A Generalised Biane Process

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Using the methods of quantum probability in a toy Fock space as outlined in [2] and the theory of finite dimensional representations of the 3-dimensional simple Lie algebra $sl(2, \mathbb{C})$ Ph. Biane [1] constructed a quantum Markov chain in discrete time and derived a classical Markov chain which is the discrete time quantum analogue of a classical Bessel process. Here exploiting the Peter-Weyl theory of representations of compact groups we extend Biane's construction to an arbitrary compact group G and derive a classical Markov chain whose state space is the space $\Gamma(G)$ of all characters of irreducible representations of G.

Let G be a compact second countable topological group and let $g \to L_g$ denote its left regular representation in the complex Hilbert space $L_2(G)$ of all square integrable functions on G with respect to its normalised Haar measure. Let $\mathcal{W}(G)$ denote the W^* algebra generated by the family $\{L_g, g \in G\}$ and let $\mathcal{Z}(G)$ be its centre. Denote by $\Gamma(G)$ the countable set of all characters of irreducible unitary representations of G. For any χ let U^χ be an irreducible unitary representation of G with character χ and dimension $d(\chi)$. If $\chi_1, \chi_2 \in \Gamma(G)$ the tensor product $U^{\chi_1} \otimes U^{\chi_2}$ decomposes into a direct sum of irreducible representations. We shall denote by $m(\chi_1, \chi_2; \chi)$ the multiplicity with which the type U^χ appears in such a decomposition of $U^{\chi_1} \otimes U^{\chi_2}$. Define

$$p^{\chi}_{\chi_1, \chi_2} = \frac{m(\chi, \chi_1; \chi_2)\, d(\chi_2)}{d(\chi)\, d(\chi_1)} \quad . \tag{1}$$

Then

$$\sum_{\chi_2 \in G} p^{\chi}_{\chi_1, \chi_2} = 1 \qquad \text{for each} \quad \chi, \chi_1 \in \Gamma(G) \quad .$$

In other words, for every fixed $\chi \in \Gamma(G)$ the matrix $P^\chi = ((p^{\chi}_{\chi_1, \chi_2}))$ is a stochastic matrix over the state space $\Gamma(G)$. In each row of P^χ all but a finite number of entries are 0 and each entry is rational. Inspired by Biane's construction in [1] we shall now combine the Peter-Weyl theorem and the methods of quantum probability in order to realise explicitly a Markov chain with transition probability matrix P^χ in the state space $\Gamma(G)$.

As a special case consider $G = SU_2$. Let χ_n denote the character of the "unique" irreducible unitary representation of G of dimension n. By (1) and

Clebsch-Gordon formula

$$p^{\chi_2}_{\chi_i,\chi_j} = \begin{cases} \dfrac{i-1}{2i} & \text{if} \quad j = i-1, \\ \dfrac{i+1}{2i} & \text{if} \quad j = i+1, \\ 0 & \text{otherwise}. \end{cases}$$

This is the case covered by Biane [1].

To realise our goal we recall that by Peter-Weyl theorem $L_2(G)$ admits the Plancherel decomposition:

$$L_2(G) = \oplus_{\chi \in \Gamma(G)} \mathcal{H}_\chi$$

where $\dim \mathcal{H}_\chi = d(\chi)^2$, L_g leaves each $\mathcal{H}_\chi$ invariant and $L_g|_{\mathcal{H}_\chi}$ is a direct sum of $d(\chi)$ copies of the representation U^χ. If π_χ denotes the orthogonal projection onto the component $\mathcal{H}_\chi$ then

$$\pi_\chi = d(\chi)^{-1} \int_G \chi(g) L_g \, dg \tag{2}$$

thanks to Schur orthogonality relations. Furthermore the abelian W^* algebra $\mathcal{Z}(G)$ is generated by the family $\{\pi_\chi, \chi \in \Gamma(G)\}$.

Fix $\chi_0 \in \Gamma(G)$. Let U^{χ_0} act in the Hilbert space $\mathcal{H}$. Denote by ρ the density matrix $d(\chi_0)^{-1} I$ in $\mathcal{H}$. Fix a positive integer N and consider the Hilbert space $\mathcal{H}^{\otimes N} = \mathcal{H} \otimes \cdots \otimes \mathcal{H}$ where the tensor product is taken N-fold. Denote by $\mathcal{B}_{n]}$ the W^* algebra generated by all operators of the form $X_1 \otimes \cdots \otimes X_n \otimes I \otimes \cdots \otimes I$ where X_i are bounded operators in $\mathcal{H}$. Then $\mathcal{B}_{1]} \subset \mathcal{B}_{2]} \subset \cdots \subset \mathcal{B}_{N]} = \mathcal{B}$ yields a finite filtration in $\mathcal{B}$ with conditional expectation maps $E_{n]} : \mathcal{B} \to \mathcal{B}_{n]}$, $1 \le n \le N$, defined by

$$E_{n]}\, X_1 \otimes \cdots \otimes X_N = \Big(\prod_{i=n+1}^{N} \operatorname{tr} \rho X_i \Big) X_1 \otimes \cdots \otimes X_n \otimes I \otimes \cdots \otimes I$$

for all $X_i \in \mathcal{B}(\mathcal{H})$ and linear extension. Thanks to Peter-Weyl theorem there exists a unique identity preserving and continuous * homomorphism $j_n : \mathcal{W}(G) \to \mathcal{B}_{n]}$ satisfying

$$j_n(L_g) = U_g^{\chi_0} \otimes \cdots \otimes U_g^{\chi_0} \otimes I \otimes \cdots \otimes I \quad \text{for all} \quad g \in G \tag{3}$$

where $U_g^{\chi_0}$ appears n-fold and I, $(N-n)$-fold. Then

$$E_{n-1]}\; j_n(L_g) = j_{n-1}(d(\chi_0)^{-1} \chi_0(g) L_g) \quad \text{for all} \quad g \in G\ . \tag{4}$$

Thus there exists a completely positive map $T : \mathcal{W}(G) \to \mathcal{W}(G)$ satisfying

$$E_{n-1]}\; j_n(X) = j_{n-1}(T(X)), \qquad X \in W(G), \tag{5}$$

$$T(L_g) = d(\chi_0)^{-1} \chi_0(g) L_g \quad \text{for all} \quad g \in G\ . \tag{6}$$

From (2) and Schur orthogonality relations it now follows that

$$
\begin{aligned}
T(\pi_\chi) &= [d(\chi_0)d(\chi)]^{-1}\int \chi_0(g)\chi(g)L_g\,dg \\
&= \sum_{\chi'\in\Gamma(G)} [d(\chi_0)d(\chi)]^{-1}d(\chi')\,m(\chi_0,\chi;\chi')\,\pi_{\chi'} \qquad (7)\\
&= \sum_{\chi'\in\Gamma(G)} p^{\chi_0}_{\chi,\chi'}\,\pi_{\chi'} \ .
\end{aligned}
$$

We now establish the following lemma.

Lemma 1. *For any $m < n$, $Z \in \mathcal{Z}(G)$, $X \in \mathcal{W}(G)$*

$$[j_m(Z), j_n(X)] = 0 \ . \qquad (8)$$

In particular, the family $\{j_m(Z),\ m = 1,2,\dots,N,\ Z \in \mathcal{Z}(G)\}$ is commutative.

Proof. We have for any $g, h \in G$,

$$
\begin{aligned}
[j_m(L_g), j_n(L_h)] &= [\underbrace{U_g^{\chi_0}\otimes\cdots\otimes U_g^{\chi_0}}_{m\text{-fold}}\otimes I\otimes\cdots\otimes I, \underbrace{U_h^{\chi_0}\otimes\cdots\otimes U_h^{\chi_0}}_{n\text{-fold}}\otimes I\otimes\cdots\otimes I] \\
&= j_m([L_g, L_h])\underbrace{I\otimes\cdots\otimes I}_{m\text{-fold}}\otimes\underbrace{U_h^{\chi_0}\otimes\cdots\otimes U_h^{\chi_0}}_{(n-m)\text{-fold}}\otimes\underbrace{I\otimes\cdots\otimes I}_{(N-n)\text{-fold}} \ .
\end{aligned}
$$

Since Z can be approximated by linear combinations of $L_g, g \in G$, it follows that $[j_m(Z), j_n(L_h)] = 0$. Since X can be approximated by linear combinations of L_h we have (8). The second part is immediate.

From (3), (5) - (7) and Lemma 1 we have the following theorem.

Theorem 2. *For any fixed $\chi_0 \in \Gamma(G)$ let the W^* homomorphisms $j_n : \mathcal{W}(G) \to \mathcal{B}$, $1 \le n \le N$, be defined by (3). In the state $\rho^{\otimes N}$, where $\rho = d(\chi_0)^{-1}I$, the sequence $\{j_n, 1 \le n \le N\}$ is a quantum Markov chain in the sense of Accardi--Frigerio-Lewis with transition operator $T : \mathcal{W}(G) \to \mathcal{W}(G)$ satisfying $T(L_g) = d(\chi_0)^{-1}\chi_0(g)L_g$ for all $g \in G$. T leaves the centre $\mathcal{Z}(G)$ of $\mathcal{W}(G)$ invariant. The family $\{j_n(Z), Z \in \mathcal{Z}(G), 1 \le n \le N\}$ is commutative. In the state $\rho^{\otimes N}$ the sequence $\{j_n|_{\mathcal{Z}(G)}, 1 \le n \le N\}$ is a classical Markov chain with state space $\Gamma(G)$ and transition probability matrix $P^{\chi_0} = ((p^{\chi_0}_{\chi,\chi'}))$, $\chi, \chi' \in \Gamma(G)$, defined by (1).*

Remark. We can replace the W^* algebra $\mathcal{W}(G)$ by the * unital algebra $\mathcal{U}(G)$ of left invariant differential operators on G if G is a compact connected Lie group. In such a case $\mathcal{Z}(G)$ can be replaced by the centre $z(G)$ of $\mathcal{U}(G)$. If we choose this infinitesimal description, put $G = SU_2$ and choose χ_0 as the character of the unique 2-dimensional irreducible unitary representation of G, then we obtain Biane's example in [1].