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A Critical Appreciation on Algebraic Image Restoration

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Abstract —**The authors recultivate the aspect of unconstrained algebraic restoration of gray level image, degraded due to defocusing, additive noise, etc., with additional nonnegative constraint on the individual pixel of gray-level image. The negative values in the gray level introduce absurdity in the entire representation. But this aspect has not been explicitly considered in the existing literatures. In the present work, the authors demonstrate that under certain conditions the constrained and unconstrained solutions for the restored image are same. It is also demonstrated that using Wolfe's algorithm the nonnegative constraint of the gray-level image can be handled very efficiently. The necessary and sufficient conditions of the optimal constrained restoration are tested very explicitly. In the existing literatures on optimal image restoration these sort of explicit tests on the necessary and sufficient conditions are absent.**

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I. Introduction

Algebraic restoration of gray-level image, degraded by defocusing, additive noise etc., is already a well established method. But the lacuna of the existing methods [1], [2] is that it does not take care of the non-negative constraint [2, p. 189] of the gray level image f. The negative values in f imply an absurdity of negative intensities of radiant energy in the original object distribution [2]. Moreover, in the existing methods of algebraic restoration (in an optimal sense) [1, p. 197–199] the necessary and sufficient conditions of optimal restoration of the degraded image are not explicitly stated. The aim of this present note is to consider all these existing drawbacks of the algebraic restoration and recultivate the aspect of optimal restoration in a more meaningful sense. We essentially consider the unconstrained restoration problem as given in [1, p. 197], [2]. The advantage of this unconstrained restoration technique is that we do not need any specific information about additive noise [1]. In this given unconstrained restoration technique we introduce the non-negative constraint $f \geqslant 0$ and thus convert it to a constrained restoration problem. Some works in this direction have also been reported in [10], [11]. We demonstrate that under certain condition the optimal solution of constrained restoration is same as unconstrained restoration, that is, same as least square estimation. We also indicate that the optimal solution of the unconstrained problem [1] becomes meaningless if at least one of the elements of the restored image f is negative. If we arbitrarily set the negative element of the restored image f to zero the solution, we obtain, is no more optimal. Hence to achieve optimal restoration of the degraded image in a systematic fashion we consider the nonnegativity constraint $f \geqslant 0$. Finally we consider the Wolfe's algorithm [5], [6], which handles the constrained restoration problem and propose some other fast computational tools [11]–[14] for the solution of the constrained restoration problem.

II. Formulation of the Constrained Problem and Analysis of the Result

Let us assume the following model of the degradation process.

$$g = Hf + \eta \tag{1}$$

where g is given $(m \times 1)$ image vector f is the original $(n \times 1)$ image vector that is to be restored in an optimal sense, H is $(m \times n)$ matrix formed by the concept of point spread function (PSF) [2], [4]. Assume that H is nonnegative, i.e., $h_{ij} \geqslant 0$ [2, p. 68]. η is the additive noise. Specific knowledge about η is absent.

Problem 2.1

$$\begin{aligned}\min J(f) &= \|g - Hf\|^2 \\ &= \{g_1 - (h_{11}f_1 + h_{12}f_2 + \cdots + h_{1n}f_n)\}^2 \\ &\quad + \{g_2 - (h_{21}f_1 + h_{22}f_2 + \cdots + h_{2n}f_n)\}^2 \\ &\quad + \cdots \\ &\quad + \{g_m - (h_{m1}f_1 + h_{m2}f_2 + \cdots + h_{mn}f_n)\}^2 \\ &\qquad s/t\ f_i \geqslant 0,\ i = 1, \cdots, n.\end{aligned}$$

The equivalent Lagrangian function of Problem 2.1 is

$$L(f, y_i, \lambda_i) = J(f) - \sum_{i=1}^{n} \lambda_i (f_i - y_i^2) \tag{2}$$

where λ_i is the Lagrangian multiplier and y_i^2 is the surplus variable.

Taking the partial derivatives of $L(f, y_i, \lambda_i)$ with respect to f, y_i and λ_i we obtain,

$$\frac{\partial L(f, y_i, \lambda_i)}{\partial f_i} = \frac{\partial J(f)}{\partial f_i} - \sum_{i=1}^{n} \lambda_i \frac{\partial f_i}{\partial f_i} = 0 \tag{3a}$$

$$\frac{\partial L(f, y_i, \lambda_i)}{\partial \lambda_i} = -f_i + y_i^2 = 0 \text{ or } f_i = y_i^2 \tag{3b}$$

$$\frac{\partial L(f, y_i, \lambda_i)}{\partial y_i} = 2\lambda_i y_i = 0. \tag{3c}$$

Equation (3a) can be rewritten as

$$\begin{aligned}\frac{\partial L(f, y_i, \lambda_i)}{\partial f_1} &= 2\{g_1 - (h_{11}f_1 + h_{12}f_2 + \cdots + h_{1n}f_n)\} \\ &\quad \times(-h_{11}) + \cdots + 2\{g_m - (h_{m1}f_1 + h_{m2}f_2 \\ &\quad + \cdots + h_{mn}f_n)\}(-h_{m1}) - \lambda_1 = 0 \\ \frac{\partial L(f, y_i, \lambda_i)}{\partial f_2} &= 2\{g_1 - (h_{11}f_1 + h_{12}f_2 + \cdots + h_{1n}f_n)\} \\ &\quad \times(-h_{12}) + \cdots + 2\{g_m - (h_{m1}f_1 + h_{m2}f_2 \\ &\quad + \cdots + h_{mn}f_n)\}(-h_{m2}) - \lambda_2 = 0 \\ &\vdots \\ \frac{\partial L(f, y_i, \lambda_i)}{\partial f_n} &= 2\{g_1 - (h_{11}f_1 + h_{12}f_2 + \cdots - h_{1n}f_n)\} \\ &\quad \times(-h_{1n}) + \cdots + 2\{g_m - (h_{m1}f_1 + h_{m2}f_2 \\ &\quad + \cdots + h_{mn}f_n)\}(-h_{mn}) - \lambda_n = 0.\end{aligned} \tag{4}$$

Thus the matrix vector form of expression (a) of (3) is as follows;

$$\frac{\partial L(f, y_i, \lambda_i)}{\partial f} = 0 = -2H^T(g - Hf) - \lambda \tag{5}$$

where $\lambda = (\lambda_1 \cdots \lambda_n)^\tau$ and τ-indicates transpose.

The Kuhn–Tucker conditions, necessary for f and λ to be the stationary points for the minimization Problem 2.1, can be summarized as follows [6], [7].

$$\begin{array}{ll} \textit{Condition-I} & \lambda_i f_i = 0 \\ \textit{Condition-II} & -2H^T(g - Hf) - \lambda = 0 \\ \textit{Condition-III} & \lambda_i \geq 0 \\ \textit{Condition-IV} & f_i \geq 0. \end{array} \tag{6}$$

From above we obtain,

$$\begin{aligned} &f = (H^TH)^{-1}H^Tg - \frac{1}{2}(H^TH)^{-1}\lambda \\ &\lambda_i f_i = 0 \quad \forall i \\ &\lambda_i \geq 0,\ f_i \geq 0. \quad \forall i. \end{aligned} \tag{7}$$

Before we proceed any further we stipulate the following results.

Lemma 2.1: The quadratic form X^tAX, where A is symmetric, is

a) positive definite if and only if every eigenvalue of A is positive; and

b) positive semidefinite if and only if every eigenvalue of A is nonnegative and at least one eigenvalue is zero.

The expression of our original objective function (refer to Problem 2.1) can be represented as follows:

$$\begin{aligned} J(f) &= \|g - Hf\|^2 \\ &= g_1^2 - 2g_1\left\{\sum_{i=1}^{n} h_{1i}f_i\right\} + \left\{\sum_{i=1}^{n} h_{1i}f_i\right\}^2 \\ &\quad + g_2^2 - 2g_2\left\{\sum_{i=1}^{n} h_{2i}f_i\right\} + \left\{\sum_{i=1}^{n} h_{2i}f_i\right\}^2 \\ &\quad \cdots + g_m^2 - 2g_m\left\{\sum_{i=1}^{n} h_{mi}f_i\right\} + \left\{\sum_{i=1}^{n} h_{mi}f_i\right\}^2 \\ &= g^Tg - 2g^THf + f^TH^THf \\ &= \hat{g} - \hat{c}f + f^TDf \end{aligned} \tag{8}$$

where $\hat{g}$ is scalar, $\hat{c} = 2g^TH$ and $D = H^TH$.

Now, to find out the global minimum of Problem 2.1 the objective function $J(f)$ has to be strictly convex, i.e., the matrix D, which is symmetric, of (8) has to be positive definite. In case D is positive semidefinite then the following condition should be satisfied.

$\hat{c}^T \in \mathscr{M}(D)$ which means there exists a vector p such that $\hat{c}^T = Dp$ [6]. The symbol $\mathscr{M}$ indicates the column space.

Now we state the following result.

Lemma 2.2: The objective function $J(f)$, subjected to the constraints $f_i \geq 0, \forall i$, has a global minimum at $f = (H^TH)^{-1}H^Tg$ provided all the components of the vector $(\boldsymbol{H}^T\boldsymbol{H})^{-1}\boldsymbol{H}^T\boldsymbol{g}$ are non-negative.

Proof: From (7) we get the stationary point for extremum, which satisfies the Kuhn–Tucker necessary conditions, as follows;

For $\lambda_i = 0$, $i = 1, \cdots, n$; $f = (H^TH)^{-1}H^Tg$. But the constraints $f_i \geq 0$, $\forall i$, become inactive for $\lambda_i = 0$, $\forall i$. Hence, the Kuhn–Tucker necessary conditions for extremum (see (7)) can be achieved when all the components of the vector $(\boldsymbol{H}^T\boldsymbol{H})^{-1}\boldsymbol{H}^T\boldsymbol{g}$ are non-negative. Q.E.D.

Observation 2.1: If all the components of the vector $(H^TH)^{-1}H^Tg$ are non-negative, the solution of the unconstrained problem [1, Page 198] is same as the solution of the constrained problem (refer to Problem 2.1).

In case the component/components of the vector $(H^TH)^{-1}H^Tg$ is/are negative we have to proceed as follows.

a) For $\lambda_i = 0$, $\lambda_j \neq 0$, $i = 1,\cdots,j-1$, $j = 1,\cdots,n$; $y_j = 0$, $f_j = 0$ and $f_1, f_2, \cdots, f_{j-1}, \lambda_j, f_{j+1}, \cdots, f_n$ can be obtained by solving the following linear equations

$$
\begin{aligned}
&2\{g_1 - (h_{11}f_1 + \cdots + h_{1j-1}f_{j-1} \\
&\quad + h_{1j+1}f_{j+1} + \cdots + h_{1n}f_n)\}(-h_{11}) \\
&\quad + \cdots + 2\{g_m - (h_{m1}f_1 + \cdots + h_{mj-1}f_{j-1} \\
&\quad + h_{mj+1}f_{j+1} + \cdots + h_{mn}f_n)\}(-h_{m1}) \\
&\quad = 0 \\
&\quad \vdots \\
&2\{g_1 - (h_{11}f_1 + \cdots + h_{1j-1}f_{j-1} \\
&\quad + h_{1j+1}f_{j+1} + \cdots + h_{1n}f_n)\}(-h_{1j}) \\
&\quad + \cdots + 2\{g_m - (h_{m1}f_1 + \cdots + h_{mj-1}f_{j-1} \\
&\quad + h_{mj+1}f_{j+1} + \cdots + h_{mn}f_n)\}(-h_{mj}) \\
&\quad - \lambda_j = 0 \\
&\quad \vdots \\
&2\{g_1 - (h_{11}f_1 + \cdots + h_{1j-1}f_{j-1} \\
&\quad + h_{1j+1}f_{j+1} + \cdots + h_{1n}f_n)\}(-h_{1n}) \\
&\quad + \cdots + 2\{g_m - (h_{m1}f_1 + \cdots + h_{mj-1}f_{j-1} \\
&\quad + h_{mj+1}f_{j+1} + \cdots + h_{mn}f_n)\}(-h_{mn}) \\
&\quad = 0
\end{aligned}
$$

b) for $\lambda_i \neq 0$, $i = 1,\cdots,n$: $f_i = 0$ for all i.

Therefore from these results we see that for the optimal solution of the Problem 2.1 we have to solve (preserving the Kuhn Tucker necessary conditions stated in the (7)) total n-number of equations for n-number of unknowns $(f_1, f_2, \cdots, f_{j-1}, \lambda_j, f_{j+1}, \cdots, f_n)$. But, if the dimension of n is large (large n is quite expected in image processing problem), then to solve simultaneous n-equations for n-unknowns, preserving the condition[1] $\lambda_i f_i = 0$ where $\lambda_i, f_i \geq 0$, becomes cumbersome. Hence, in the next section we propose an alternative approach for the entire constrained problem (refer Problem 2.1).

III. Application of Wolfe's Algorithm for Solving the Restoration Problem

In this section we essentially solve Problem 2.1 in a more efficient way using Wolfe's algorithm [6]. We visualize Problem 2.1 in the following quadratic programming form.

[1]The condition $\lambda_i f_i = 0$, where λ_i and f_i are non-negative, implies that if $\lambda_i = 0$, $f_i \geq 0$ and vice versa.

We consider (8) as the final expression of the objective function of Problem 2.1. Hence, we write

$$\min J(f) = \hat{g} - \hat{c}f + f^TDf \tag{9}$$

subject to,

$$f \geq 0.$$

The function f^TDf defines a quadratic form where D is symmetric. Now for global minimum D has to be positive definite. In case D is positive semidefinite we have to put restriction on $\hat{c}^T$ (refer Lemma 2.1 and the related discussion). In this case constraints are linear which guarantees a convex solution space. The solution to this problem is secured by direct application of the Kuhn–Tucker necessary conditions. Now the entire problem may be written as,

$$\min J(f) = \hat{g} - \hat{c}f - f^TDf$$

subject to,

$$G(f) = (-I)f \leq 0. \tag{10}$$

Let $\lambda = (\lambda_1, \lambda_2, \cdots, \lambda_n)^T$ be the Lagrangian multipliers corresponding to $-f \leq 0$. Application of the Kuhn–Tucker conditions immediately yields

$$
\begin{gathered}
\lambda \geq 0, \\
\nabla J(f) + \lambda^T \nabla G(f) = 0 \\
\lambda_j f_j = 0, \quad j = 1,\cdots,n \\
-f \leq 0.
\end{gathered} \tag{11}
$$

Now

$$
\begin{gathered}
\nabla J(f) = -\hat{c} + 2f^TD \\
\nabla Gf = -I.
\end{gathered} \tag{12}
$$

Thus the conditions stated above reduce to

$$
\begin{gathered}
-2f^TD + \lambda^T = -\hat{c} \\
\lambda_j f_j = 0, \quad \forall j \\
\lambda, f \geq 0.
\end{gathered} \tag{13}
$$

Since $D^T = D$, the transpose of the first expression of (13) yields,

$$-2Df + \lambda = -\hat{c}^T$$

or

$$2Df - \lambda = \hat{c}^T.$$

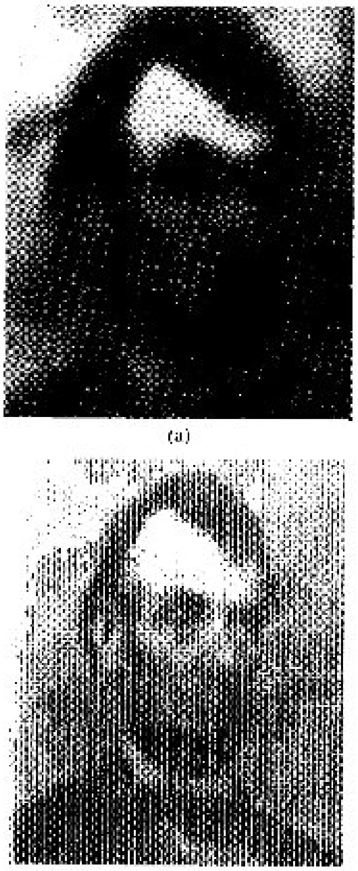
Hence the necessary conditions stated above may be combined as

$$
\begin{gathered}
(2D | -I)\begin{pmatrix} f \\ \lambda \end{pmatrix} = (\hat{c}^T) \\
\lambda_j f_j = 0, \quad \forall j \\
\lambda, f \geq 0.
\end{gathered} \tag{14}
$$

Except for the condition $\lambda_j f_j = 0$, the remaining equations are linear function in f and λ. The problem is thus equivalent to solving a set of linear equations, while satisfying the additional conditions $\lambda_j f_j = 0$. To solve (14) we use the efficient algorithm proposed by Wolfe [5], [6]. Wolfe's algorithm is essentially a modified simplex for quadratic programming problem. The Fortran-IV software of Wolfe's algorithm; which we use for the solution of (14), is given in [9, pp. 106–119]. Before we conclude we state the following result.

Lemma 3.1: If, in the final iteration of Wolfe's algorithm, all the basic variables, in the restrictive sense (restriction due to $\lambda_j f_j = 0$), are f_j then the solution for $f = (H^TH)^{-1}H^Tg$.

Proof: If the f_j are basic variables, then the corresponding λ_j's in (14) are nonbasic. Hence in the final solution of (14), the λ_j are set to zero. These zero values of λ_j along with positive values of f_j also satisfy the condition $\lambda_j f_j = 0$, for all j. There-



(a)

(b)

Fig. 1. (a) Degraded image with signal-to-noise ratio 10 dB. (b) Restored image.

fore from (14) we obtain

$$2Df = c^T = 2H^T g$$

therefore;

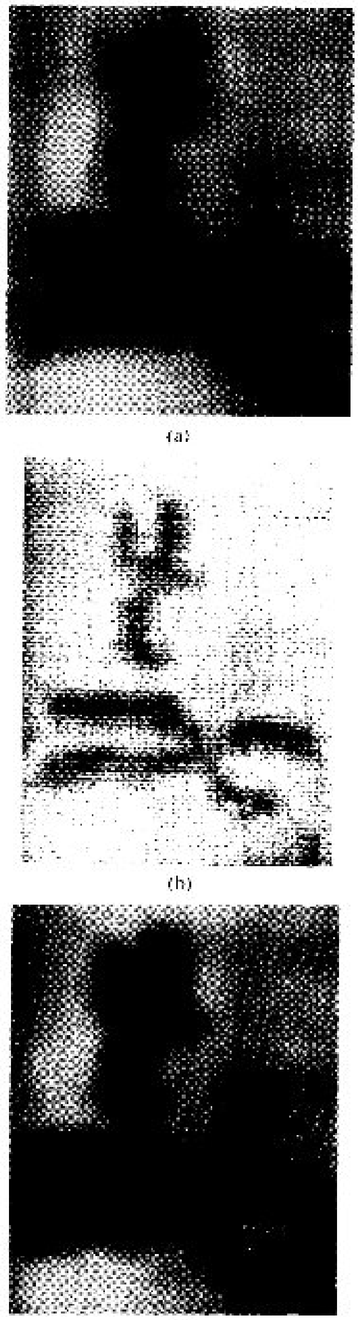
$$f = D^{-1}H^T g$$
$$= (H^T H)^{-1} H^T g. \qquad \text{Q.E.D.}$$

Observation 3.1: Solution of Problem 2.1 using Wolfe's algorithm is same as the solution of the unconstrained problem (i.e., minimize $J(f) = \|g - Hf\|^2$) if all the components of the vector $(\boldsymbol{H}^T\boldsymbol{H})^{-1}\boldsymbol{H}^T\boldsymbol{g}$ are non-negative. But if any f_i is negative, then using Wolfe's algorithm we can obtain, very efficiently, the optimal non-negative solution of f.

Observation 3.2: Unconstrained least square solution of f is a special case of constrained (the non-negativity constraint $f_i \geqslant 0$) least square solution.

IV. Conclusion

To test the effectiveness of the Wolfe's algorithm for image restoration under the non-negative constraint of f, we consider the degraded gray-level images as shown in Figs. 1(a) and 2(a). For degradation of the gray-level images we initially form the degradation matrix H using the concept of point spread function (PSF) [2], [4]. Then through lexicographic ordering we form the column vector $\boldsymbol{f}$ of the original image matrix and compute the degraded column vector $\boldsymbol{Hf}$. Finally we generate the noise vector η through random number generation software and add the noise vector η with $\boldsymbol{Hf}$ to form the degraded image vector $\boldsymbol{g}$. Of course from the degraded image vector g we can further generate the image matrix, as shown in Figs. 1(a) and 2(a), through the reverse process of the lexicographic ordering. During the addition of the noise vector we maintain the signal-to-noise ratio (SNR) within 10 to 20 dB. To select the starting basic variables of our iteration, we introduce artificial variables R_j, $j = 1, \cdots, n$ to the first set of (15). If the set of (15) has a feasible solution, then at the end of first phase of iteration [7] the sum of all artificial variables will be



(a)

(b)

(c)

Fig. 2. (a) Degraded image with signal-to-noise ratio 18 dB. (b) Restored image. (c) Restored image using conventional unconstrained least square estimate (i.e. without considering the non-negativity constraint $f_i \geqslant 0$).

zero. If all f_i's are positive then they will be the basic variables of the last table of our iteration. Otherwise a particular f_i, which is negative will be replaced from the list of basic variables yielding place to the corresponding λ_i. Thus we obtain the restored results as shown in Figs. 1(b) and 2(b). The results we obtain using Wolfe's algorithm is quite satisfactory in nature. The advantage of the present technique of image restoration over the conventional unconstrained least square solution is that the present technique can very efficiently handle the non-negativity constraint $f_i \geq 0$. Whereas in the conventional least square solution $f = (H^T H)^{-1} H^T g$, if any of the components of the vector $(H^T H)^{-1} H^T g$ is negative then in ultimate representation of the gray levels of the restored image we have to arbitrarily set that negative component to zero. Thus the solution of the problem is shifted from its optimum level. This type of phenomenon may be visualized as an addition of the random noise to the restored image. Hence the restored image as shown in Fig. 2(c) is degraded. The CPU-time (in EC 1033 computer) required for restoration of the degraded images (Fig. 1(a) and Fig. 2(a)) (restoration using the efficient simplex method for quadratic programming [5], [9]) are 5.55 min and 9.35 min respectively. The CPU-time required for the solution of the unconstrained problem (solution is shown in Fig. 2(c)) is 3.28 min. Therefore it is obvious that with the addition of the non-negativity constraints which force the entire problem in the quadratic programming domain the computational time required for the restoration of the degraded images increases substantially. Hence to reduce the computational burden associated with the quadratic programming solution we may look for linear complementary programming approach which turns out to be computationally superior [12, p. 117] than the approach proposed in [5], [9] for convex quadratic programming. The conjugate gradient method [14], [15] for the quadratic programming problem may be thought of as an alternative computational tool. Of course in this direction a definite approach has been reported in [11]. The computational times required for the restoration of the Figs. 1(a) and 2(a) using the method proposed in [11] are 4.45 min and 7.75 min respectively. So depending upon the complexity (size of the image matrix and the SNR etc.) of the problem, for the off-line simulation of a restoration problem we may suitably choose one of the methods [11], [12], [14] mentioned above. Even in the most-time critical situation that is completely beyond the scope of the present discussion and which may arise during on-line implementation of the proposed algorithm the quadratic programming computation may also be tackled very efficiently provided we adopt the distributive computation of the constrained optimization problems by network of locally interconnected simple processors as suggested in [13].

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