

A QUALITATIVE UNCERTAINTY PRINCIPLE FOR SEMISIMPLE LIE GROUPS

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*Dedicated to Robert Edwards in recognition of
25 years' distinguished contribution to mathematics in Australia,
on the occasion of his retirement*

Abstract

Recently M. Benedicks showed that if a function $f \in L^2(\mathbf{R}^d)$ and its Fourier transform both have supports of finite measure, then $f = 0$ almost everywhere. In this paper we give a version of this result for all noncompact semisimple connected Lie groups with finite centres.

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0. Introduction

Let G be a locally compact group equipped with left Haar measure dm and \hat{G} its unitary dual (that is, a maximal set of pairwise inequivalent unitary irreducible continuous representations of G). For $f \in L^1(G)$ and $\pi \in \hat{G}$, define the operator $\pi(f) = \int_G f(x)\pi(x) dm(x)$ (which acts on the underlying Hilbert space for π). The assignment $\pi \rightarrow \pi(f)$ can be thought of as the (group theoretic) analogue of the classical Fourier transform \hat{f} of an integrable function on \mathbf{R} . It has long been recognized that if f is 'concentrated' near a point, then \hat{f} has to be 'spread out' and vice versa. A quantitative expressions of this principle leads to the Heisenberg uncertainty principle—see for example [5].

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Another expression of this principle is the following result of Benedicks [2]: If $f \in L^2(\mathbf{R}^d)$ with $m(\text{supp } f) < \infty$ and $m(\text{supp } \hat{f}) < \infty$, then $f = 0$ a.e. (If $\text{supp } f$ is compact then the above result collapses to an easy exercise in introductory Fourier analysis. However with only the assumption $m(\text{supp } f) < \infty$, the result quoted above is more substantial.) In view of this a natural question to ask is whether the above principle can be formulated for a locally compact group G . In this paper we show that a principle very close to the one of Benedicks holds for all noncompact semisimple connected Lie groups with finite centres. Earlier this kind of principle had been established for a wide variety of groups including $\text{SL}(2, \mathbf{R})$ ([7]). However, for general semisimple Lie groups rather severe restrictions had to be placed on the kind of L^1 functions being dealt with. For quantitative versions of this principle for certain groups see [4], [8] and [9].

1. Notation and preliminaries

Throughout this paper G will denote a connected noncompact semisimple Lie group with finite centre. (For unexplained terminology and results, see [11].) Fix a maximal compact subgroup K of G . Let \hat{G} denote the unitary dual of G and \hat{K} the unitary dual of K . Fix a Haar measure m on G —as is well known G is unimodular—and let μ be the (corresponding) Plancherel measure on \hat{G} . In this section we describe the structure and representation theory of G that will be needed in the next section.

Let $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ be a Cartan decomposition of the Lie algebra \mathfrak{g} of G with Cartan involution θ . Here \mathfrak{k} is the Lie algebra of K . Let \mathfrak{a} be a maximal abelian subalgebra of \mathfrak{p} and let

$$\mathfrak{g} = \mathfrak{m} \oplus \mathfrak{a} \oplus \sum_{\alpha \in R} \mathfrak{g}_{\alpha}$$

be a decomposition of \mathfrak{g} into real root spaces for \mathfrak{a} , where \mathfrak{m} is the centralizer of \mathfrak{a} in \mathfrak{k} and R is the set of nonzero real roots. Fix once and for all a set of positive roots R^+ , and let S be the set of simple positive roots. We write \mathfrak{n} for $\sum_{\alpha \in R^+} \mathfrak{g}_{\alpha}$. At the group level, we write K, A and N for the connected subgroups of G with Lie algebras $\mathfrak{k}, \mathfrak{a}$ and \mathfrak{n} respectively, and M for the centralizer of A in K . Then MAN is a so-called minimal parabolic subgroup, hereafter denoted P_0 . (It is unique up to conjugation.)

The other “parabolic subgroups” of G (up to conjugation) all arise in the following way. Pick a subset S_i of S and let R_i be the set of roots which are linear combination of roots in S_i . There is a unique closed subgroup of G , denoted by P_i (known as a parabolic subgroup), which contains P_0 and whose

Lie algebra is

$$\bigoplus_{\alpha \in R_i} \mathfrak{g}_\alpha + \mathfrak{m} + \mathfrak{a} + \mathfrak{n}.$$

This group has a Langlands decomposition $P_i = M_i A_i N_i$, where M_i is reductive, A_i is abelian and N_i is nilpotent. The Lie algebra \mathfrak{m}_i of M_i is generated by $\mathfrak{m} \oplus \sum_{\alpha \in R_i} \mathfrak{g}_\alpha$; A_i and N_i are $\exp(\mathfrak{a}_i)$ and $\exp(\mathfrak{n}_i)$ respectively (exp denoting the exponential map), where \mathfrak{a}_i is the orthogonal complement to $\mathfrak{m}_i \cap \mathfrak{a}$ in \mathfrak{a} , relative to the inner product on \mathfrak{a} induced by the Killing form, and $\mathfrak{n}_i = \sum_{\alpha \in R^+ \setminus R_i} \mathfrak{g}_\alpha$. If M_i contains a compact Cartan subgroup, then P_i is said to be cuspidal. We let $\{P_j; j \in J\}$ be a maximal set of (nonconjugate) cuspidal parabolic subgroups constructed as above.

Harish-Chandra showed that sufficiently many irreducible unitary representations of G to decompose $L^2(G)$ may be obtained by taking a cuspidal parabolic subgroup P_j , a discrete series representation δ of M_j (that is, $\delta \in (\hat{M}_j)_d$) and a character $\chi_\lambda: \exp(H) \rightarrow \exp(i\lambda(H))$ of A_j (where $\lambda \in \mathfrak{a}_j^*$, the real dual of \mathfrak{a}_j), forming the unitary representation (denoted abusively) $\delta \otimes \chi_\lambda \otimes 1$ of P_j (where $\delta \otimes \chi_\lambda \otimes 1(man) = \delta(m)\chi_\lambda(a)$, $m \in M_j$, $a \in A_j$, $n \in N_j$) and inducing unitarily to G . We write $\pi_{\delta,\lambda}^{(j)} = \text{ind}_{P_j}^G \delta \otimes \chi_\lambda \otimes 1$. The representations $\pi_{\delta,\lambda}^{(j)}$ and $\pi_{\varepsilon,\mu}^{(i)}$ can be equivalent only if $P_i = P_j$, and then if and only if (δ, λ) and (ε, μ) are conjugate under an appropriate (finite) Weyl group action. In [6] (see in particular Sections 25 and 36) Harish-Chandra calculated explicitly the Plancherel measure associated with the various series of representations of G . Except in the case when $P_i = G$ (that is, when G is a cuspidal parabolic subgroup of itself), A_i is a nontrivial vector group, and then for fixed δ in $(\hat{M}_i)_d$, the Plancherel measure $\mu(i, \delta, \lambda)$ is a smooth function of λ , which actually extends to an analytic function in a tube containing \mathfrak{a}_i^* in $(\mathfrak{a}_i^*)_{\mathbb{C}}$ (the complexification of \mathfrak{a}_i) and is of polynomial growth in λ in almost all directions in \mathfrak{a}_i^* . An easy consequence of Harish-Chandra's calculation is $\mu(i, \delta, \mathfrak{a}_i^*) = \infty$.

Now we need to study the representations $\pi_{\delta,\lambda}^{(i)}$ in more detail. We fix a *proper* parabolic subgroup P_i of G and $\delta \in (\hat{M}_i)_d$. Let H_δ be the Hilbert space of δ . Define $H_\delta^{(i)}$ as the space of measurable H_δ -valued functions v on K which satisfy the conditions

$$v(km) = \delta(m^{-1})v(k), \quad k \in K, m \in K \cap M_i,$$

and

$$\int_K |v(k)|^2 dk < \infty.$$

The induced representations $\pi_{\delta,\lambda}^{(i)}$ may be considered to act unitarily on $H_\delta^{(i)}$ by the formula

$$[\pi_{\delta,\lambda}^{(i)}(g)v](k) = \delta(m^{-1})v(k') \exp[(i\lambda + \rho_i)H_i(g^{-1}k)]$$

where

$$\rho_i = \frac{1}{2} \sum_{\alpha \in R^+ \setminus R_i} (\dim \mathfrak{g}_\alpha) \alpha \quad \text{and} \quad g^{-1}k = k'm \exp(H_i(g^{-1}k))n,$$

with $k' \in K$, $m \in M_i \cap AN$, $n \in N_i$ and $H_i(g^{-1}k) \in \mathfrak{a}_i$. (It should be noted that every element of G can be expressed uniquely in the form $kman$ where $k \in K$, $m \in M_i \cap AN$, $a \in A_i$ and $n \in N_i$. Indeed the 'Iwasawa decomposition' gives a unique decomposition of the form kan with $k \in K$, $a \in A$, $n \in N$. Furthermore an then factorizes uniquely as $a'n'a_i n_i$ with $a'n' \in AN \cap M_i$ and $a_i n_i \in A_i N_i$.) We note that the action of K on $H_\delta^{(i)}$ is just left translation and this is independent of λ in \mathfrak{a}_i^* .

Fix a basis $\{e_j: j \in \mathbf{N}\}$ of $H_\delta^{(i)}$ consisting of K -finite vectors. We have the following result:

PROPOSITION. *Fix a proper cuspidal parabolic subgroup P_i and choose δ in $(\hat{M}_i)_d$. Given $f \in L^1(G)$ and $j, h \in \mathbf{N}$, the function*

$$\lambda \rightarrow \int_G f(x) \langle \pi_{\sigma, \lambda}^{(i)}(x) e_j, e_h \rangle dx$$

on \mathfrak{a}_i^ extends to a holomorphic function in a tube in $(\mathfrak{a}_i^*)_{\mathbb{C}}$ which contains \mathfrak{a}_i^* .*

PROOF. Using the techniques of Cowling [3] and of Anker [1], it is straightforward to show that, if the imaginary part of λ in $(\mathfrak{a}_i^*)_{\mathbb{C}}$ is not too big, then the representation $\pi_{\delta, \lambda}^{(i)}$ of the analytic continuation acts isometrically on a mixed L^p -space which we denote $L^p(K)$. The basis vectors e_j and e_h being smooth lie in $L^p(K)$ and its dual $L^q(K)$ for all p . In fact

$$\begin{aligned} |\langle \pi_{\delta, \lambda}^{(i)}(x) e_j, e_h \rangle| &\leq \|\pi_{\delta, \lambda}^{(i)}(x) e_j\|_p \|e_h\|_q = \|e_j\|_p \|e_j\|_q \leq \|e_j\|_\infty \|e_h\|_\infty \\ &= \sup_{k \in K} \|e_j(k)\|_{HS} \sup_{k \in K} \|e_h(k)\|_{HS} < \infty. \end{aligned}$$

The proposition follows immediately.

2. The main results

We are now in a position to state and prove the following theorem.

THEOREM. *Let G, \hat{G}, K, μ and m be as in the introduction. Let $f \in L^1(G)$ and let $A_f = \{x: f(x) \neq 0\}$ and $B_f = \{\pi \in \hat{G}: \pi(f) \neq 0\}$. If $m(KA_fK) < \infty$ and $\mu(B_f) < \infty$, then $f = 0$ a.e.*

PROOF. The statement that $\mu\{\pi: \pi(f) \neq 0\} < \infty$ implies that the Plancherel measure of each set $\{\tau \in \hat{A}_i: \pi_{\sigma,\tau}^{(i)}(f) \neq 0\}$ is finite. Since the Plancherel measure of $\{\pi_{\sigma,\tau}\}_{\tau \in \hat{A}_i}$ for fixed σ and i is infinite, it follows that $\{\tau \in \hat{A}_i: \pi_{\sigma,\tau}^{(i)}(f) = 0\}$ has positive Plancherel measure and hence has positive Lebesgue measure. This last statement follows from the fact that the Plancherel measure on the series $\{\pi_{\sigma,\tau}^{(i)}\}_{\tau \in \hat{A}_i}$, σ, i fixed, is absolutely continuous with respect to Lebesgue measure on \hat{A}_i . Now one knows that (at least in the sense of distributions) $f = \sum_{\nu \in \hat{K}} \sum_{\mu \in \hat{K}} d(\mu)d(\nu)\chi_\mu * f * \chi_\nu$. (Here we are identifying the characters χ_μ and χ_ν of the representations μ and ν of the compact group K with the (singular) measures $\chi_\mu dk$ and $\chi_\nu dk$ on G . Also for each $\mu \in \hat{K}$, $d(\mu)$ is its dimension.) Fix $\delta_1, \delta_2 \in \hat{K}$ and consider $h = \chi_{\delta_1} * f * \chi_{\delta_2}$. Let $E_\sigma^{(i)} = \{\tau \in \hat{A}_i: \pi_{\sigma,\tau}^{(i)}(f) = 0\}$. From what we said above $E_\sigma^{(i)}$ has positive Lebesgue measure. Now notice that if $\tau \in E_\sigma^{(i)}$, then $\pi_{\sigma,\tau}^{(i)}(h)$ is also zero. Thus $\pi_{\sigma,\tau}^{(i)}(h)$ is zero on a set of positive Lebesgue measure in \hat{A}_i . Let u_1, \dots, u_m be those basis vectors in $H_\sigma^{(i)}$ which transform according to δ_1 for $\pi_{\sigma,\tau}^{(i)}|_K$ and w_1, \dots, w_n be those basis vectors in $H_\sigma^{(i)}$ which transform according to δ_2 for $\pi_{\sigma,\tau}^{(i)}|_K$. (Notice these are independent of τ for σ and i fixed.) Since h satisfies $h = d(\delta_1)d(\delta_2)\chi_{\delta_1} * h * \chi_{\delta_2}$, it follows that $\pi_{\sigma,\tau}^{(i)}(h)$ is completely determined by the scalars $\langle \pi_{\sigma,\tau}^{(i)}(h)w_s, u_t \rangle$, $1 \leq s \leq n$ and $1 \leq t \leq m$. However $\langle \pi_{\sigma,\tau}^{(i)}(h)w_s, u_t \rangle = \int_G \langle \pi_{\sigma,\tau}^{(i)}(x)w_s, u_t \rangle h(x) dx$. By the proposition, as a function of τ the above function is holomorphic in a strip containing \hat{A}_i . Thus the fact that this vanishes in a set of positive Lebesgue measure on \hat{A}_i forces it to be identically zero on \hat{A}_i . Hence for fixed i and fixed $\sigma \in (\hat{M}_i)_d$, $\pi_{\sigma,\tau}^{(i)}(h) = 0$ for all τ . Thus for all i and $\sigma \in (\hat{M}_i)_d$, $\pi_{\sigma,\tau}^{(i)}(h) = 0$ for all $\tau \in \hat{A}_i$. This means that $\pi_{\sigma,\tau}^{(i)}(h) = 0$ unless $P_i = G$ so that the Fourier transform of h is supported by the discrete series of G . From D. Vogan's theory of minimal K -types [10], it is clear that only finitely many discrete series representations of G when restricted to K can contain the representations δ_1 and δ_2 . On the other hand, it is routine to show that if $\sigma \in \hat{G}_d$, then $\sigma(h) = 0$ unless $\sigma|_K$ contains δ_1 and δ_2 . Consequently h is a finite linear combination of matrix elements of the discrete series of G , and is therefore real analytic on G . Now since $h = \chi_{\delta_1} * f * \chi_{\delta_2}$, if $x \notin KA_fK$, $h(x) = 0$. But by our assumption $m(KA_fK) < \infty$ and so $m(KA_fK)^c > 0$. Thus since G is connected and h is real analytic, this forces $h \equiv 0$. However $f \sim \sum \sum d(\delta)d(\nu)\chi_\delta * f * \chi_\nu$ and we have just shown that each term on the right side is zero. Hence $f = 0$ as a distribution, that is, $f = 0$ a.e.

An examination of the proof shows we have actually proved the following stronger result.

COROLLARY (to proof of Theorem). *Let $f \in L^1(G)$ with $m(KA_fK)^c > 0$. Assume for each fixed i, σ that the set $\{\tau \in \hat{A}_i, \pi_{\sigma,\tau}^{(i)}(f) = 0\}$ has positive Lebesgue measure. Then $f = 0$ a.e.*

REMARK. The property discussed in this paper (and in [2] and [7]) fails completely in many situations. For example, the Fourier transform of the characteristic function of a compact open subgroup of the p -adic numbers is another such characteristic function. Also the existence of supercuspidal representations for reductive p -adic groups gives rise to further counterexamples [7].

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