(R) \cap PO(R)$.

Example 2. Consider a two-person exchange economy in which individual preferences are strictly monotonic and such that indifference curves never touch the axes (generated, for example, by Cobb–Douglas utility functions). Consider a (Maskin) monotonic choice rule which always allocates strictly positive vectors of commodities. An example is the Walrasian correspondence. Such a choice rule will in this framework satisfy Condition β^*.

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