Topological aspects of a fermion and the chiral anomaly

Ashim Roy and Pratul Bandyopadhyay
Indian Statistical Institute, Calcutta-700035, India

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It is here shown that the chiral anomaly is related to the topological properties of a fermion. The quantization procedure of a relativistic particle requires that the particle be an extended one, and to quantize a Fermi field, it is necessary to introduce an anisotropic feature in the internal space of the particle so that it gives rise to two internal helicities corresponding to a particle and an antiparticle. This specific quantum geometry of a Dirac particle gives rise to the solitonic feature as envisaged by Skyrme and the Skyrme term appears as an effect of quantization. When in the Lagrangian formulation the effect of this topological property is taken into account, it is found that the anomaly vanishes.

I. INTRODUCTION

In recent times, the old idea of the topological origin of the baryon number proposed by Skyrme[1] and Finkelstein and Rubinstein[2] has been revived. These authors put forward the idea that conserved quantum numbers arise as a consequence of the topological properties and that particles that carry conserved quantum numbers are built up from classical fields of nontrivial topology. In this picture baryons appear as solitons, commonly known as skyrmions. In a recent paper[3] it has been shown that the Skyrme term, which is necessary for the stability of a soliton, may appear as a consequence of the anisotropic feature of the internal space-time, where we have assumed that there is a fixed axis corresponding to a "direction vector" and this property of internal space-time helps us to have a consistent quantization of a Fermi field. In this scheme all fermions appear as solitons and the Skyrme term may be considered as an effect of quantization.

It may be added that Sternberg[4] has studied in detail the operation of charge conjugation and has argued that geometrically charged conjugation is induced by the Hodge star operator acting on a twistor space. It has been pointed out elsewhere[5] that the geometrical formulation of conformal inversion, which is induced by a charge conjugation acting on a spinor, in effect, corresponds to the inversion of the internal helicity for a spinor. This internal helicity may be taken to correspond to a fixed direction vector in the internal space of a massive spinor or a direction vector (vortex line) attached to the space-time point of a massless or massive spinor in a composite system of hadrons. The Hodge star operation in twistor space eventually inverts the orientation of the direction vector. In view of this, the internal helicity may be taken to represent the fermion number and can be taken to be of topological origin.

Jackiw[6] first pointed out the significance of topological effects in gauge field theories and its relationship with anomalies in quantum field theory. In a very elegant way he has shown how anomalies arise due to quantum mechanical symmetry breaking. Alvarez-Gaume and Ginsparg[7] studied non-Abelian anomalies from topological considerations. In this paper we shall show that the topological aspect of the stochastic quantization procedure of a Fermi field, where a direction vector is attached to a space-time point corresponding to the anisotropic feature of the internal space giving rise to the fermion number, helps us to find out the origin of the chiral anomaly in quantum field theory. This anomaly is avoided when we take into account this quantum geometry to study interactions involving gauge fields.

II. CONFORMAL GEOMETRY, TWISTOR SPACE, AND TOPOLOGICAL ASPECTS OF A FERMION

It is well known that the wave function of the form $\Psi(X_\mu, Y_\mu)$, where Y_μ is an attached vector that extends the Lorentz group SO(3,1) to the de Sitter group SO(4,1). Now in the stochastic quantization procedure for a fermion, it has been shown that a massive fermion is characterized by a fixed direction vector in the internal space that helps us to derive the fermionic propagator in Minkowski space from the two-point correlation of the stochastic fields $\varphi(Z_\mu) = \varphi(X_\mu) + i\varphi(Y_\mu)$, where the coordinate is given by $Z_\mu = X_\mu + iY_\mu$ in a complex manifold.[8] This indicates that the internal space of a massive fermion is disconnected in nature. This disconnectedness of the internal space gives rise to an internal helicity of the particle that corresponds to the fermion number. This follows from the fact that since the group structure is now given by SO(4,1), the irreducible representations of SO(4), the maximal compact subgroup of SO(4,1), are characterized by two numbers (k,n), where k is an integer or half-integer and n is a natural number. These two numbers are related to the eigenvalues of the Casimir operators by

$$\begin{aligned} \tfrac{1}{2} S^{\alpha\beta} S_{\alpha\beta} &= k^2 + (|k| + n)^2 - 1, \\ \tfrac{1}{8} \epsilon^{\alpha\beta\gamma\delta} S_{\alpha\beta} S_{\gamma\delta} &= k(|k| + n), \end{aligned} \tag{1}$$

where $S_{\alpha\beta}$, $\alpha,\beta = 1,2,3,4$, are the generators of the group. Barut and Bohm[9] have shown that the representations of SO(4,1) given by $S = \frac{1}{2}$ and $k = \pm\frac{1}{2}$ can be fully extended to two inequivalent representations of the conformal group SO(4,2). In fact these values actually correspond to the eigenvalues of the operator $K = \frac{1}{2}(a^\dagger a - b^\dagger b)$ in the oscillator representation of the $SO(3)_1 \times SO(3)_2$ basis of SO(4). The value of k as well as its signature is an SO(4,2) invariant. The representation $(s = 0, k = 0)$ in the conformal interpretation of SO(4,2) describes a massless spin-0 particle. The representation $s=\frac{1}{2}, k=\pm\frac{1}{2}$ describes the helicity state of a massless spinor. Now for a massive particle, the conformal invariance breaks down and $k=\pm\frac{1}{2}$ cannot be represented as helicity states in the conventional sense, but represents an "internal helicity" or orientation so that the mutually opposite orientations are equivalent to particle and antiparticle states.

Since these representations can be fully extended to the conformal group SO(4,2), we can now deal with eight-component conformal spinors. The simplest conformally covariant spinor field equation postulated as an O(4,2) covariant equation in a pseudo-Euclidean manifold $M^{4,2}$ is of the form

$$\left(\Gamma_a \frac{\partial}{\partial \eta_a} + m\right)\xi(\eta) = 0, \quad a = 0,1,2,3,5,6, \tag{2}$$

where the elements of the Clifford algebra Γ_a are the basis unit vectors of $M^{4,2}$, m is a constant matrix, and $\xi(\eta)$ is an eight-component spinor field. Cartan[10] has shown that in the fundamental representation where the unit vectors are represented by 8×8 matrices of the form

$$\Gamma_a = \begin{vmatrix} 0 & \Xi \\ H & 0 \end{vmatrix}, \tag{3}$$

the conformal spinors ξ are of the form

$$\xi = \begin{vmatrix} \phi_1 \\ \phi_2 \end{vmatrix}, \tag{4}$$

where ϕ_1 and ϕ_2 are Cartan semispinors. The characteristic property of these spinors is that for any reflection ϕ_1 and ϕ_2 interchange. In this basis, Eq. (2) becomes equivalent to the coupled equations in the Minkowski space

$$\begin{aligned} i\partial\!\!\!/ \phi_1 &= m\phi_2, \\ i\partial\!\!\!/ \phi_2 &= m\phi_1. \end{aligned} \tag{5}$$

However it is also possible to obtain from Eq. (2) a pair of standard Dirac equations in Minkowski space. To this end, we have to work with a unitary transformation C_1 given by

$$C_1 = \begin{vmatrix} L & R \\ R & L \end{vmatrix}, \tag{6}$$

where $L=\frac{1}{2}(1+\gamma_5)$, $R=\frac{1}{2}(1-\gamma_5)$ with

$$\gamma_5 = \begin{vmatrix} 1 & 0 \\ 0 & -1 \end{vmatrix}.$$

With this, we have

$$C_1\xi = \xi^D = \begin{vmatrix} \psi_1 \\ \psi_2 \end{vmatrix} \tag{7}$$

and

$$C_1^{-1}\Gamma_\mu C_1 = \Gamma_\mu{}^D = \begin{vmatrix} \gamma_\mu & 0 \\ 0 & \gamma_\mu \end{vmatrix}.$$

This suggests that Eq. (2) is equivalent in Minkowski space to the pair of standard Dirac equations

$$\begin{aligned} (i\partial\!\!\!/ + m)\psi_1 &= 0, \\ (i\partial\!\!\!/ + m)\psi_2 &= 0. \end{aligned} \tag{8}$$

It is to be noted that space or time reflection interchanges φ_1 and φ_2 and transforms ψ_1 and ψ_2 into themselves; conformal reflection interchanges both $\varphi_1 \leftrightarrow \varphi_2$ and $\psi_1 \leftrightarrow \psi_2$. It should be added that ψ_1 and ψ_2 may represent physical free massive fermions whereas φ_1 and φ_2 do not unless they are massless since they obey coupled equations. However, in the case $m\neq 0$, if we define φ_1 and φ_2 such that they represent two different "internal helicity" states given by $k=+\frac{1}{2}$ and $-\frac{1}{2}$, i.e., $\varphi_1=\psi(k=\frac{1}{2})$ and $\varphi_2=\psi(k=-\frac{1}{2})$, Eqs. (5) can be reduced to a single equation with two internal degrees of freedom when the linear combination of $\psi(k=+\frac{1}{2})$ and $\psi(k=-\frac{1}{2})$ represents an eigenstate. Now, to retain the four-component nature of the spinor in Minkowski space, these two internal degrees of freedom should be associated with particle–antiparticle states. Evidently this property of φ_1 and φ_2 satisfies the criteria that space, time, or conformal reflection transforms one into the other. This follows from the facts that (a) the parity operator changes the sign of k; (b) the time reversal operator T changes the orientation of the internal helicity and hence changes the sign of k; (c) as φ_1 and φ_2 are related here to particle–antiparticle states, conformal reflection changes one into the other. Thus each member of the doublet of massive spinors having the internal helicity $k=+\frac{1}{2}$ and $-\frac{1}{2}$, corresponding to particle and antiparticle states, represents a Cartan semispinor.

To have a geometrical interpretation of the doublet of Cartan semispinors it may be noted that it is possible to regard the components of the semispinor as the homogeneous coordinates of a point in three-dimensional projective space whereas those of another semispinor are regarded as the homogeneous coordinates of a plane in P^3 (Ref. 11). Moreover, a point-plane correspondence exists in P^3 that reflects the conjugation relation of semispinors. On the other hand, according to the analysis of Penrose,[12] there also exists a 1-1 correspondence between twistors of valence $\binom{1}{0}$ $\binom{0}{1}$ and a point plane in P^3. Thus the semispinors into which an eight-component spinor splits in the Cartan basis are identical to Penrose twistors. This reflects the analysis of Sternberg[4] that charge conjugation corresponds to Hodgestar operation in twistor space.

This analysis along with the fact that the anticommutation relation of the eight-component conformal spinors gives rise to supersymmetry algebra[13] suggests that we can introduce a spinor structure at each space-time point so that we have additional degrees of freedom to our space-time manifold E parametrized by $(x_\mu,\theta,\bar\theta)$, where $\theta=\binom{\theta_1}{\theta_2}$ is a two-component spinor. This effectively corresponds to a superspace. Indeed, the additional degrees of freedom θ, $\bar\theta$ in the space-time structure may be related here with the internal helicity given by the values $k=+\frac{1}{2}$ and $-\frac{1}{2}$ in the representation space of $SO(4)=SO(3)_1\otimes SO(3)_2$. To this end, we choose the chiral coordinates in the superspace as

$$Z^\mu = x^\mu + (i/2)\lambda^\mu_\alpha\theta^\alpha \quad (\alpha=1,2), \tag{9}$$

where we identify the coordinate in the complex manifold $Z_\mu = X_\mu + iY_\mu$ with $Y^\mu = \frac{1}{2}\lambda^\mu_\alpha\theta^\alpha$. We now replace the chiral coordinates by the matrices

$$Z^{AA'} = X^{AA'} + (i/2)\lambda^{AA'}_\alpha\theta^\alpha, \tag{10}$$

where $\lambda^{AA'}_\alpha(\alpha=1,2)\in SL(2,c)$. With these relations the twistor equation is now modified as

$$\bar Z_\alpha Z^\alpha + \lambda^{AA'}_\alpha\theta^\alpha\bar\pi_A\pi_{A'} = 0, \tag{11}$$

where $\bar\pi_A$ ($\pi_{A'}$) is the spinorial variable corresponding to the four-momentum variable p^μ, the conjugate of X^μ, and is given by in the matrix representation

$$p^{AA'} = \bar{\pi}^A \pi^{A'} \tag{12}$$

and

$$Z^\alpha = (\omega^A, \pi_{A'}), \quad \bar{Z}_\alpha = (\bar{\pi}_A, \bar{\omega}^{A'}), \tag{13}$$

with

$$\omega^A = i(X^{AA'} + (i/2)\lambda_\alpha^{AA'}\theta^\alpha)\pi_{A'}. \tag{14}$$

Equation (11) now involves the helicity operator

$$S = -\lambda_\alpha^{AA'}\theta^\alpha \pi_A \pi_{A'}. \tag{15}$$

It may be noted that in the complex manifold, we have taken the matrix representation of P_μ, the conjugate of X_μ in the complex coordinate $Z_\mu = X_\mu + iY_\mu$, as $p^{AA'} = \bar{\pi}^A \pi^{A'}$ implying $P_\mu^2 = 0$ and so the particle will attain its mass due to the nonvanishing characteristic of the quantity Y_μ^2. In the null plane where $Y_\mu^2 = 0$, we can write the chiral coordinates as follows:

$$Z^{AA'} = X^{AA'} + (i/2)\bar{\theta}^A\theta^{A'}, \tag{16}$$

where the coordinate Y^μ is replaced by $Y^{AA'} = \frac{1}{2}\bar{\theta}^A\theta^{A'}$. In this case, the helicity operator is given by

$$S = -2Y^{AA'}\bar{\pi}_A\pi_{A'} = -\bar{\theta}^A\theta^{A'}\bar{\pi}_A\pi_{A'} = \bar{\epsilon}\epsilon, \tag{17}$$

with $\epsilon = i\theta^{A'}\pi_{A'}$, $\bar{\epsilon} = -i\bar{\theta}^A\bar{\pi}_A$. In this case, following Shirafuji[14] we can apply the canonical quantization procedure where $i\bar{Z}_\alpha$ and $i\bar{\epsilon}$ are canonically conjugate to Z^α and ϵ, respectively, and we can postulate the canonical commutation and anticommutation relations given by

$$[Z^\alpha, \bar{Z}_\beta] = \delta^\alpha{}_\beta, \tag{18}$$

$$\{\epsilon^i, \bar{\epsilon}_j\} = \delta^i{}_j. \tag{19}$$

Symmetrizing $\bar{Z}_\alpha$ and Z^α and antisymmetrizing $\bar{\epsilon}$ and ϵ we require that the state vectors should satisfy

$$(\{\bar{Z}_\alpha, z^\alpha\} + [\bar{\epsilon}, \epsilon])|\psi\rangle = 0. \tag{20}$$

From this we find

$$(\bar{S} + \tfrac{1}{2}\bar{\epsilon}\epsilon - \tfrac{1}{4})|\psi\rangle = 0, \tag{21}$$

where

$$\bar{S} = \tfrac{1}{4}\{\bar{Z}_\alpha, Z^\alpha\}. \tag{22}$$

Now defining the operators

$$S_i^\alpha = \bar{\epsilon}_i Z^\alpha, \quad S_\alpha^i = \bar{Z}_\alpha \epsilon^i, \tag{23}$$

we have the commutation relations

$$[\bar{S}, S_i^\alpha] = -\tfrac{1}{2}S_i^\alpha, \qquad [\bar{S}, S_\alpha^i] = \tfrac{1}{2}\bar{S}_\alpha^i, \tag{24}$$

which indicates that $\bar{S}_\alpha^i$ and S_i^α are the helicity raising and lowering operators, respectively. The state with the internal helicity $+\frac{1}{2}$ is the vacuum state of the fermion operator

$$\epsilon|S = +\tfrac{1}{2}\rangle = 0. \tag{25}$$

Similarly, the state with the internal helicity $-\frac{1}{2}$ is the vacuum state of the fermion operator

$$\bar{\epsilon}|S = -\tfrac{1}{2}\rangle = 0. \tag{26}$$

In case of a massive spinor, we can define a negative definite plane D^- where for the coordinate $Z = X + iY$, Y belongs to the interior of the forward light cone ($Y \gg 0$) and as such represents the upper half-plane with the condition $\det Y > 0$ and $\frac{1}{2}\,\mathrm{Tr}\,Y > 0$. The positive definite plane D^+ is given by the set of all coordinates Z with Y in the interior of the backward light cone ($Y \ll 0$). The map $Z \to Z^*$ sends a negative definite plane to a positive definite plane. The space M of null space ($\det Y = 0$) is the Shilov boundary so that a function holomorphic in $D^-(D^+)$ is determined by its boundary values. Thus if we consider that any function $\varphi(Z) = \varphi(X) + i\phi(Y)$ is holomorphic in the whole domain, we note that the helicity $+\frac{1}{2}$ $(-\frac{1}{2})$ given by the operator $i\theta^A\pi_A$ $(-i\bar{\theta}^A\bar{\pi}_A)$ in the null plane may be taken to be the limiting value of the "internal helicity" in the upper (lower) half-plane. This indicates that in the massive spinor case, we can consider that the helicity given by

$$S = -\lambda_\alpha^{AA'}\theta^\alpha\bar{\pi}_A\pi_{A'} \tag{27}$$

represents the internal helicity $+\frac{1}{2}$ where we have $Y \gg 0$. Since the map $Z \to Z^*$ transforms a negative definite plane to a positive definite plane, we will have an opposite internal helicity $-\frac{1}{2}$ with the coordinate $Z^\mu = X_\mu - iY_\mu$ replaced by the matrices $Z^{AA'} = X^{AA'} - (i/2)\lambda_\alpha^{AA'}\bar{\theta}\alpha$ having $\frac{1}{2}\,\mathrm{Tr}\,Y < 0$. In the null plane we will have the condition $Y^{AA'} = \frac{1}{2}\bar{\theta}^A\theta^{A'}$ so that we can have the simultaneous existence of two helicities $+\frac{1}{2}$ and $-\frac{1}{2}$ corresponding to the spin projections on the z axis for a massless spinor. In this way, we can relate the spinorial variables θ and $\bar{\theta}$ in the superspace given by the coordinate $(X_\mu, \theta, \bar{\theta})$ with the internal helicity of a massive spinor. Evidently, this corresponds to the values $k = +\frac{1}{2}$ and $-\frac{1}{2}$ in the representation space of $SO(4) = SO(3)_1 \otimes SO(3)_2$ in the de Sitter space.

Now we want to point out that when the extension of a particle is given by the coordinate $(X_\mu, \theta, \bar{\theta})$, we can have a gauge field theoretic description of this extension when the corresponding gauge fields have the group structure $SL(2,c)$. Indeed, the metric tensor $g_{\mu\nu}^{AA'}(X,\theta,\bar{\theta}) = g_{\mu\nu}(x)\bar{\theta}^A\theta^{A'}$ can be taken to be described by the $SL(2,c)$ gauge fields in Minkowski space-time with the gauge field strength tensor given by

$$F_{\mu\nu} = \partial_\mu B_\nu - \partial_\nu B_\mu + [B_\mu, B_\nu], \tag{28}$$

where B_μ is the matrix-valued potential and belongs to $SL(2,c)$ (Ref. 3). The asymptotic zero curvature condition then implies $F_{\mu\nu} = 0$ so that we can write the non-Abelian gauge field as

$$B_\mu = U^{-1}\partial_\mu U, \quad \text{where} \quad U \in SL(2,c).$$

With the substitution, we note that the corresponding Lagrangian is given by

$$L = M^2\,\mathrm{Tr}(\partial_\mu U^{\prime}\partial_\mu U) + \mathrm{Tr}[\partial_\mu UU^+, \partial_\nu UU^+]^2, \tag{29}$$

where M is a suitable constant having the dimension of mass.

Thus we find that the quantization of a Fermi field considering an anisotropy in the internal space leading to an internal helicity description corresponds to the realization of a nonlinear sigma model—where the Skyrme term in the Lagrangian ($L_{\mathrm{Skyrme}} = \mathrm{Tr}[\partial_\mu UU^+, \partial_\nu UU^+]^2$) automatically arises for stabilizing the soliton. Thus in this picture, fermions appear as solitons and the fermion number is found to have a topological origin. Indeed, for the Hermitian representation, we can take the group manifold as SU(2) and this leads to a mapping from the space three-sphere S^3 to the group space S^3 [$SU(2) = S^3$] and the corresponding winding number is given by

$$q = \frac{1}{24\pi^2} \int_{S^3} ds_\mu \, \epsilon^{\mu\nu\alpha\beta} \mathrm{Tr}[U^{-1}\partial_\nu U U^{-1} \partial_\lambda U U^{-1} \partial_\beta U]. \tag{30}$$

Evidently q is a topological index and represents the fermion number.

III. TOPOLOGICAL ASPECTS OF A FERMION AND THE CONSERVED CURRENT

The above analysis can be used to link up the topological origin of fermion number with the internal helicity. Then the wave function for a particle and an antiparticle is implicitly represented as $\psi(x,\theta)$ and $\psi(X,\bar{\theta})$, $\theta,\bar{\theta}$ indicating the internal helicity $+\frac{1}{2}$ and $-\frac{1}{2}$, respectively, and the metric tensor is given by $g_{\mu\nu}(X,\theta,\bar{\theta})$. That is, spinor structures are introduced to each space-time point and we have a superspace. This geometry effectively gives rise to the SL(2,c) gauge fields (as the spinor-affine connection) having the field strength

$$F_{\mu\nu} = \partial_\mu B_\nu - \partial_\nu B_\mu + [B_\mu, B_\nu],$$

where B_μ is the matrix-valued potential. In superspace a given covariant tensor $F_{\mu\nu}$ does not have contravariant components $F^{\mu\nu}$. Therefore, following Carmeli and Malin[15] we choose the simplest Lagrangian density which is invariant under SL(2,c) transformations

$$\alpha = -\tfrac{1}{4}\mathrm{Tr}(\epsilon^{\alpha\beta\gamma\delta} F_{\alpha\beta} F_{\gamma\delta}) \tag{31}$$

where $\epsilon^{\alpha\beta\gamma\delta}$ is the completely antisymmetric tensor density in four dimensions with $\epsilon^{0123} = 1$. Applying the usual procedure of variational calculus, we get the field equations

$$\partial_\delta(\epsilon^{\alpha\beta\gamma\delta} F_{\alpha\beta}) - [B_\delta, \epsilon^{\alpha\beta\gamma\delta} F_{\alpha\beta}] = 0. \tag{32}$$

Taking the infinitesimal generators of the group SL(2,c) as

$$g_1 = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \quad g_2 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \quad g_3 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \tag{33}$$

we can write

$$\begin{aligned} B_\mu &= b^a_\mu g^a = \mathbf{b}_\mu \cdot \mathbf{g}, \\ F_{\mu\nu} &= F^a_{\mu\nu} g^a = \mathbf{f}_{\mu\nu} \cdot \mathbf{g} \quad (a = 1,2,3). \end{aligned} \tag{34}$$

Evidently in this space, these SL(2,c) gauge fields will appear as background fields.

Thus to describe a matter field in this geometry, the Lagrangian will be modified by the introduction of this SL(2,c) invariant Lagrangian density (31). Hence for a massless spinor field we write for the Lagrangian

$$L = -\bar{\psi}\gamma^\mu D_\mu \psi - \tfrac{1}{4}\mathrm{Tr}\,\epsilon^{\alpha\beta\gamma\delta} F_{\alpha\beta} F_{\gamma\delta}, \tag{35}$$

where D_μ is the SL(2,c) gauge covariant derivatives defined by

$$D_\mu = \partial_\mu - igB_\mu,$$

where g is some coupling strength. It is to be observed that by the introduction of the SL(2,c) gauge field Lagrangian in (35), we are effectively taking into account the effect of the extension of the fermionic particle giving rise to the internal helicity in terms of the gauge fields.[16] That is, writing the space-time coordinate and the four-momentum variables as

$$\begin{aligned} Q_\mu &= q_\mu + i\hat{Q}_\mu, \\ P_\mu &= p_\mu + i\hat{p}_\mu, \end{aligned} \tag{36}$$

where q_μ (p_μ) corresponds to the mean position (momentum) relating to the external space-time and $\hat{Q}_\mu$ ($\hat{P}_\mu$) corresponds to the internal stochastic extension, we can write, following Brooke and Prugovecki,[17] the following representation of Q_μ/ω and P_μ/ω, $\omega = \hbar/lmc$ (l being a fundamental length) acting on functions defined in phase space:

$$\begin{aligned} \frac{Q_\mu}{\omega} &= -i\left(\frac{\partial}{\partial p_\mu} + \phi_\mu\right), \\ \frac{P_\mu}{\omega} &= i\left(\frac{\partial}{\partial q_\mu} + \psi_\mu\right), \end{aligned} \tag{37}$$

where ϕ_μ and ψ_μ are matrix-valued functions. Thus identifying ϕ_μ with the SL(2,c) gauge field B_μ, we note that this spatial extension will give rise to a Lagrangian density given by (31) in addition to the point-particle spinorial Lagrangian density $\bar{\psi}\gamma_\mu\partial_\mu\psi$. Besides we can conceive of a coupling with this backgroundfield with the spinor and this leads to Eq. (35) for the effective Lagrangian of the spinorial matter field.

From this, we can now construct a conserved current corresponding to this Lagrangian and we get (neglecting the coupling with the gauge field)

$$\mathbf{j}^\mu = \bar{\psi}\gamma^\mu\psi - \epsilon^{\mu\nu\alpha\beta}\mathbf{b}_\nu \times \mathbf{f}_{\alpha\beta} = \mathbf{j}^\mu_x + \mathbf{j}^\mu_\theta. \tag{38}$$

Indeed from the properties of SL(2,c) generators we find from (32) that

$$\epsilon^{\mu\nu\alpha\beta}(\partial_\nu \mathbf{f}_{\alpha\beta} - \mathbf{b}_\nu \times \mathbf{f}_{\alpha\beta}) = 0.$$

This suggests that

$$\mathbf{j}^\mu_\theta = \epsilon^{\mu\nu\alpha\beta}\mathbf{b}_\nu \times \mathbf{f}_{\alpha\beta} = \epsilon^{\mu\nu\alpha\beta}\partial_\nu \mathbf{f}_{\alpha\beta}. \tag{39}$$

Then using the antisymmetric property of the Levi-Civita tensor density $\epsilon^{\mu\nu\alpha\beta}$ we get

$$\partial_\mu \mathbf{j}^\mu_\theta = \epsilon^{\mu\nu\alpha\beta}\partial_\mu \partial_\nu \mathbf{f}_{\alpha\beta} = 0. \tag{40}$$

Now noting that for spinor field, the vector current density is conserved, we finally have

$$\partial_\mu \mathbf{j}^\mu = \partial_\mu(\mathbf{j}^\mu_x + \mathbf{j}^\mu_\theta) = 0. \tag{41}$$

However, in the Lagrangian (35), if we split the Dirac massless spinor in chiral forms and identify the internal helicity ($+\frac{1}{2}$) ($-\frac{1}{2}$) with left (right) chirality corresponding to θ and $\bar{\theta}$, we can write

$$\begin{aligned} \bar{\psi}\gamma_\mu D_\mu\psi &= \bar{\psi}\gamma_\mu\partial_\mu\psi - ig\bar{\psi}\gamma_\mu B^a_\mu g^a\psi \\ &= \bar{\psi}\gamma_\mu\partial_\mu\psi - (ig/2)\{\bar{\psi}_R\gamma_\mu B^1_\mu\psi_R - \bar{\psi}_R\gamma_\mu B^2_\mu\psi_R \\ &\quad + \bar{\psi}_L\gamma_\mu B^2_\mu\psi_L + \bar{\psi}_L\gamma_\mu B^3_\mu\psi_L\}. \end{aligned} \tag{42}$$

Then the three SL(2,c) gauge field equations give rise to the following three conservations laws,

$$\begin{aligned} &\partial_\mu[\tfrac{1}{2}(-ig\bar{\psi}_R\gamma_\mu\psi_R) + j^1_\mu] = 0, \\ &\partial_\mu[\tfrac{1}{2}(-ig\bar{\psi}_L\gamma_\mu\psi_L + ig\bar{\psi}_R\gamma_\mu\psi_R) + j^2_\mu] = 0, \\ &\partial_\mu[\tfrac{1}{2}(-ig\bar{\psi}_L\gamma_\mu\psi_L) + j^3_\mu] = 0. \end{aligned} \tag{43}$$

These three equations represent a consistent set of equations if we choose

$$j_\mu^1 = -j_\mu^2/2, \quad j_\mu^3 = j_\mu^2/2,$$

which evidently guarantees the vector current conservation. Then we can write

$$\partial_\mu(\bar\psi_R\gamma_\mu\psi_R + j_\mu^2) = 0,$$
$$\partial_\mu(\bar\psi_L\gamma_\mu\psi_L - j_\mu^2) = 0. \tag{44}$$

From these, we find

$$\partial_\mu(\bar\psi\gamma_\mu\gamma_5\psi) = \partial_\mu j_\mu^5 = -2\partial_\mu j_\mu^2. \tag{45}$$

Thus the anomaly is expressed here in terms of the second SL(2,c) component of the gauge field current j_μ^2. However, since in this formalism the chiral currents are modified by the introduction of j_μ^2, we note from Eq. (44) that the anomaly vanishes.

From these equations, two separately conserved charges emerge, viz.,

$$\tilde{Q}_L = \int \psi_L^\dagger \psi_L\, d^3x - \int j_0^2\, d^3x,$$
$$\tilde{Q}_R = \int \psi_R^\dagger \psi_R\, d^3x + \int j_0^2\, d^3x. \tag{46}$$

The charge corresponding to the gauge field part is

$$q = \int j_0^2\, d^3x = \int_{\text{surface}} \epsilon^{ijk}\, da_i\, F_{jk}^2 \quad (i,j,k = 1,2,3). \tag{47}$$

Visualizing F_{jk}^2 to be the magnetic fieldlike components for the vector potential B_i^2, we see that ($i = 1,2,3$) is actually associated with the magnetic pole strength for the corresponding field distribution.

The term $\epsilon^{\alpha\beta\gamma\delta}\,\mathrm{Tr}\,F_{\alpha\beta}F_{\gamma\delta}$ in the Lagrangian can be actually expressed as a four-divergence of the form $\partial_\mu\Omega^\mu$, where

$$\Omega^\mu = -(1/16\pi^2)\epsilon^{\mu\alpha\beta\gamma}\,\mathrm{Tr}\left[\tfrac{1}{2}B_\alpha F_{\beta\gamma} - \tfrac{2}{3}(B_\alpha B_\beta B_\gamma)\right]. \tag{48}$$

We recognize that the gauge field Lagrangian is related to the Pontryagin density

$$P = -(1/16\pi^2)\,\mathrm{Tr}^*F_{\mu\nu}F^{\mu\nu} = \partial_\mu\Omega^\mu \tag{49}$$

and Ω^μ is the corresponding Chern–Simons secondary characteristic class. The Pontryagin index

$$q = \int P d^4x \tag{50}$$

is then a topological invariant. If we consider Euclidean four-dimensional space-time, then the above integral may be reduced to a three-surface integral where the three-surface is topologically equivalent to S^3. Now it is noted that we must have $F_{\alpha\beta} = 0$ at all spatial and temporal infinity points so that the action $S = \int L\, d^4x$ gives rise to a finite energy gauge field configuration. Then the gauge potentials tend to a pure gauge at large distances in all four directions, i.e., we have

$$B_\mu \underset{x_\mu\to\alpha}{\to} U^{-1}\partial_\mu U. \tag{51}$$

This then helps us to write

$$q = \frac{1}{24\pi^2}\int_{S^3} ds_\mu\, \epsilon^{\mu\alpha\beta\gamma}$$
$$\times \mathrm{Tr}[U^{-1}\partial_\alpha U U^{-1}\partial_\beta U U^{-1}\partial_\gamma U]. \tag{52}$$

We observe that this is nothing but the fermion number as discussed in the previous section. In four-dimensional space-time, if we assume B_μ to go faster than $1/r$, B_μ being zero at negative infinity of the time coordinate but tends to a pure gauge at positive infinity of the time coordinate, we can write

$$q = \int dr\, \Omega^0 \Big|_{x^0 = \alpha}. \tag{53}$$

From this, it appears that the axial vector current is now modified as

$$\tilde{j}_\mu^5 = j_\mu^5 + 2\hbar\Omega_\mu \tag{54}$$

and though $\partial_\mu j_\mu^5 \neq 0$, we have $\partial_\mu \tilde{j}_\mu^5 = 0$. That means, when the topological properties of a fermion related to the origin of fermion number is taken into account, we are not confronted with the chiral anomaly. The origin of the chiral anomaly is thus found to be due to the naive form of the point particle current without any topological structure, which turns out to be essential for the quantization of a Fermi field.

IV. FERMIONS AND THE INTERACTION WITH AN EXTERNAL ABELIAN GAUGE FIELD

The chiral description of the matter field in terms of the spinorial variables $\theta,\bar\theta$ in the metric tensor $g_{\mu\nu}(x,\theta,\bar\theta)$ giving rise to the SL(2,c) gauge field currents necessitates the introduction of a disconnected gauge group for the external Abelian field interacting with the matter field in a chiral symmetric way. In the case where the external Abelian gauge field is the electromagnetic field, the Lagrangian density is given by

$$L = -\bar\psi\gamma_\mu D_\mu\psi - \tfrac{1}{4}\,\mathrm{Tr}(\epsilon^{\alpha\beta\gamma\delta}\tilde{F}_{\alpha\beta}\tilde{F}_{\gamma\delta})$$
$$-\tfrac{1}{4}\,\mathrm{Tr}(F_{\mu\nu}F^{\mu\nu}) + \mathrm{Tr}(j_\mu A^\mu). \tag{55}$$

Here D_μ is the SL(2,c) gauge covariant derivative and, considering the order of $\psi - B_\mu$ coupling to be negligible compared to the matter current electromagnetic field coupling, we can replace it by ∂_μ,

$$\tilde{F}_{\alpha\beta} = \partial_\alpha B_\beta - \partial_\beta B_\alpha + [B_\alpha, B_\beta], \quad B_\mu \in \mathrm{SL}(2,c)$$

and $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$, A_μ being the electromagnetic gauge potential and j_μ is the matter current matrix given by

$$j_\mu = \begin{bmatrix} \bar\psi_R\gamma_\mu\psi_R + j_\mu^2 & 0 \\ 0 & \bar\psi_L\gamma_\mu\psi_L - j_\mu^2 \end{bmatrix}, \tag{56}$$

where j_μ^2 is the second component of the SL(2,c) gauge field current as discussed in the previous section. It is evident that this matrix structure of j_μ exhibiting the chiral form suggests that for A_μ we should take the disconnected gauge group $U_{1L}\times U_{1R} = U_1 x\{1,d\}$ where d is the orientation reversing operation. Evidently in such an interaction the field strength and current are not gauge invariant but only gauge covariant, each changing sign under d. This is similar to the non-Abelian theories where field strengths and currents are only gauge covariant even under gauge transformations connected to the identity. The internal symmetry group here is O(2) which is given by the relation

$$O(2) = SO(2)\times\{1,d\} = U_1\times\{1,d\}, \tag{57}$$

where d is the orientation reversing operator. Indeed, we can take

$$A_\mu = \begin{bmatrix} A_{\mu+} & 0 \\ 0 & A_{\mu-} \end{bmatrix}. \tag{58}$$

Kiskis[18] has studied the interactions having disconnected gauge group. Following Kiskis, we can think of a large system of observers each responsible for a small open region U_i of the connected space-time manifold M. Let us consider that all the frames in U_i have the same orientation. Physically this means that the space is simply connected and the observer can give an unambiguous definition of positive charge everywhere. This suggests that we can introduce the connection (gauge field) in the Lagrangian

$$L^i = L_g^{(i)} + L_M^{(i)}, \tag{59}$$

where i identifies quantities associated with the region U_i, $L_M^{(i)}$ is the matter field Lagrangian, and $L_g^{(i)}$ is the kinetic energy term for the connection. The gauge symmetry of the $L_g^{(i)}$ is given by

$$A \to g^{-1}(\partial + A)g, \tag{60}$$

with g a smooth map

$$g = U_i \to O(2) \tag{61}$$

which may lie in either component of O(2). A transformation that reverses the orientation at each point can be written as

$$g = dg_0,$$
$$g_o = U_i \to SO(2), \tag{62}$$
$$d = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

This gives

$$A \to g_0^{-1}(\partial - A)g_0. \tag{63}$$

We see that it is a combination of charge conjugation and orientation preserving gauge rotation. Evidently in this formalism the chiral currents interact with the gauge field in a disconnected form. Indeed, writing

$$A_\mu = \begin{bmatrix} A_{\mu+} & 0 \\ 0 & A_{\mu-} \end{bmatrix}$$

we find the interaction term is given by

$$\begin{bmatrix} (\bar\psi_L \gamma_\mu \psi_L - j_\mu^2)A_{\mu+} & 0 \\ 0 & (\bar\psi_R \gamma_\mu \psi_R + j_\mu^2)A_\mu \end{bmatrix}. \tag{64}$$

Evidently there is no term like $A_{\mu+}A_{\mu-}$ in the Lagrangian.

As Kiskis[18] has discussed, in the overlap region

$$U_{ij} = U_i \cap U_j$$

there are two observers studying the same physical system where each observer has set up his own basis in the internal symmetry space over U_{ij}. The relation between these bases is a gauge transformation

$$g_{ij}: U_{ij} \to O(2),$$

where the map lies in either component of O(2). That is, observers i and j may have opposite charge convention. If they have opposite convention about charge, they will have opposite convention about field. In fact, if we designate *a priori* what is a particle and what is an antiparticle, the left and right directions can be determined by any parity violating interaction. On the other hand, if we designate what is left and what is right the particle–antiparticle designation remains fixed. Thus any parallel transport from a region U_i to U_j of the manifold will be such that either the orientation remains the same and the observer will see the same charge or the orientation is opposite when by reversing the orientation of U_j the observer will see the same charge. Thus any path from any region U_i to U_j will be such that either this will give the same orientation for U_j or it is opposite when reversing the orientation, the observer will identify a left-handed or a right-handed particle.

V. DISCUSSION

We have shown above that the chiral anomaly is connected with the topological properties of a fermion. Indeed, the topological property of a fermion gives rise to the fermion number which is always conserved and helps us to treat fermions as solitons. The Skyrme term here arises just as an effect of quantization of a fermion[3] and is related to the quantum geometry of a relativistic particle. The relativistic generalization of a quantum particle necessitates the particle to be an extended one and to attain the fermionic property, we need to introduce an anisotropic feature in the internal space of the particle so that it gives rise to two internal helicities corresponding to a particle and an antiparticle. This specific quantum geometry of a Dirac particle gives rise to the solitonic feature as envisaged by Skyrme[1] as well as by Finkelstein and Rubinstein.[2] When in the Lagrangian formulation the effect of this topological property is taken into account, we find that the anomaly vanishes.

This analysis suggests that the origin of anomaly lies in the fact that fermions are conventionally treated as localized point particles devoid of any specific geometrical and topological feature. But when this topology is taken into account anomaly vanishes implying that when we study quantum mechanical symmetry breaking, we must take into account the geometrical features involved in the quantization procedure. That is, quantum mechanical effects have their origin in quantum geometry and need to be studied in this perspective.

[1]T. H. R. Skyrme, Proc. R. Soc. London Ser. A **260**, 127 (1961). Nucl. Phys. **31**, 556 (1962).
[2]D. Finkelstein, J. Math. Phys. **7**, 1218 (1966); D. Finkelstein and J. Rubinstein, *ibid.* 1762 (1968).
[3]P. Bandyopadhyay and K. Hajra, J. Math. Phys. **28**, 711 (1987).
[4]S. Sternberg, Comm. Math. Phys. **109**, 649 (1987).
[5]P. Bandyopadhyay, "Holomorphic Quantum Mechanics, Conformal Reflection, and the Internal Symmetry of Hadrons," to be published in Int. J. Mod. Phys.
[6]R. Jackiw, *Topological Investigations of Quantized Gauge Theories* (Relativity, Groups and Topology, Les Houches 1983), edited by B. S. Dewitt and R. Stora (North-Holland, Amsterdam, 1984).
[7]L. Alvarez-Gaume and P. Ginsparg, Nucl. Phys. B **243**, 449 (1984).
[8]P. Bandyopadhyay and K. Hajra, "Stochastic Quantization in Minkowski Space" (submitted for publication).
[9]A. O. Barut and A. Bohm, J. Math. Phys. **11**, 2938 (1970).

[10]E. Cartan, *The Theory of Spinors* (Paris, 1966).
[11]M. Daniel, Phys. Lett. B **65**, 246 (1976).
[12]R. Penrose, Int. J. Theor. Phys. **1**, 61 (1968).
[13]A. Haag, J. Lopusanski, and M. Sohnius, Nucl. Phys. B **88**, 257 (1975); M. Daniel and C. N. Ktorides, Nucl. Phys. B **115**, 313 (1976).
[14]T. Shirafuji, Prog. Theor. Phys. **70**, 18 (1983).
[15]S. Malin and M. Carmeli, Ann. Phys. **103**, 208 (1977).
[16]P. Bandyopadhyay and K. Hajra, "Stochastic Quantization, Localizability of a Relativistic Particle and Quantum Geometry" (submitted for publication).
[17]J. A. Brooke and E. Prugovecki, Nuovo Cimento A **89**, 126 (1985).
[18]J. Kiskis, Phys. Rev. **17**, 3196 (1978).