Intersection Pattern of the Classical Ovoids in Symplectic 3-Space of Even Order

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For $s = 2^e$, $e > 1$ odd, we determine how the copies of the Suzuki group $Sz(s)$ in the symplectic group $Sp(4, s)$ intersect. Using this information we determine how the classical ovoids in symplectic 3-space $W(s)$ meet and obtain a complete set of double coset representatives of $Sz(s)$ in $Sp(4, s)$. We also note that the permutation representation of $Sp(4, s)$ on the cosets of $Sz(s)$ is multiplicity free, and its irreducible constituents are explicitly determined. Indeed, we show that the complex Hecke algebra of this permutation representation is isomorphic to the center of the complex group algebra of $Sz(s)$. A combinatorial offshoot of this study is the construction of several new series of Buckenhout diagram geometries of type ◦—◦—◦—◦ which are embedded as subgeometries of miquelian and Suzuki–Tits inversive planes. © 1989 Academic Press, Inc.

1. INTRODUCTION AND STATEMENT OF RESULTS

1.1. Let F_s denote the field of order s, $s = 2^e$, $e \geq 1$. Let G denote the symplectic group $Sp(4, s) = B_2(s)$ consisting of the linear automorphisms of the four-dimensional vector space V over F_s leaving a fixed non-degenerate symplectic bilinear form on V invariant. The symplectic 3-space (or equivalently [4] the regular generalized quadrangle of order (s, s)) $W(s)$ is the linear geometry whose points are the points of the projective 3-space $PG(3, s)$ and whose lines are the totally isotropic lines of $PG(3, s)$ with respect to the given symplectic form.

It is well known that G has a unique conjugacy class of maximal subgroups, each isomorphic to $SL(2, s^2) \cdot 2$. In addition, if $s = 2^e$ with $e > 1$ odd, then G has a unique conjugacy class of maximal subgroups, each isomorphic to the Suzuki group $Sz(s) = {}^2B_2(s)$. Under the natural action of G on the points of $PG(3, s)$, each of these groups has precisely two point orbits one of which is an ovoid of $W(s)$. Recall [13, p. 69] that an *ovoid*

of $W(s)$ is a set of $s^2 + 1$ points no two of which are collinear in $W(s)$. From the transitivity of G on the points and the maximality of $SL(2, s^2) \cdot 2$ and $Sz(s)$, it follows that these subgroups are the full stabilizers in G of their orbits. An ovoid of $W(s)$ whose full set-wise stabilizer in G is isomorphic to $SL(2, s^2) \cdot 2$ (respectively $Sz(s)$) is called an *elliptic ovoid* (respectively *Tits ovoid*) of $W(s)$. These are the only known ovoids of $W(s)$, and are called the *classical ovoids* of $W(s)$.

1.2. It is proved in [1, Corollary 2, p. 139] that any ovoid of $W(s)$ meets a classical ovoid of $W(s)$ in an odd number of points. Recall from [3] that a *conic of* $W(s)$ is by definition the intersection of an elliptic ovoid of $W(s)$ with one of its secant planes. It is known (see, for instance, the discussion in [2, Section 3]) that any two elliptic ovoids of $W(s)$ meet in a point or in a conic of $W(s)$. These hold for $s = 2^e$ for all $e \geqslant 1$.

1.3. For the rest of the paper (unless otherwise mentioned) we take $s = 2^{2r+1} = 2r^2$, H to be a subgroup of G which is isomorphic to $Sz(s)$ and θ to be the Tits ovoid of $\tilde{W}(s)$ fixed by H.

For any $x \in \theta$, the stabiliser of x in H contains a unique Sylow 2-subgroup S of H and the action of H on $\theta \setminus \{x\}$ is regular. The union of x with a non-trivial orbit in $\theta \setminus \{x\}$ of the center $Z(S)$ of S is a *circle* (see Lemma 2.8(b) below) of the Suzuki–Tits inversive plane $I(\theta)$ associated with θ as in [12, p. 126]. The union of any two such $Z(S)$-fixed circles through x will be called a *figure of eight* at x.

By [6], the centralizer of any cyclic subgroup of G of order $s \pm 2r + 1$ is a cyclic subgroup of order $s^2 + 1$. Therefore by [11, Lemma 2], any cyclic subgroup of order $s - 2r + 1$ of G acts semi-regularly on the points of $W(s)$. A set of points of $W(s)$ will be called a *cap* (respectively a *cup*) if it is a point-orbit contained in θ of a cyclic subgroup of H of order $s + 2r + 1$ (respectively $s - 2r + 1$).

Finally, any (cyclic) subgroup T of H of order $s - 1$ fixes precisely four points of $W(s)$, two of them on θ, and acts semi-regularly on the remaining points of $W(s)$ (see [3]). A set of points of $W(s)$ will be called a *pseudo-circle* if it is the union of the two T-fixed points on θ with a nontrivial T-orbit contained in θ provided it is not one of the two T-fixed circles of $I(\theta)$.

In terms of these definitions, we prove:

THEOREM 1. *Let* $s = 2^{2n+1}$, $n \geqslant 1$. *Then*

(a) *The intersection of an elliptic ovoid of $W(s)$ and a Tits ovoid of $W(s)$ is either a cup or a cap.*

(b) *The intersection of any two distinct Tits ovoids of $W(s)$ is either a point, a pseudo-circle, a figure of eight, a cup, or a cap.*

Now, Theorem 1, together with the second result mentioned in 1.2 above, implies:

COROLLARY 2.   Let $s = 2^e$, $e \geqslant 1$. Then the cardinality of the intersection of any two distinct classical ovoids of $W(s)$ is one of 1, $s - (2s)^{1/2} + 1$, $s + 1$, $s + (2s)^{1/2} + 1$, or $2s + 1$.

1.4.   In Theorem 3 below we present a construction of a semi-biplane from any pair of ovoids of $W(s)$. Recall that a semi-biplane ([8, 15]) is a finite connected incidence system in which through any two distinct points pass 0 or 2 blocks and any two distinct blocks have 0 or 2 points in common. It follows that there are constants v and k such that the number of points $= v =$ the number of blocks and the number of points in each block $= k =$ the number of blocks through each point. The semi-biplane is said to have parameters (v, k). In other words, a semi-biplane is a Buekenhout diagram geometry of type $\circ\!\!-\!\!\!\!\overset{c}{-}\!\!\!\!-\!\!\circ\!\!-\!\!\!\!\overset{2}{-}\!\!\!\!-\!\!\circ$. Apparently the only known general constructions of semibiplanes are: (a) the doubling construction due to Hughes and Dickey (see [8]) and (b) the construction from involutions in projective planes due to Hughes in [8]. For an exhaustive enumeration of semi-biplanes with $k \leqslant 6$ see [15]. Here we prove:

THEOREM 3.   Let $s = 2^e$, $e \geqslant 1$, and let θ_1 and θ_2 be two distinct ovoids of $W(s)$ meeting in $s^2 + 1 - v$ points. Then the incidence system, with the "point" set $\theta_1 \backslash \theta_2$, the "block" set $\theta_2 \backslash \theta_1$, and with collinearity in $W(s)$ as the incidence, is a semi-biplane with parameters $(v, s + 1)$.

Theorem 3 and Corollary 2 yield:

COROLLARY 4.   If $s = 2^e$, then there exist semi-biplanes with parameters $(v, k) = (s^2, s + 1)$ and $(s^2 - s, s + 1)$. If, further, $e > 1$ is odd then there exist semi-biplanes with parameters $(v, k) = (s^2 - 2s, s + 1)$, $(s^2 - s - (2s)^{1/2}, s + 1)$, and $(s^2 - s + (2s)^{1/2}, s + 1)$.

1.5.   The basic idea in the proof of Theorem 1 is that the intersection of any two classical ovoids has odd size [1, Corollary 2, p. 239] and it is the union of some of the point orbits of the intersection of the full stabilisers in G of these two ovoids. To implement this idea, we require:

THEOREM 5.   Let $s = 2^{2n+1} = 2r^2$ and let $H < G = \mathrm{Sp}(4, s)$ be isomorphic to $\mathrm{Sz}(s)$. Then

(a) H meets each copy of $SL(2, s^2) \cdot 2$ in G in a subgroup of order $4(s \pm 2r + 1)$. Indeed, $K \to K \cap H$ defines a bijection between the set of sub-

groups K of G isomorphic to $SL(2, s^2) \cdot 2$ and the set of subgroups of H of order $4(s + 2r + 1)$ and $4(s - 2r + 1)$:

(b) G contains:

(1) $(s-1)(s^2+1)$ copies of Sz(s) intersecting H in a Sylow 2-subgroup of H,

(2) $s(s-1)(s^2+1)$ copies of Sz(s) intersecting H in a subgroup of H of order $2s$ which contains the center of a Sylow 2-subgroup of H,

(3) $s^2(s-2)(s^2+1)/2$ copies of Sz(s) intersecting H in a cyclic subgroup of H of order $s-1$,

(4) $s^2(s-1)(s-2r)(s-2r+1)/4$ copies of Sz(s) intersecting H in a cyclic subgroup of H of order $s + 2r + 1$, and

(5) $s^2(s-1)(s+2r)(s+2r+1)/4$ copies of Sz(s) intersecting H in a cyclic subgroup of H of order $s - 2r + 1$.

Together, these account for all the copies of Sz(s) in G different from H.

The authors would like to thank Professor John Thompson whose question (personal conversation with the second-named author)—"Does Sp$(4, s)$ contain two copies of Sz(s) with trivial intersection?"—led to Theorem 5(b). Of course, this theorem answers the question in the negative when $s > 2$.

1.6. Towards the study of the permutation representation (by conjugation) of Sp$(4, s)$ on the copies of Sz(s) in it (equivalently, the permutation representation of Sp$(4, s)$ (by right multiplication) on the left cosets of a fixed copy of Sz(s) in it), we prove:

THEOREM 6. Let G and H be as in Theorem 5. Further, for $1 \leqslant i \leqslant 5$, choose and fix a subgroup H_i of H as in part (i) of Theorem 5(b), and let A_i (respectively B_i) be the normaliser in H (respectively in G) of H_i. For part (c) below we also assume that our choice is such that $H_1 > H_2$. Then

(a) The complex character 1_H^G of the permutation representation of G on the cosets of H is the sum of $s + 3$ irreducible complex characters of G, each appearing with multiplicity one. Indeed, using Enomoto's notation in [6] for the irreducible complex characters of G, we have

$$1_H^G = \theta_0 + \theta_2 + \theta_3 + \theta_4 + \sum_{k=1}^{s/2-1} \chi_1(k, (s+2r)k)$$

$$+ \frac{1}{4}\left[\sum_{k=1}^{s-2r} \chi_5((s+2r+1)k) + \sum_{k=1}^{s+2r} \chi_5((s-2r+1)k)\right]. \quad (1.1)$$

(b) *The restriction to H of the character 1_H^G of G decomposes as follows:*

$$1_H^G|_H = 1_H + 1_{H_1}^H + 2 \cdot 1_{H_2}^H + (s/2 - 1) 1_{H_3}^H$$

$$+ \frac{(s - 2r)}{4} \cdot 1_{H_4}^H + \frac{(s + 2r)}{4} \cdot 1_{H_5}^H. \qquad (1.2)$$

(c) *For $1 \leqslant i \leqslant 5$, $i \neq 2$, let π_i be a complete set of representatives of the non-trivial (i.e., $\neq A_i$) (A_i, A_i)-double cosets in B_i. Let π_2 consist of representatives of the (A_2, A_2)-double cosets in B_2 which are contained in the set theoretic difference $B_2 \backslash B_1$. Finally, let $\pi_0 = \{1\}$ and $\pi = \bigcup_{i=0}^5 \pi_i$. Then π is a complete set of representatives of the (H, H)-double cosets in G.*

COROLLARY 7. *Let G and H be as in Theorem 5. Then the complex Hecke algebra of the permutation representation of G on the left cosets of H is isomorphic to the center of the complex group algebra of H.*

1.7. *Remarks.* (a) We are intrigued by the isomorphism in Corollary 7 and wonder if it can be proved by exhibiting a natural and explicit isomorphism. We conjecture that Corollary 7 holds for the pairs $(G, H) = (G_2(s), {}^2G_2(s))$ and $= (F_4(s), {}^2F_4(s))$ as well. (See the Note added in proof). Computations are under way to verify this for the first-named pair. It may be speculated that these three cases, as well as the similar (though not entirely analogous) phenomenon observed by Gow in [7] may be instances of a general theorem involving a finite simple group G of Chevalley type and the subgroup H consisting of the fixed points of an involutional outer automorphism of G.

(b) The proof of Theorem 5 actually shows the following: if A is a subgroup of $H = \mathrm{Sz}(s) < \mathrm{Sp}(4, s) = G$ with $|A| \in \{s^2, s - 1, s \pm 2r + 1\}$, then the copies H^* of Sz(s) in G with $H \cap H^* = A$ constitute (together with H) a single orbit under the action of $N_G(A)$ under conjugation. If, on the other hand, $|A| = 2s$ and A contains the center of a Sylow 2-subgroup B of H, then the copies H^* of Sz(s) in G with $H \cap H^* = A$ or B (together with H) constitute a single $N_G(A)$ orbit.

(c) In view of the identifications mentioned in the footnote to the character table of Sp(4, s) in [6], each of the characters $\chi_5(\cdot)$ occuring in (1.1) above occurs four times in the sum. The factor $1/4$ outside the square bracket in (1.1) indicates that these characters are to be taken once each.

(d) Note that Eq. (1.2) in Theorem 6(b) may be thought of as the algebraic counterpart of the geometric content of Theorem 5(b).

(e) In Theorem 6(c) we have $|\pi_0| = |\pi_1| = 1$, $|\pi_2| = 2$, $|\pi_3| = s/2 - 1$, $|\pi_4| = (s - 2r)/4$, and $|\pi_5| = (s + 2r)/4$ (see Lemma 2.5 below).

(f) We expect to use Theorem 1 to settle the following conjecture in [3]: if an automorphism of order $s - 1$ in an inversive plane of even order

s fixes exactly two circles through its two fixed points then the inversive plane is of Suzuki–Tits type.

2. PRELIMINARIES

2.1. We use the description of $G = \mathrm{Sp}(4, s)$ and $H = \mathrm{Sz}(s)$ given in [5, pp. 234–237]. For $t, u \in F_s$, we let

$$\alpha(t) = X_a(t^r)\, X_b(t)\, X_{a+b}(t^{r+1}),$$

$$\beta(u) = X_{a+b}(u)\, X_{2a+b}(u^{2r}),$$

$$S = \{\alpha(t)\,\beta(u) : t, u \in F_s\},$$

$$S_0 = \{\beta(u) : u \in F_s\}.$$

Then S is a subgroup of G of order s^2 and S_0 is the center of S.

Let L be the subgroup of G of order $(s-1)^2$ defined in [5, p. 218] (where the notation H is used for what we call L) and let T be the subgroup of L of order $s-1$ defined by

$$T = \{h(\chi) \in L : \chi(a) = \chi(b)^r\}.$$

Then

$$H = ST \cup ST(w_a w_b)^2 \cdot S$$

is a subgroup of G of order $s^2(s-1)(s^2+1)$ which is isomorphic to $\mathrm{Sz}(s)$ (see [5, p. 234]) and S is a Sylow 2-subgroup of H.

For any subgroup A of the additive group of F_s, let C_A be the subgroup of the multiplicative group F_s^* of F_s defined by

$$C_A = \{t \in F_s^* : tA = A\}$$

and let $S_A \leqslant S$, $T_A \leqslant T$ be the subgroups given by

$$S_A = \{\alpha(t)\,\beta(u) : t \in A, u \in F_s\}$$

$$T_A = \{h(\chi) \in T : \chi(b) \in C_A\}.$$

Then T_A normalises S_A. Note that in particular $S_A = S_0$ when $A = 0$ and $S_A = S$ when $A = F_s$. Also, $T_A = T$ for $A = F_s$.

For $t \in F_s$ let P_t denote the subgroup of G consisting of the elements $X_a(u)\, X_b(v)\, X_{a+b}(w)\, X_{2a+b}(x)$ satisfying

$$(u + v^r)^{2r-1} = 0 \text{ or } t^{1-r}.$$

Clearly $P_0 = S \cdot X_{2a+b}$ is of order s^3 and is a subgroup of index two of each P_t with $t \neq 0$.

Finally let M^+ (respectively M^-) be a subgroup of H of order $s + 2r + 1$ (respectively $s - 2r + 1$).

2.2. LEMMA. *The only proper subgroups of H containing S_0 are of the form $S_A U$ for some subgroup A of the additive group of F, and for some subgroup U of T_A.*

Proof. From the list [14, p. 137] of maximal subgroups of H, the only maximal subgroup containing S_0 is ST. So it suffices to examine the subgroups of ST/S_0. Since ST/S_0 is isomorphic to $F_s \cdot F_s^*$ (with F_s^* acting on the additive group of F_s by multiplication), this is easy.

2.3. LEMMA. (a) *Let A be an additive subgroup of F_s. Then*

(i) *for each nontrivial subgroup U of T_A, $N_G(S_A \cdot U) = S_A T_A = N_H(S_A \cdot U)$;*

(ii) *if $A = \{0\}$, then $N_G(S_A) = X_a X_b X_{c+b} X_{2a+b} T$ and $N_H(S_A) = ST$;*

(iii) *if $|A| > 2$ then $N_G(S_A) = P_0 T_A$, $N_H(S_A) = S \cdot T_A$;*

(iv) *if $|A| = 2$, say $A = \{0, t\}$, then $N_G(S_A) = P_t$, $N_H(S_A) = S$.*

(b) *$N_G(T) = L \cdot \langle (w_a w_b)^2 \rangle$, $N_H(T) = T \cdot \langle (w_a \omega_b)^2 \rangle$, and $N_H(T)$ is self-normalising in G.*

(c) *$N_G(M^+) = M^+ N^\pm \langle t^+ \rangle$, $N_H(M^+) = M^+ \langle t^\pm \rangle$, where $M^\pm N^\pm = Z_{s^2+1}$, and $t^\pm$ is an element of H of order 4 acting semi-regularly on the non-identity elements of $M^\pm N^\pm$. Further, for any subgroup $K^\pm$ of H with $M^\pm < K^- \leqslant N_H(M^\pm)$, $N_G(K^+) = N_H(K^+) = N_H(M^+)$.*

Proof. (a) and (b) are verified by routine computations using the relations in G given in [5, pp. 213 and 235]. Since the $(s+1)$th power of a suitable Singer cycle in $PGL(3, s)$ is in G, (c) follows from Wielandt's theorem [9, Satz 5.8, p. 285] and [9, Satz 7.3, p. 188].

2.4 LEMMA. (a) *For $d = s^2$, $s-1$, $s+2r+1$, or $s-2r+1$, any two subgroups of H of order d are conjugate in H.*

(b) *Any two subgroups of H of order $2s$ containing the centre of some Sylow 2-subgroup of H are conjugate in H.*

Proof. (a) follows from [10, Theorem 3.10, p. 190] and (b) from the regular action of T on S/S_0.

2.5. LEMMA. (a) *The number of $(N_H(X), N_H(X))$-double cosets in $N_G(X)$ is 2, $s/2$, $1 + (s-2r)/4$, or $1 + (s+2r)/4$ according as X is S, T, M^+, or M^-.*

(b) *If $A = \{0, 1\} \subseteq F_s$, then*

$$P_1 \backslash P_0 = \{ X_a(u^r + 1) X_b(u) X_{a+b}(v) X_{2a+b}(w) : u, v, w \in F_s \}$$

is the union of two $(N_H(S_A), N_H(S_A))$-double cosets, namely the set of those elements for which $\text{tr}(u^{r+1} + v + w) = 0$ and the set of those for which this trace (from F_s to F_2) is 1; in particular, $X_a(1)$ and $X_b(1)$ may be chosen as representatives of these two double cosets.

Proof. (a) for $X = S$ and (b) are verified by routine computations. To see (a) for the remaining cases we use Lemma 2.6 below and the fact that $(w_a w_b)^2$ and t^+ acts semi-regularly on the non-identity elements of T and $M^\pm$ respectively.

2.6 LEMMA. *Let A be an abelian group and C a group of automorphisms of A leaving a subgroup B of A invariant. Then $x, y \in A$ are in the same (BC, BC) double coset of AC iff x and y^c are in the same coset of B for some $c \in C$.*

Proof. Trivial.

2.7. We identify G with $SP(4, s)$ via the isomorphism taking $X_a(1)$, $X_b(1)$, $X_{a+b}(1)$, $X_{2a+b}(1)$, and $h(\chi)$ to $I_4 + E_{2,1} + E_{4,3}$, $I_4 + E_{3,1} + E_{4,2}$, $I_4 + E_{4,1}$, and $\text{diag}(\alpha, \beta, \beta^{-1}, \alpha^{-1})$ respectively, where $\chi(2a + b) = \alpha^2$, $\chi(b) = \beta^2$, E_{ij} is the 4×4 matrix with 1 at the (i, j)th place and 0 elsewhere, and I_4 is the identity matrix of order 4 (see [6, p. 76]).

Under this isomorphism, the Tits ovoid of $W(s)$ stabilised by H is

$$\theta = \{p_\infty\} \cup \{p(x, y): x, y \in F_s\},$$

where $p_\infty = (1, 0, 0, 0)'$ and

$$p(x, y) = (xy + x^{2^{r+1}+2} + y^{2^r+1}, y, x, 1)'.$$

(See [10, pp. 182–189].)

2.8. LEMMA. (a) *H, with its natural faithful permutation representation on θ, is a Zassenhaus group. The stabilizer in H of p_∞ is ST and the action of S on $\theta \setminus \{p_\infty\}$ is regular.*

(b) *The union of each non-trivial $Z(S)$ orbit in θ with $\{p_\infty\}$ is a circle of the inversive plane $I(\theta)$.*

(c) *The intersection of θ with its image under x is $\{p_\infty\}$ for half of the elements x in $P_1 \setminus P_0$ and is a figure of eight at p_∞ for the remaining half.*

Proof. For (a) see [10, Theorem 3.3, p. 184].

$Z(S)$ stabilises $\{p_\infty\} \cup \{p(0, y): y \in F_s\}$ which is clearly the intersection of θ with a plane of $PG(3, s)$, and hence is a circle of $I(\theta)$. Since H acts transitively on the circles of $I(\theta)$ and since the full stabilizer of such a circle in H is of order $s(s-1)$, it follows that each circle of $I(\theta)$ is stabilized by

a unique conjugate of $Z(S)$ in H. A two-way count shows that $Z(S)$ fixes exactly s circles of $I(\theta)$, hence (b). Let $x = X_a(u^r + 1) X_b(u) X_{a+b}(v) X_{2a+b}(w) \in P_1 \backslash P_0$. By computation, we see that $|\theta \cap \theta^x| = 2s + 1$ or 1 according as the traces (from F_1 to F_2) of $v + w$ and $(u^{r+1} + 1)^{2r+2}$ are equal or not. If $\theta \cap \theta^x$ is of size $2s + 1$ then it is the union of $\{p_\infty\}$ with two non-trivial $Z(S)$-orbits, and hence by (b) it is then a figure of eight at p_∞. This proves (c).

3. Proofs

3.1. *Proof of Theorem* 5. (a) By Lemma 2.3(c), the normaliser L in G of any subgroup $M^\perp$ of G of order $s \pm 2r + 1$ is an extension of a cyclic group L_0 of order $s^2 + 1$ by a cyclic group of order 4. Of the $s + 1$ point orbits of L_0, exactly one is an elliptic ovoid θ_0 of $W(s)$ (see [1, Lemma 2, p. 141]). Since L acts on the L_0-orbits, it follows that L stabilizes θ_0. That is, $L \subseteq N$ where N is the full stabilizer in G of θ_0, whence $N = SL(2, s^2) \cdot 2$. We have $N_H(M^\pm) = L \cap H \subseteq N \cap H$. But $N_H(M^\pm)$ is maximal in H, so that $N_H(M^\pm) = N \cap H$. Thus for each subgroup $M^\pm$ of order $(s \pm 2r + 1)$ in H we have exhibited a copy N of $SL(2, s^2) \cdot 2$ in G with $N_H(M^\pm) = N \cap H$. But the total number of choices for $M^\perp$ is $s^2(s-1)(s-2r+1)/4 + s^2(s-1)(s+2r+1)/4 = s^2(s^2-1)/2$ which is also the total number of copies of $SL(2, s^2) \cdot 2$ in G. This proves (a).

(b) Let $Y \leqslant H$ be a subgroup of H such that any G-conjugate of Y contained in H is in fact an H-conjugate of Y. We note that the subgroups denoted as either A or X in this paragraph have this property. Let $\mu(Y)$ denote the index of $N_H(Y)$ in $N_G(Y)$. Since the number of conjugates of Y in G (respectively in H) is $[G : N_G(Y)]$ (respectively $[H : N_H(Y)]$) and since the number of conjugates of H in G is $[G : H]$, a two-way count shows that $\mu(Y)$ is the number of copies of $Sz(s)$ in G (including H) which contain Y. By Lemmas 2.3 and 2.4, $\mu(X) = 1$ if $A < X < H$ and $|A| \in \{s^2, s-1, s \pm 2r + 1\}$; hence in each of these cases, the $\mu(A) - 1$ copies of $Sz(s)$ (other than H) containing A intersect H precisely in A. On the other hand, if $|A| = 2s$ and A contains the center of a Sylow 2-subgroup B of H, then $A < B$ and Lemmas 2.2, 2.3, and 2.4 imply that if $A < X \leqslant B$, then $\mu(X) = \mu(B)$, while if $A < X \leqslant H$, $X \nleqslant B$, then $\mu(X) = 1$. Thus, the $\mu(A) - \mu(B)$ copies of $Sz(s)$ in G which contain A but not B intersect H precisely in A in this case.

Now from Lemmas 2.3 and 2.4 we see that the numbers of subgroups A of H with $|A| = s^2$, $s-1$, $s + 2r + 1$, or $s - 2r + 1$, or with $|A| = 2s$ and A containing the center of a Sylow 2-subgroup of H, are respectively $s^2 + 1$, $s^2(s^2 + 1)/2$, $s^2(s-1)(s-2r+1)/4$, $s^2(s-1)(s+2r+1)/4$, and $(s^2+1)(s-1)$. The corresponding values of $\mu(A)$ are s, $s-1$, $s - 2r + 1$, $s + 2r + 1$, and $2s$.

This yields a total of $(s-1)\cdot(s^2+1)+(s-2)\cdot s^2(s^2+1)/2 + (s-2r)\cdot s^2(s-1)(s-2r+1)/4 + (s+2r)s^2(s-1)(s+2r+1)/4 + s\cdot(s^2+1)(s-1)$ copies of Sz(s) (other than H) in G whose intersections with H are as described in (b). But this sum adds up to $s^2(s+1)^2(s-1)-1$, which is the total number of copies of Sz(s) (other than H) in G. This proves (b).

3.2. *Proof of Theorem* 1.   (a)   Fix a Tits ovoid θ in $W(s)$. Let H be its stabilizer in $G=$ Sp(4, s). Thus $H \simeq$ Sz(s). If θ^* is any elliptic ovoid of $W(s)$, then letting N denote the stabiliser of θ^* in G, it follows from Theorem 5(a) that for $s^2(s-1)(s-2r+1)/4$ (respectively $s^2(s-1)(s+2r+1)/4$) of the choices for θ^*, $H \cap N$ contains a cyclic subgroup of order $s+2r+1$ (respectively $s-2r+1$) acting semi-regularly on the points of $W(s)$, whence each $H \cap N$-orbit has at least $s+2r+1$ (respectively $s-2r+1$) points in it; since $\theta \cap \theta^*$ is a nonempty (by [1, Corollary 2, p. 139]) union of $H \cap N$-orbits, it follows that $|\theta \cap \theta^*| \geqslant s+2r+1$ (respectively $\geqslant s-2r+1$).

Let n be the number of ordered pairs (x, θ^*), where θ^* is an elliptic ovoid of $W(s)$ and $x \in \theta \cap \theta^*$. The number of choices for x is s^2+1 and through each point x of $W(s)$ pass $s^2(s-1)/2$ elliptic ovoids of $W(s)$, hence $n = s^2(s^2+1)(s-1)/2$. On the other hand, the previous paragraph yields the estimate $n \geqslant (s+2r+1) \cdot s^2(s-1)(s-2r+1)/4 + (s-2r+1) \cdot s^2(s-1)(s+2r+1)/4 - s^2(s^2+1)(s-1)/2$. Since equality holds here, the inequalities in the previous paragraph must also be equalities. This implies (a).

(b)   We exploit the bijection between the Tits ovoids of $W(s)$ and the copies of Sz(s) stabilizing them. Fix a Tits ovoid θ with stabilizer H in G. If θ^* is any other Tits ovoid of $W(s)$ with stabilizer H^* then $\theta \cap \theta^*$ is a union of $H \cap H^*$-orbits, and by [1, Corollary 2], $\theta \cap \theta^*$ has odd size (in particular it is non-empty). From Theorem 5(b) it now follows immediately that (i) for the $(s-1)(s^2+1)$ Tits ovoids θ^* with $|H \cap H^*| = s^2$ we have $|\theta \cap \theta^*| = 1$ (use Lemma 2.8(a)); (ii) for the $s^2(s-1)(s \mp 2r)(s \mp 2r+1)/4$ copies of θ^* with $|H \cap H^*| = s \pm 2r+1$ we have $|\theta \cap \theta^*| \geqslant s \pm 2r+1$. Also, by Lemma 2.8(c) and Remark 1.7(b), we see that (iii) for half of the $s(s-1)(s^2+1)$ copies of θ^* with $|H \cap H^*| = 2s$ we have $\theta \cap \theta^*$ a figure of eight (and in particular $|\theta \cap \theta^*| = 2s+1$) while for the remaining half of them $|\theta \cap \theta^*| = 1$. Finally, for any $A \leqslant H$ with $|A| = s-1$, $N_G(A)$ fixes the two fixed points of A in θ, so that Remark 1.7(b) implies that all the Tits ovoids θ^* for which $H \cap H^* = A$ contains these two fixed points. Since $\theta \cap \theta^*$ has odd size and since A acts semiregularly on θ minus its two fixed points in θ, it follows that (iv) for each of the $s^2(s-2)(s^2+1)/2$ copies of θ^* for which $|H \cap H^*| = s-1$, we have $|\theta \cap \theta^*| \geqslant s+1$.

Now we count the number m of ordered pairs (x, θ^*), where $\theta^* \neq \theta$ is a Tits ovoid of $W(s)$ and $x \in \theta \cap \theta^*$. There are s^2+1 choices for x, and

for each point x of $W(s)$ the number of Tits ovoids of $W(s)$ containing x is $s^2(s^2-1)$ (which number includes θ when $x \in \theta$). Hence $n = (s^2+1)(s^4-s^2-1)$. On the other hand, the previous paragraph yields the estimate

$$
\begin{aligned}
n \geqslant{} & 1 \cdot (s-1)(s^2+1) + (s+2r+1) \cdot s^2(s-1)(s-2r)(s-2r+1)/4 \\
& + (s-2r+1) \cdot s^2(s-1)(s+2r)(s+2r+1)/4 + 1 \cdot s(s-1)(s^2+1)/2 \\
& + (2s+1) \cdot s(s-1)(s^2+1)/2 + (s+1) \cdot s^2(s-2)(s^2+1)/2 \\
={} & (s^2+1)(s^4-s^2-1).
\end{aligned}
$$

Since equality holds here, all the inequalities in the previous paragraph must also be equalities. This implies (b).

3.3. *Proof of Theorem 3.* This is immediate from : (i) any two points in an ovoid of $W(s)$ are non-collinear; (ii) given any two non-collinear points of $W(s)$, the set of points collinear with both is a non-isotropic line of $PG(3, s)$ (with respect to the symplectic polarity of $PG(3, s)$ defining $W(s)$); and (iii) any non-isotropic line of $PG(3, s)$ meets any ovoid of $W(s)$ in 0 or 2 points.

3.4. *Proof of Theorem 6.* (a) Of the $s+3$ conjugacy classes of $Sz(s)$ (see [14]), only two fuse in $Sp(4, s)$ (namely the two classes of elements of order 4). In the character table of $Sp(4, s)$ given in [6, p. 93], A_1; A_{32}; A_{42}; $B_1((2^e+1)i, 2^e \cdot i)$, $1 \leqslant i \leqslant s/2-1$; and $B_5((s \pm 2r+1)i)$, $1 \leqslant i \leqslant (s \mp 2r)/4$ are the representatives, respectively, of the conjugacy classes of elements of $Sz(s)$ of order 1; order 2; order 4; (the $s/2-1$ classes of non-identity elements of) order dividing $s-1$; (the $(s \mp 2r)/4$ classes of non-identity elements of) order dividing $s \mp 2r+1$.

A computation, which makes use of the character table of $Sp(4, s)$ in [6] and the Frobenius reciprocity, now yields (1.1), proving (a).

(b) Using the character table of H as given in [14, p. 141] (with the notation for the irreducible characters of H as in [14]) we see the restrictions to H of irreducible characters which appear in 1_H^G decompose as

$$
\theta_2|_H = \theta_3|_H = \mathrm{Id} + X + \sum_{i=1}^{s/2-1} X_i;
$$

$$
\begin{aligned}
\theta_4|_H ={} & \mathrm{Id} + (s+1)X + (s+1)\sum_{i=1}^{s/2-1} X_i \\
& + (s-2r+1)\sum_{i=1}^{(s+2r)/4} Y_i + (s+2r+1)\sum_{i=1}^{(s-2r)/4} Z_i \\
& + r(W_1 + W_2);
\end{aligned}
$$

$$\chi_1(k, (s+2r)k)|_H = \mathrm{Id} + (s+4)X + (s+3) \sum_{i=1}^{s/2-1} X_i$$

$$+ X_{\pm 2k(1-2r)} + 2X_{\pm 2k(1+r)}$$

$$+ (s-2r+3) \sum_{i=1}^{(s+2r)/4} Y_i + (s+2r+3) \sum_{i=1}^{(s-2r)/4} Z_i$$

$$+ r(W_1 + W_2)$$

(here the suffixes $\pm 2k(1+2r)$ and $\pm 2k(1+r)$ in the two X terms are to be taken modulo $s-1$, with the sign so chosen that, reduced modulo $s-1$, they lie between 1 and $s/2-1$);

$$\chi_S(k)|_H = \mathrm{Id} + s \cdot X + (s+1) \sum_{i=1}^{s/2-1} X_i + (s-2r+1) \sum_{i=1}^{(s+2r)/4} Y_i$$

$$+ (s+2r+1) \sum_{i=1}^{(s-2r)/4} Z_i - Z_k + (r+1)(W_1 + W_2)$$

$$\text{if } k \equiv 0 \ (\mathrm{mod}\ s+2r+1);$$

and

$$\chi_S(k)|_H = \mathrm{Id} + sX + (s+1) \sum_{i=1}^{s/2-1} X_i + (s-2r+1) \sum_{i=1}^{(s+2r)/4} Y_i - Y_k$$

$$+ (s+2r+1) \sum_{i=1}^{(s-2r)/4} Z_i + (r-1)(W_1 + W_2)$$

$$\text{if } \ k \equiv 0 \ (\mathrm{mod}\ s-2r+1).$$

Adding, we get, in view of (a),

$$1_H^G|_H = (s+3)\,\mathrm{Id} + (s^2 + 2s - 1)X$$

$$+ r(s-1)(W_1 + W_2) + (s^2 + 2s + 3) \sum_{i=1}^{s/2-1} X_i$$

$$+ (s^2 - 2rs + 2s - 3) \sum_{i=1}^{(s+2r)/4} Y_i$$

$$+ (s^2 + 2rs + 2s - 3) \sum_{i=1}^{(s-2r)/4} Z_i.$$

Now (b) follows from the computations

$$1^H_{H_1} = \mathrm{Id} + X + 2 \sum_{i=1}^{s/2-1} X_i;$$

$$1^H_{H_2} = \mathrm{Id} + s/2 X + (s/2+1) \sum_{i=1}^{s/2-1} X_i$$
$$+ (s/2-1) \sum_{i=1}^{(s+2r)/4} Y_i + (s/2-1) \sum_{i=1}^{(s-2r)/4} Z_i;$$

$$1^H_{H_3} = \mathrm{Id} + (s+2)X + (s+1) \sum_{i=1}^{s/2-1} X_i + (s-2r+1) \sum_{i=1}^{(s+2r)/4} Y_i$$
$$+ (s+2r+1) \sum_{i=1}^{(s-2r)/4} Z_i + r(W_1 + W_2);$$

$$1^H_{H_4} = \mathrm{Id} + (s-2r)X + (s-2r+1) \sum_{i=1}^{s/2-1} X_i$$
$$+ (s-4r+3) \sum_{i=1}^{(s+2r)/4} Y_i$$
$$+ (s-1) \sum_{i=1}^{(s-2r)/4} Z_i + r(W_1 + W_2);$$

and

$$1^H_{H_5} = \mathrm{Id} + (s+2r)X + (s+2r+1) \sum_{i=1}^{s/2-1} X_i$$
$$+ (s-1) \sum_{i=1}^{(s+2r)/4} Y_i$$
$$+ (s+4r+3) \sum_{i=1}^{(s-2r)/4} Z_i + r(W_1 + W_2).$$

(c) From the proof of Theorem 5(b) (also see Remarks 1.7(b)) we see that for $x \in \pi_i$, $H \cap H^x = H_i$. Since the H_i's have distinct orders, it follows that the (H, H) double coset in G represented by any $x \in H_i$ has size different from that represented by any $y \in \pi_j$ if $i \neq j$. Also, if $x, y \in \pi_i$ represent the same (H, H)-double coset, then we have $HxH = HyH$ and $H \cap H^x = H_i = H \cap H^y$. Thus $y = h_1 x h_2$ with h_1, h_2 in H. Hence $H \cap H^y = (H \cap H^x)^{h_2} = (H \cap H^y)^{h_2}$, whence $h_2 \in B_i$. Since $x, y \in B_i$, it follows that $h_1 \in B_i$. Thus $h_1, h_2 \in A_i$ and hence x and y represent the same (A_i, A_i) double coset in B_i. From the choice of π_i, it follows that $x = y$. Thus the elements of π represent distinct (H, H)-double cosets. From Lemmas 2.4 and 2.5 (also see Remark 1.7(e)), $|\pi| = s + 3$ while, from

Theorem 6(a), $s + 3$ is also the total number of (H, H) double cosets in G. This proves (c).

3.5. *Remarks.* We note that in the character table of Sp(4, s) in [6] the degrees of θ_1 and θ_2 have been interchanged. We actually have $\deg(\theta_1) = s(s + 1)^2/2$ and $\deg(\theta_2) = s(s^2 + 1)/2$.

3.6. *Proof of Corollary* 7. Let **A** be the complex Hecke algebra of 1_H^G and let B be the center of the complex group algebra $\mathbb{C}[H]$ of H. From Theorem 6(a) and (c) we have dim $\mathbf{A} = s + 3$ while from [14] we have dim $\mathbf{B} = s + 3$. **B** is trivially commutative, while by Theorem 6(a) 1_H^G is multiplicity free and hence **A** is commutative. Thus **A** and **B** are both $(s + 3)$-dimensional commutative semi-simple complex algebras, and hence both are isomorphic (as algebras) to $\mathbb{C}^{s+3}$. This proves the Corollary.

Note added in proof. Recent computations by the second named author ("Intersection pattern of the Ree groups in $G_2(3^{2n+1})$," preprint) shows that the conjecture in 1.7(a) is false for the pair $(G_2, {}^2G_2)$, but just barely so. Indeed, only one irreducible complex character of G_2 (namely the unique one of degree $s(s + 1)(s^3 + 1)/2$) appears in this permutation character with multiplicity two, while the remaining $s + 2$ irreducible constituents appear with multiplicity one each. The Conjecture is still open for the pair $(F_4, {}^2F_4)$.

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