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Water waves generated at an inertial surface by an axisymmetric initial surface disturbance

by B. N. MANDAL and S. MUKHERJEE
Department of Applied Mathematics, Calcutta University, 92, A.P.C. Road, Calcutta-700009, India

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The problem of generation of surface waves in a liquid with an inertial surface due to an initial axially symmetric surface disturbance is discussed. The form of the inertial surface is obtained asymptotically for large values of time and distance.

1. Introduction

The problems of generation of water waves due to an explosion above or within the water can be formulated as an initial value problem within the framework of linearized theory of water waves. When the explosion occurs above water, we can assume the initial condition to be an initial impulsive pressure distributed over a certain region of the free surface. However, when the explosion occurs within the water, the initial condition is taken as an initial displacement (elevation or depression) distributed over a region of the free surface, resulting from the explosion. For the case of initial disturbance in the form of an impulse or displacement concentrated at the origin, the potential functions as well as the free surface elevations were given in [1, 2]. The axially symmetric disturbance was considered in brief by Kranzer and Keller [3] who compared the theory with experimental results, Choudhuri [4] and Wen [5] considered the case where the disturbance is over any arbitrary region of the free surface and the water is of uniform finite depth by the method of multiple Fourier transforms. In all these cases the method of stationary phase was applied to obtain the approximate expression for the potential function and free surface elevation for large values of time and distance. Problems of generation of surface waves in a liquid covered by an inertial surface composed of a thin uniform distribution of non-interacting materials have attracted the attention of mathematicians recently. Rhodes-Robinson [6], Mandal and Kundu [7], Mandal [8] considered in a number of papers the generation of water waves due to different types of sources with time-dependent strength submerged in a liquid with an inertial surface.

In this paper we consider the generation of surface waves in a liquid with an inertial surface due to surface disturbances. The liquid is assumed to be incompressible, inviscid and is at rest. The motion starts due to an initial disturbance over the surface. Since the motion starts from rest, it is irrotational. Within the framework of linearized theory it can be described by a potential function whch satisfies an initial value problem. Taking Laplace transform in time, the transformed potential then satisfies a boundary value problem. Due to axially symmetric disturbance, Hankel transform is employed to solve this boundary value problem. Finally Laplace inversion produces the potential function in terms of an integral. Hence the depression of the inertial surface at any time t is obtained as an integral. For large time and distance this integral is evaluated asymptotically. Known results are recovered for the case of initial disturbance (impulse or displacement) concentrated at the origin in the absence of an inertial surface.

2. Statement and solution of the problem

We consider a liquid of volume density ρ covered by an inertial surface of area density $\varepsilon\rho(\varepsilon\geqslant 0)$. $\varepsilon=0$ corresponds to a liquid with a free surface.

We choose a cylindrical coordinate system (r,θ,y) in which the y axis is taken vertically downwards with the plane $y=0$ coinciding with the rest position of the inertial surface. The motion is generated by an initial disturbance (impulse or displacement) distributed over inertial surface. Assuming the motion to be irrotational, within the framework of linearized theory, the motion is described by a velocity potential $\varphi(r,y,t)$ which satisfies the Laplace equation

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\varphi}{\partial r}\right)+\frac{\partial^2\varphi}{\partial y^2}=0,\quad y\geqslant 0,\quad t\geqslant 0 \tag{2.1}$$

the boundary conditions,

$$\frac{\partial^2}{\partial t^2}(\varphi-\varepsilon\varphi_y)-g\varphi_y=0 \text{ on } y=0,\quad t>0, \tag{2.2}$$

$$\nabla\varphi\to 0 \text{ as } y\to\infty \tag{2.3}$$

and the initial conditions

$$\varphi-\varepsilon\varphi_y=-\frac{1}{\rho}F(r) \text{ on } y=0,\quad t=0, \tag{2.4 a}$$

$$\frac{\partial}{\partial t}(\varphi-\varepsilon\varphi_y)=0 \text{ on } y=0,\quad t=0 \tag{2.5 a}$$

when an initial axially symmetric impulse $F(r)$ is applied per unit area of the inertial surface at a distance r from the origin, or

$$\varphi-\varepsilon\varphi_y=0\quad y=0,\quad t=0, \tag{2.4 b}$$

$$\frac{\partial}{\partial t}(\varphi-\varepsilon\varphi_y)=gG(r)\quad y=0,\quad t=0 \tag{2.5 b}$$

when an initial axially symmetric depression $G(r)$ of the inertial surface at a distance r from the origin is prescribed.

Let $\bar\varphi(r,y;p)$ be the Laplace transform of $\varphi(r,y,t)$ defined by

$$\bar\varphi(r,y,p)=\int_0^\infty\varphi(r,y,t)\exp(-pt)\,dt,\quad p>0. \tag{2.6}$$

Then the transformed potential $\bar\varphi(r,y;p)$ satisfies the boundary value problem

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\bar\varphi}{\partial r}\right)+\frac{\partial^2\bar\varphi}{\partial y^2}=0\quad y\geqslant 0, \tag{2.7}$$

$$p^2\bar{\varphi}-(g+\varepsilon p^2)\bar{\varphi}_y=\begin{cases}-\dfrac{p}{\rho}F(r) \text{ on } y=0 & (2.8\text{a})\\ \text{or} & \\ gG(r) \text{ on } y=0, & (2.8\text{ b})\end{cases}$$

and

$$\nabla\bar{\varphi}\to 0 \text{ as } y\to\infty.$$

(2.8 a) holds for an initial surface impulse while (2.8b) holds for an initial surface ¨isplacement.

Let $\Psi(y;k)$ be the Hankel Transform of $\bar{\varphi}(r,y)$ given by

$$\Psi(y;k)=\int_0^\infty r\bar{\varphi}(r,y)J_0(kr)\,dr, \quad k>0. \tag{2.9}$$

hen Ψ satisfies

$$\frac{d^2\Psi}{dy^2}-k^2\Psi=0, \quad y\geqslant 0, \tag{2.10}$$

$$p^2\Psi-(g+\varepsilon p^2)\Psi_y=\begin{cases}-\dfrac{p}{\rho}\hat{F}(r) \text{ on } y=0\\ \text{or}\\ g\hat{G}(r)\end{cases} \tag{2.11}$$

ᵀ here $\hat{F}$ and $\hat{G}$ denote the Hankel transform of $F(r)$ and $G(r)$ respectively. Then

$$\Psi(k,y)=\frac{\exp(-ky)}{p^2(1+\varepsilon k)+gk}\begin{cases}-\dfrac{p}{\rho}\hat{F}(k)\\ \\ g\hat{G}(k).\end{cases} \tag{2.12}$$

Thus,

$$\bar{\varphi}(r,y,p)=\int_0^\infty \frac{k\exp(-ky)}{p^2(1+\varepsilon k)+gk}J_0(kr)\begin{Bmatrix}-\dfrac{p}{\rho}\hat{F}(k)\\ \\ g\hat{G}(k)\end{Bmatrix}dk \tag{2.13}$$

For the case of an initial impulse, we obtain

$$\varphi(r,y,t)=-\frac{1}{\rho g}\int_0^\infty \hat{F}(k)\Omega^2\cos\Omega t\exp(-ky)J_0(kr)\,dk \tag{2.14}$$

where

$$\Omega^2=\frac{gk}{1+\varepsilon k} \tag{2.15}$$

Hence the inertial surface depression is

$$\zeta(r,t)=\frac{1}{g}\left[\frac{\partial}{\partial t}(\varphi-\varepsilon\varphi_y)\right]_{y=0}$$

$$=\frac{1}{\rho g}\int_0^{\infty}k\hat{F}(k)\Omega J_0(kr)\sin\Omega t\,dk. \tag{2.16}$$

he absence of inertial surface ($\varepsilon=0$), this reduces to the result given in [2] for an al impulse of unit strength concentrated at the origin. To obtain the approximate n of ζ for large values of t and r, we adopt the method of stationary phase using

$$J_0(kr)=\frac{2}{\pi}\int_0^{\pi/2}\cos(kr\cos\beta)\,d\beta.$$

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$$\zeta(r,t)=\frac{1}{\pi\rho g}\,\mathrm{Im}\int_0^{\infty}\int_0^{\pi/2}k\hat{F}(k)\Omega\{\exp(\mathrm{i}(\Omega t+kr\cos\beta))$$

$$+\exp(\mathrm{i}(\Omega t-kr\cos\beta))\}\,d\beta\,dk. \tag{2.17}$$

e main contribution to ζ for large t and r such that t/r remains finite comes from second term. Using the method of stationary phase first we evaluate the β integral d then we evaluate the k integral and finally we obtain

$$\zeta(r,t)\simeq\frac{2^{1/2}}{\rho g^{1/2}r}\hat{F}(k_0)\frac{k_0^{3/2}}{(1+4\varepsilon k_0)^{1/2}}\sin\left\{\frac{gt^2}{4r}\frac{1+2\varepsilon k_0}{(1+\varepsilon k_0)^3}\right\} \tag{2.18}$$

where k_0 is the real positive root of the biquadratic equation

$$k(1+\varepsilon k)^3=\frac{gt^2}{4r^2} \tag{2.19}$$

In the absence of inertial surface

$$k_0=K(r,t)\equiv gt^2/4r^2 \tag{2.20}$$

and in that case we obtain

$$\zeta(r,t)\simeq\frac{gt^3}{2^{5/2}\rho r^4}\hat{F}(K)\sin\left(\frac{gt^2}{4r}\right) \tag{2.21}$$

For an impulse concentrated at the origin we can take $\hat{F}(K)=1/2\pi$ so that this coincides with the result given in [2], p. 166.

For small εK we can approximate k_0 as $k_0\simeq K(1-3\varepsilon K)$ so that in this case

$$\zeta(r,t)=\frac{gt^3}{2^{5/2}\rho r^4}\hat{F}(k(1-3\varepsilon k))\sin\left\{\frac{gt^2}{4r}\left(1-\varepsilon\frac{gt^2}{4r^2}\right)\right\} \tag{2.22}$$

For the case of an initial axially symmetric displacement we can similarly obtain

$$\varphi(r,y,t)=\int_0^{\infty}\hat{G}(k)\Omega\exp(-ky)\sin\Omega t\,J_0(kr)\,dk \tag{2.23}$$

so that

$$\zeta(r,t)=\int_0^\infty \hat{G}(k)kJ_0(kr)\cos\Omega t\,\mathrm{d}k. \tag{2.24}$$

For large t and r such that r/t remains finite we can similarly obtain

$$\zeta(r,t)\simeq\frac{1}{2^{3/2}}\frac{gt^2}{r^3}\hat{G}\left\{\frac{gt^2}{4r^2}\left(1-3\varepsilon\frac{gt^2}{4r^2}\right)\right\}\left(1-\frac{9}{8}\varepsilon\frac{gt^2}{4r^2}\right)\times\cos\left\{\frac{gt^2}{4r}\left(1-\varepsilon\frac{gt^2}{4r^2}\right)\right\} \tag{2.25}$$

For displacement concentrated at the origin

$$\hat{G}=\frac{1}{2\pi}$$

so that in this case

$$\zeta(r,t)=\frac{1}{2^{3/2}}\frac{gt^2}{r^3}\left(1-\frac{9}{8}\varepsilon\frac{gt^2}{4r^2}\right)\cos\left\{\frac{gt^2}{4r}\left(1-\varepsilon\frac{gt^2}{4r^2}\right)\right\} \tag{2.26}$$

In the absence of the inertial surface ($\varepsilon=0$), (2.26) reduces to the known result for an initial displacement concentrated at the origin.

3. Discussion

From (2.22) and (2.25) it is apparent that the inertial surface affects both oscillatory and non-oscillatory factors of $\zeta(r,t)$ in both cases (for an initial impulse and for an initial displacement). When $\varepsilon(gt^2/4r^2)$ is small, at a fixed point the amplitude increases as t^3 in case of an initial impulse concentrated at the origin and t^2 in case of initial displacement concentrated at the origin while for a fixed time the amplitude increases as r^{-4} and r^{-3} respectively. For known $F(r)$ and $G(r)$ similar types of conclusion might be arrived at. Also for finite uniform depth of water, a similar type of asymptotic method can be followed to obtain the form of the inertial surface.

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