ESSAYS ON GROUP DEVIATION

CONAN MUKHERJEE

A DISSERTATION SUBMITTED TO THE INDIAN STATISTICAL INSTITUTE IN PARTIAL FULFILMENT OF THE REQUIREMENT FOR THE AWARD OF THE DEGREE OF DOCTOR OF PHILOSOPHY

INDIAN STATISTICAL INSTITUTE

KOLKATA

JULY 2011

Abstract

In this thesis, we analyze the issues of group formation and design of mechanisms immune to group deviation. We focus on two types of allocation problems, (i) indivisible goods allocation problem, and (ii) queueing problem. In the first chapter, we state the need for a careful study of these issues.

In the second chapter, we use the n-agent version of the Rubinstein's bargaining game to predict the coalitions/groups that form at a Vickrey auction. We state the asymptotic results as $\delta \to 1$. For single goods, we show that when the highest valuation agent proposes first, the resultant bidding ring must be the largest average maximizing coalition. For multiple identical goods, we focus on *coalition structures* to account for the inherent externalities across coalitions, in this setting. In particular, we focus on the peculiar class of coalition structures; where any one winner generates the gains from cooperation (by colluding all the losers) while other winners (who stay alone or form pairs among themselves) free ride and get a good at the base price. We state the sufficient condition for a member of this class to form, *irrespective of the protocol function.*

In the third chapter, we address the issue of mechanism design. We look to implement fair (no-envy) allocation rules through mechanisms immune to group deviation/misreporting. We find that there are no such mechanism that block group deviations where the total group utility increases. To get a possibility, we weaken our requirement to weak group strategyproofness and completely characterize the class of fair and weak group strategyproof

mechanisms. We also find that the *Pivotal* mechanism is the only feasible mechanism in this class satisfying a zero transfer condition.

In the final chapter, we analyze the implementation of decision efficient and fair allocations using weak group strategyproof mechanisms. Interestingly, in this context, fairness manifests itself as the continuity of the mechanisms, as the latter turns out to be equivalent to "equal treatment of equals". We specify a necessary condition for weak group strategyproof mechanisms to implement fair and decision efficient allocations. We also specify a sufficient class of mechanisms in this regard.

Acknowledgement

This thesis is the culmination of my journey over the last five years. It was a dizzying ride all throughout; which saw days of abject dejection and a few (and far in between) days of joyous delirium. Looking back on these years, I can only humbly thank the people who made this possible.

I must start with the my supervisor Dr. Manipushpak Mitra who taught me all that I know about logical thinking. It was under his tutelage that I learned that reputation is built, and diminished, with a "multiplier effect". Not to mention his generosity in providing me with a laptop, which considerably sped up my thesis work. I will forever be grateful to him for initiating the joint project that forms the second chapter of this thesis.

I must also express my gratitude to Dr. Kalyan Chatterjee, for agreeing to co-author the afore-mentioned chapter. Dr. Chatterjee, in spite of being extremely busy, has always found time to help, guide and educate me. I express my thanks to Dr. Satya. R. Chakravarty whose comments and suggestions were indispensable. My sincere thanks go to Dr. Arunava Sen and Dr. Suresh Mututswami, whose comments made me confident in my academic outlook and shaped my research.

I thank the watchful faculty of Economic Research Unit (ERU), under whose aegis, this work was completed. I cannot but acknowledge the incredible support and companionship that all (present as well as past) my colleagues, research fellows at ERU, provided me. I should mention the ERU staff, too, who provided the required research infrastructure, not to mention the zillion cups of tea.

On the personal front, I thank my parents; especially my mother, for her unwavering faith in me, that continues to amaze and inspire. Thanks are also due to my mother-in-law who helped me (and my wife) at some of our lowest ebbs. I must also mention my little sister, who is not so little anymore.

However, of all people, my deepest gratitude goes to my wife, Nabanita; who *bled* the most, in my struggle for the last five years. She kept believing in me, even when I had lost all faith. I can only hope that I have done some justice to her perseverence and sacrifice.

July 18, 2011
Kolkata

Conan Mukherjee

Gō*d*es.

Sólo Por Ti, mi Amor!

Contents

[1]Co-authored with Kalyan Chatterjee and Manipushpak Mitra

Chapter 1

Introduction

Much of *noncooperative game theory* has focussed on individual incentives. The seminal concept of Nash Equilibrium, for example, focuses only on unilateral deviations. However, the very concept of equilibrium involves ruling out the possibility of any kind of departure from the decided behaviour, individual or otherwise. Hence, an in-depth study of refinements of equilibrium concepts with regard to group deviations, is in order.

Now, any group must first decide which members to include; and only then, can it decide how to deviate and divide the proceeds from the deviation. Therefore, any group deviation is subsequent to group formation, and so, a study of group deviation is incomplete without an explicit description of group formation. In this thesis, we analyze how coalitions or groups form and how mechanisms immune to such deviations can be designed. We concern ourselves with implementation of fair and efficient decisions. In particular, we focus on problems of allocation of indivisible good and positions on a queue. Our endeavour is to look for incentives, benefits, and barriers to group deviation, in such problems.

A group of agents, colluding and playing, as a single unit, is not a novel concept in game theory. In fact, what we call today *cooperative game theory*, has always dealt with this issue. The chief tool of studying coalitions in this literature, is the "characteristic function", which assigns a particular worth to each coalition. However, this approach is fraught with two particular difficulties. One, the worths ascribed by the characteristic function, are derived under the assumption of the groups playing zero sum game amongst them. In many settings,[1] this assumption turns out to be untrue. Second, the focus of cooperative game theory is on the allocations which can not be undermined by *any* coalition. This subsumes the coalition formation process and hence, ignores the possibility of certain coalitions never forming.

The former difficulty was alleviated to a certain extent by modifying the characteristic function to depend on the ambient *coalition structure*[2]. The latter difficulty continued to persist till Chatterjee, Dutta, Ray and Sengupta [6] and Ray and Vohra [39] extended the 2 player Rubinstein's bargaining game to an n player setting, and specified the coalition that would form along with the realized payoff distribution. Given a characteristic function (a "partition function" in Ray and Vohra [39]), they modelled coalition formation as a complete information dynamic game; and used the notion of sequential rationality to capture the *farsightedness* that agents employ while forming groups.

This thesis starts on the lines of Ray and Vohra [39], with the classic iden-

[1] For example, the n-person Cournot market.

[2] See Ray and Vohra [39].

tical indivisible goods allocation problem where a set of agents report their valuations for the good, based on which, the seller (or a planner) decides the recipient of the good. As shown in the seminal paper Vickrey [47], implementation of the efficient decision (allocating the good to the highest valuation agent) in dominant strategies requires that the highest valuation agent pay the second highest valuation to the seller. This translates into the second price auction, more generally known as, the Vickrey auction. We apply the analysis of Chatterjee, Dutta, Ray and Sengupta [6], and Ray and Vohra [39] to the Vickrey auction both for single and multiple identical indivisible goods. We show how coalitions can be formed to undermine the implementation of the efficient decision in dominant strategies, and hence, the seller revenue; *even in absence* of institutions facilitating binding contract. In particular, we show what coalitions are formed by farsighted agents; with the intra-group contract being enforced by sequential rationality itself.

Having delved into the mechanics of coalition formation, the study progresses to the search for barriers to coalitional deviation. In line with the second chapter, we take up the issue of indivisible goods allocation in the third chapter. We look for *group strategyproof* mechanisms that implement fair decisions, for single as well as multiple identical goods. We capture "fairness" by the popular concept of *no-envy* (introduced by Foley [11] and Varian [46]). This notion of fairness turns out to imply decision efficiency, in this setting.

However, we find that there is no mechanism that implements efficient decision; and ensures a fall in group (total) utility at all possible group deviations, for all possible groups. Therefore, to get a possibility result, we relax our group strategyproofness requirement a bit. Instead of ruling out all coalitional deviations in which the group (total) utility increases; we require the mechanism to rule out only those group deviations that strictly benefit all its members. This relaxation turns out to be intuitive too; any collusion at an auction would be deemed illegal and hence, contracts written by the groups with regard to the division of proceeds are not enforceable by judiciary. In such an environment, it is only natural for agents to take part in *only* those collusive ventures that strictly benefit them; without the need for any post-deviation redistribution. We call this *weak group strategyproofness* and; completely characterize the class of no-envy and weak group strategyproof mechanisms for the single (and multiple identical) indivisible good(s) allocation problem. This class includes the Vickrey mechanism (up to a constant).

In the fourth chapter, we take up another type of allocation problem: the *Queueing* problem. We consider a situation where there is a set of agents with identical jobs to process; and a set of machines which can only process these jobs *sequentially* (one by one). The machines are non-identical, and so, consume different amounts of time to complete a job. These machine-specific speeds, remain constant across jobs. The agents dislike waiting; and the planner wants to minimize the aggregate disutility that the agents incur in getting their jobs done. In particular, we look at the problem of assigning positions on queues for the machines to a set of agents having the same job but different costs of waiting. Maniquet [28] discusses many interesting applications of this problem.

We study the queueing problem in the lines of Chapter 3. That is, we attempt to characterize the weak group strategyproof and continuous mechanisms that implement decision efficiency (in this case, queue-efficiency). We achieve complete characterization for two players, but fail to do so for the general case. For the general case, we specify a necessary condition and a sufficient one. Continuity turns out to be a fairness restriction. This is because, in this setting, restriction of continuity is equivalent to that of *Equal Treatment of Equals*; which is a famous fairness axiom.

Chapter 2

Coalition Formation: Vickrey Auction[1]

2.1 Introduction

This chapter discusses coalition formation at a Vickrey auction. Such coalitions are popularly known as bidding rings. Existence of such a ring is a prevalent phenomenon and its incidence in second price auctions (Vickrey auctions with a single good) is well documented. Robinson [40] shows that collusion among bidders is easier to sustain in a second price auction than in a first price auction. Milgrom [31] also shows that repeated second price auction are more vulnerable to bidder collusion than repeated first price auctions. More recently, Marshall and Marx [29] compare the profitability and viability of collusion in first and second price auctions. They discuss two types of arrangements within a coalition: one where the coalition rec-

[1] Co-authored with Kalyan Chatterjee and Manipushpak Mitra

ommends bids to its members (the “Bid Coordination Mechanism” (BCM)) and another, where the coalition has power to control their bids (the “Bid Submission Mechanism” (BSM)). They show that efficient collusion can be sustained in the first price auction only under the BSM arrangement while both arrangements are feasible in the second price setting.

Given the difficulty of sustaining bidding rings in a first price auction in comparison to the second price auction, we revisit the issue of bidding rings only in a second price auction. In particular we model the formation of a bidding ring in second price auction. We view bidding rings as an outcome of a sequential bargaining game with irreversible commitments. This game is layered by all the bidders and it is assumed that bidders know each other’s valuation. Before participating in the auction, the bidders play a Rubinstein [41] bargaining game whose equilibrium outcome constitutes of a coalition structure and a sharing rule among members within each member coalition of the coalition structure. Throughout this analysis we assume a non-strategic seller who simply sets a minimum price for the indivisible object (or identical indivisible objects) to be sold. We also rule out resale possibilities across bidders.

Marshall and Marx [29] (and Graham and Marshall [15]) assume that bidding rings can employ taxes to ensure truthful revelation of agent valuations amongst the members of the ring; and then decide on the optimal bid recommendation (or submission) for each agent. To avoid ascribing such coercive power to the coalition, we model ring formation as a dynamic game so that any ring formed in equilibrium is sustained only by the incentive considerations, that is, all members agree to cooperate because there is no credible benefit from any type of deviation.

McAfee and McMillan [30] show that if all the bidders form a cartel and their private values are drawn from the same distribution, then they can conduct a simple pre-auction and efficiently designate a winner and divide the spoils by making appropriate incentive compatible side payments. Graham and Marshall [15] address similar issues and show that any subset of bidders can achieve efficient collusion if an external banker is available to achieve ex-post budget balance (see also Graham, Marshall and Richard (1990)). Mailath and Zemsky [27] show how the same can be done in a second price auction, without any external agent, even when the agents are not ex-ante identical. Hendricks, Porter and Tan [20] derive the necessary and sufficient condition for an efficient, incentive compatible cartel in a common value setting.

These papers analyze collusion in an ex-ante sense where, at the beginning of the ring formation process, the bidders are yet to know the valuations of their colluding partners. Our analysis adopts an ex-post approach in this regard where, right from the onset, bidders know the exact valuations of their potential collusion partners. This, however, does not amount to assuming complete information because the seller does not know the valuations of the bidders. Such information structure has been widely used with respect to all-pay auctions (Baye, Kovenock, De Vries [3]), lobbying (Baye, Kovenock, De Vries [2]), and can be observed in real life at (repeated) government auctions of timber contracts and railway sleepers. In any case, our work serves as a complete information benchmark.

In all the papers mentioned above, only the single unit auction is analyzed, that too, with the size of coalition chosen exogenously. We discuss the issue of collusion in single as well as multi-unit second price auction and focus on endogenous determination of the equilibrium coalition structure. These coalition structures are relevant because they capture the externality between the bidders, especially in the multi-unit auctions. This sort of externality has not received much attention in the literature.[2]

We look for stationary subgame perfect equilibria in the pre-auction bargaining game. In the single goods case, we provide the necessary and sufficient conditions for formation of any bidding ring when agents are sufficiently patient and the highest valuation agent proposes. In the multiple goods case, we specify the sufficient conditions for formation of an interesting class of coalition structures where (a) exactly one winner (any one agent out of those who would win a good in the non-cooperative play) colludes with all the losers (the agents who would not win any of the goods in non-cooperative play) and, (b) depending on the protocol, the remaining winners either stay alone or collude in pairs.

We present the results for the single good and the multiple goods models separately, because of the qualitative difference that arises due externality in the latter case. The bargaining game for the single goods case follows Chatterjee, Dutta, Ray and Sengupta [6]. The bargaining over the externality present in the multiple good case, is modelled as in the Ray and Vohra [39]

[2]Some papers which incorporate this issue are Jehiel and Moldovanu [22], Jehiel, Moldovanu and Stacchetti [23], Caillaud and Jehiel [5], Fullerton and McAfee [14]. The paper relevant to our work is Caillaud and Jehiel [5] who show that externality across bidders tends to make collusion harder.

paper. However, we study the two cases in the same general unified structure. This structure has a fundamental difference with respect to Chatterjee, Dutta, Ray and Sengupta [6], and Ray and Vohra [39]. They assume that members of a coalition can write binding contracts among themselves but not across coalitions. In our case, any cooperation is sustained by the payoff considerations only, without the need for any external institution.

Section 2.2 states the general unified structure under which we analyze both the single and multiple good case. Sections 2.3.1 and 2.3.2 state the results for the single and the multiple goods case respectively. Section 2.4 states the conclusion.

2.2 Model

We consider the model of multiple identical indivisible goods auction where $N = \{1, \ldots, n\}$ is the set of agents and k is the number of objects. Each agent has need for only one object. Let $V = (V_1, V_2, \ldots, V_n)$ represent the valuation vector and s be the reservation price of the seller. We assume that V is common knowledge among the bidders but the seller has no information about it. Let $v_l = \max\{V_l - s, 0\}$, $\forall\, l \in N$. Arrange the v values in a descending order, and rename the agents so that the first ranked agent in such an order, is now called 1, the second is called 2 and so on. Further, assume that the players have non-identical valuations.[3] Thus we now have a vector $v = (v_1, \ldots, v_n)$ such that $v_1 > v_2 > \ldots > v_n > v_{n+1} = 0$. Let $K = \{1, \ldots, k\}$ denote the

[3]Allowing identical valuations would only lead to multiplicity of equilibria without adding to the qualitative analysis.

set of agents who win a good at the non-cooperative play; henceforth, called as *winners*. Similarly, let $L = \{k+1, \ldots, n\}$ denote the set of agents who do not win a good at the non-cooperative play; henceforth, called as *losers*. Hence, $N = K \cup L$ with $K \cap L = \emptyset$.

For each non-empty $S \subseteq N$, define the set of all possible partitions on S as $\Pi(S)$. Thus, each $\pi_S \in \Pi(S)$ is a collection of mutually exclusive and exhaustive subsets of S. Pick any $\pi_S \in \Pi(S)$ and define $L(\pi_S) := \{T \in \pi_S \mid T \cap K = \emptyset\}$ and $\bar{L}(\pi_S) := \cup_{T \in L(\pi_S)} T$. Therefore $\bar{L}(\pi_S)$ denotes the union of those members of π_S that do not contain any winner. Define the following *partition function* which assigns a worth to each $S \in \pi_N$, $\forall\, \pi_N \in \Pi(N)$,

$$\bar{w}(S; \pi_N) = \sum_{j \in S \cap K} \left\{ v_j - \max_{l \in \bar{L}(\pi_N)} v_l \right\}. \tag{2.1}$$

The partition function specifies the maximum payoff that any member coalition S of a partition π_N of agent set N, can achieve.

The particular functional form of the partition function in this setting, follows from the desire of the winners to manipulate the price that they end up paying for the good. That is, winners want to collude with losers to persuade them to bid lower than their true valuations; and thereby, ensure procurement of the good at a lower price, in the auction. The extra payoff that accrues to the winner out of this enterprise, is used to compensate the losers suitably. Hence, any worthwhile collusive venture must involve at least one winner, while the losers that are not included in any such venture, cannot benefit by forming coalitions amongst themselves, and so, play non-cooperatively. So, for any $\pi_N \in \Pi(N)$, members of $\bar{L}(\pi_N)$ bid their true valuations. Therefore, the going price at the auction, when a coalition structure π_N has formed, turns out to be $\max_{l \in \bar{L}(\pi_N)} v_l$. The coalitional worth of each coalition in π_N, then, is simply the sum of the payoffs of the winners in that coalition. The following example illuminates on this.

Example 2.2.1. Consider $N = \{1, 2, 3\}$, $k = 2$. Therefore, $K = \{1, 2\}$ and $L = \{3\}$. Then,

$$
\begin{array}{ll}
\bar{w}(\{1\}; \{1\}, \{2, 3\}) = v_1 & \bar{w}(\{2, 3\}; \{1\}, \{2, 3\}) = v_2 \\
\bar{w}(\{1, 2\}; \{1, 2\}, \{3\}) = v_1 + v_2 - 2v_3 & \bar{w}(\{3\}; \{1, 2\}, \{3\}) = 0 \\
\bar{w}(\{1, 3\}; \{1, 3\}, \{2\}) = v_1 & \bar{w}(\{2\}; \{1, 3\}, \{2\}) = v_2 \\
\bar{w}(\{1, 2, 3\}; \{1, 2, 3\}) = v_1 + v_2 &
\end{array}
$$

while the non-cooperatively play payoffs are given as $\bar{w}(\{1\}; \{1\}, \{2\}, \{3\}) = v_1 - v_3$, $\bar{w}(\{2\}; \{1\}, \{2\}, \{3\}) = v_2 - v_3$ and $\bar{w}(\{3\}; \{1\}, \{2\}, \{3\}) = 0$.

With respect to coalition structures π_N such that $\bar{L}(\pi_N) \neq \emptyset$, we assume that any coalition $S \subseteq N$ formed at the bargaining game, bids at the auction according to the following rule;

R: For any $S \subseteq N$,

$$
b_i = \begin{cases} v_i & \text{if } i \in \underset{i' \in S}{\operatorname{argmax}}\ v_{i'} \\ 0 & \text{otherwise} \end{cases}
$$

Observe that this bidding rule **R** guarantees coalition S, the highest possible worth, no matter what *substructure* $\pi \in \Pi(N \setminus S)$ the remaining agents organize themselves into. We could have endogenised the bid arrangements within a coalition, by requiring the proposals in our bargaining game to specify not only the division of the realized coalitional worth (contingent upon the final coalition structure) but also the bids recommended to each member of the coalition. However, bidding rule **R** would always turn out to be a weakly dominant strategy. Our assumption therefore, simply rules out multiple SSPE profile of strategies where the outcome coalition structure and the agent specific payoffs are same, but the bid arrangements are different. This is in consonance with our primary objective of focussing on the coalition formation aspect of collusion at (second price) auctions.

In Ray and Vohra [39], the assumption of binding contracts served two purposes. First, to make irreversible the commitment that an agent makes by agreeing to be the member of a coalition. Second, to ensure that no member of any coalition plays differently at the underlying game than the play agreed upon by the coalition (that is, cheating is ruled out). In our work, the bidding arrangement takes care of the second purpose by ensuring that any deviation from the agreed upon strategy of a coalition, gives the same payoff as that resulting from any non-cooperative strategy. We assume, here, that the payoffs to the loser members of any collusive venture are paid by the winner members, *before* the auction begins. This eliminates the possibility of the winner members of reneging their commitments, post-auction, after the gains from cooperation have accrued to the winners.

For all $i \in K$ and $S \subseteq [L \cup \{i\}]$, define $w^i(S)$ to be the maximum worth that coalition S can attain at a single good second price auction with the agent set $L \cup \{i\}$. It is easy to see that

$$w^i(S) = \begin{cases} v_i - \max\limits_{l \in L \setminus S} v_l & \text{if } i \in S \\ 0 & \text{otherwise} \end{cases}$$

Using this, we state an obvious property of the partition function given by (2.1) is the following;

Proposition 2.2.1. For all $i \in K$ and non-empty $S \subseteq L$,

$$\bar{w}(S \cup \{i\}; S \cup \{i\}, \pi_{L\setminus S}, \pi_{K\setminus\{i\}}) = w^i(S \cup \{i\}) > 0$$

for all $\pi_{L\setminus S} \in \Pi(L \setminus S)$ and all $\pi_{K\setminus\{i\}} \in \Pi(K \setminus \{i\})$.

Proof: It follows from the definition that $\forall\, \pi_{L\setminus S} \in \Pi(L \setminus S)$ and $\forall\, \pi_{K\setminus\{i\}} \in \Pi(K \setminus \{i\})$, $\bar{w}(S \cup \{i\}; S \cup \{i\}, \pi_{L\setminus S}, \pi_{K\setminus\{i\}}) = v_i - \max_{l \in L\setminus S} v_l$. □

The pre-auction ring formation is captured by a Rubinstein [41] bargaining game $G \equiv (N, \bar{w}, p, \delta)$. The function $p : 2^N \mapsto N$ is the protocol function which assigns to each set of *active* agents (agents who are yet to form coalitions), a proposer from that set, who carries the game forward. Therefore, $p(T) \in T, \forall\, T \subseteq N$. $\delta \in (0, 1)$ is the common discount factor, that is, any agent receiving payoff x in period t gets a utility $\delta^{t-1}x$. A *stage* in G is given by a *substructure* (partition defined on a strict subset of N) constituting of the coalitions who have formed and left the game. The set of all possible stages is $\mathcal{P} := \cup_{S\subset N}\Pi(S)$, which is the set of all possible partitions of all possible strict subsets of N. Define $R(\pi) := N \setminus \{\cup_{T\in\pi}T\}$, $\forall\, \pi \in \mathcal{P}$. Therefore, $R(\pi)$ is the set of remaining (active) agents after coalitions in substructure π have formed and left the game.

We assume that agents follow stationary Markovian strategies which depend on a small set of state variables in a way that is insensitive to past history. In particular, they depend on current set of active agents, coalition (sub)structure that has already formed and, in case of response, the on-going proposal.

At any stage $\pi \in \mathcal{P}$, an agent j must choose (i) a pair (T, z) with $T \subseteq R(\pi)$, $j \in T$ and $z \in \Re_+^{|T|}$; and (ii) an $a_j(\pi) \in \Re_+$. The choice of (i) signifies the proposal decision of j at stage π; where j proposes some subset T of the current set of active agents $R(\pi)$ containing j and offers the members of T a payoff z.[4] The choice (ii) signifies the response decision of j at stage π; where j accepts any proposal (T', z') with $j \in T'$ only if $z'_j \geq a_j(\pi)$. At any stage, if a proposal gets rejected then all active agents at that stage incur a utility loss due to delay of *one* period which is captured by the common discount factor $\delta \in (0, 1)$. We make the following assumptions.

Assumption 2.2.1. Any group of agents agree to cooperate only if the payoff from cooperation exceeds that by staying alone.

Assumption 2.2.2. All agents get 0 utility in case of perpetual disagreement.

At any stage in G, we call a proposal *acceptable* if all the members of the coalition proposed, are offered a payoff no less than their respective acceptance thresholds of that stage. It, then, follows that the coalition mentioned in any acceptable proposal will form and leave the game. If an *unacceptable* proposal is made, it will be rejected by any one of the members at whom the proposal is directed. As mentioned before, this will cause a period of delay to all the agents. In the next period, the *rejector will propose.*

Define $S(L) := \{S_k(m)\}_{m=k+1}^{m=n}$ where $S_k(k+1) := \{k+1\}$ and $S_k(r) := \{k+1, \ldots, r\}$ for all integers $r = k+2, \ldots n$. We call the partition of N

[4] In equilibrium, this payoff division z must exhaust the coalitional worth of T, no matter what the finally realized coalition structure is.

having all singleton members, $\underline{\pi}$.

2.3 Results

2.3.1 Single Good

There is now single winner, and so, $K = \{1\}$ and $L = N \setminus \{1\}$. Note that the worth of partition function, now, reduces to $w^1(.)$, that is,

$$\bar{w}(S; S, \pi_{N\setminus S}) = w^1(S) = \begin{cases} v_1 - \max\limits_{l \in N\setminus S} v_l & \text{if } 1 \in S \\ 0 & \text{otherwise} \end{cases}$$

$\forall\, S \subseteq N, \forall\, \pi_{N\setminus S} \in \Pi(N \setminus S)$. This occurs because the worth of a coalition S no longer depends on the coalition structure.

Proposition 2.3.1. For any $G = (N, w^1, p, \delta)$, if the SSPE outcome π^* is such that $\pi^* \neq \underline{\pi}$, then

(i) $\exists\, S_1(m) \in \mathcal{S}(L)$ such that $\{S_1(m) \cup \{1\}\} \in \pi^*$, and

(ii) $\forall\, l \in L \setminus S_k(m), \{l\} \in \pi^*_N$

Proof: Suppose that the equilibrium coalition structure π^* is such that there exists $X \in \pi^*$ with the property that $1 \in X$ and $X \setminus \{1\} \notin \mathcal{S}(L)$. Then there must exist an $m' > 2$ such that $m' \in X$ with $X \setminus \{1\} \subset S_1(m')$. Therefore, $w^1(X) = w^1(X \setminus \{m'\})$, that is, the marginal contribution of agent m' to the coalition X is zero. But, given Assumption 2.2.1, $X \in \pi^*$ implies that in equilibrium m' gets a positive payoff. This is clearly suboptimal and hence, contradiction. So any such X will never be formed in equilibrium. Thus (i) follows. Finally, (ii) follows from Assumption 2.2.1 that all other losers form singleton coalitions. □

It is important to observe that at any stage of the game $G = (N, w^1, p, \delta)$, *no active agent makes an unacceptable proposal.* The reason is provided in the next two paragraphs.

The $w^1(.)$ function implies that a coalition S generates positive payoff only if $S \ni 1$. Therefore at any stage where 1 is not active then, from Assumption 2.2.1, all the active agents stay alone. Consider any stage where 1 is active and the proposal power is with some $l \in L$. Agent l will never make an unacceptable proposal because any such proposal, given stationarity, does not change the stage of the game. It simply passes the power of proposal to some other active agent (because rejector proposes in our bargaining game). This rejector can *either* make an acceptable proposal (which must contain 1 to have a positive worth) and leave the game; *or* propose unacceptably, in which case, the stage of the game remains unchanged even after two periods of delay. The latter possibility is undesirable to l as it causes delay without changing the stage of the game. The former possibility gives l zero payoff *if* l is not one of the members to whom acceptable proposal is made. Even if l is a member of the said coalition, he always could have proposed the same thereby saving the cost of delay.

Now, consider the stage where 1 has the proposal power. Suppose agent 1 can get a payoff of x by making an acceptable proposal. As before, 1 observes that given stationarity, an unacceptable proposal will not change the stage of the game and will only pass the proposal power to some $l \in L$ in the present stage. By previous argument l will never propose unacceptably. Moreover l

will never leave the game alone (as it will give 0 payoff). So l must propose a coalition containing 1 acceptably. This can be done by offering at least δx to 1. Therefore, we see that an unacceptable proposal by 1 gives $\delta^2 x < x$. Hence making acceptable proposal strictly dominates any unacceptable proposal.

Therefore, at any SSPE outcome, for any $G = (N, w^1, p, \delta)$, there is no delay.

Define $AV(i, l) = \frac{v_i - v_l}{l-k}$ where $i = 1, 2, \ldots, k$ and $l = k+1, k+2, \ldots, n$. Observe that for single good game $G = (N, w^1, p, \delta)$, $w^1(\{1\} \cup S_1(m)) = mAV(1, m+1)$ for all $m = 2, \ldots, n$. We refer to a coalition $\{1, 2, \ldots, r-1, r\}$ as an r-ring, for every $r \in N$. For any $T \subseteq N$, let $2^T := \{T' : T' \subseteq T\}$.

Proposition 2.3.2. For any $G = (N, w^1, p, \delta)$ with $p(N) = 1$, $\exists\, \delta' \in (0, 1)$ such that $\forall\, \delta \in (\delta', 1)$ the SSPE outcome is an r-ring without any delay, if and only if

1. $AV(1, r+1) \geq AV(1, t+1)$, $\forall\, t \in \{1, 2, \ldots, r-1\}$ and

2. $AV(1, r+1) > AV(1, t+1)$, $\forall\, t \in \{r+1, r+2, \ldots, n\}$.

Proof:

Only If: Consider a stage π in the game such that $1 \in R(\pi)$. Since there can be no delay in equilibrium (since no active agent at any stage proposes unacceptably), the equilibrium acceptance threshold of any $i \in R(\pi)$ must be the one period discounted payoff that i can generate by making the equilibrium proposal, at the stage π itself. Therefore, for a given δ; from Proposition 2.3.1 it follows that the equilibrium acceptance thresholds $\{a_i^\delta(\pi)\}_{i \in R(\pi)}$ must satisfy the following equality[5]

$$\frac{a_i^\delta(\pi)}{\delta} = \max_{T\in[2^{R(\pi)}\cap S(L)],i\in T}\left\{w(\{1\}\cup T) - \sum_{j\in[\{1\}\cup T]\setminus\{i\}} a_j^\delta(\pi)\right\}$$

$\forall\, i \neq 1$ and,

$$\frac{a_1^\delta(\pi)}{\delta} = \max_{T\in[2^{R(\pi)}\cap S(L)]}\left\{w(\{1\}\cup T) - \sum_{j\in T} a_j^\delta(\pi)\right\}$$

From Chatterjee, Dutta, Ray and Sengupta [6], it follows that $\forall\, \pi \in \mathcal{P}$ with $1 \in R(\pi)$; the acceptance thresholds are obtained by the following recursion;

(i) $a_1^\delta(\pi) = \max_{T\in[2^{R(\pi)}\cap S(L)]} \frac{\delta w^1(\{1\}\cup T)}{1+\delta|T|}$ and $a_i^\delta(\pi) = a_1^\delta(\pi)$ for all $i \in \bar{H}_1^\delta(\pi)$ where $\bar{H}_1^\delta(\pi) := \left[\cup_{T\in H_1^\delta(\pi)} T\right]$ with $H_1^\delta(\pi) := \operatorname*{argmax}_{T\in[2^{R(\pi)}\cap S(L)]} \frac{\delta w^1(\{1\}\cup T)}{1+\delta|T|}$.

(ii) Suppose $(\bar{H}_1^\delta, \bar{H}_2^\delta, \ldots, \bar{H}_q^\delta)$ is well defined. If $R(\pi)\setminus[\{1\}\cup\bar{H}_q^\delta(\pi)] \neq \emptyset$, then define

$$H_{q+1}^\delta(\pi) := \operatorname*{argmax}_{T\in[2^R(\pi)\cap S(L)],\bar{H}_q^\delta(\pi)\subset T} \frac{\delta\left\{w^1(\{1\}\cup T) - \sum_{j\in\bar{H}_q^\delta(\pi)} a_j^\delta(\pi) - a_1^\delta(\pi)\right\}}{1+\delta(|T| - |\bar{H}_q^\delta(\pi)| - 1)}$$

As before, $\bar{H}_{q+1}^\delta(\pi) := \left[\cup_{T\in H_{q+1}^\delta(\pi)} T\right]$. For all $i \in \bar{H}_{q+1}^\delta(\pi)$, $a_i^\delta(\pi)$ is the maximized value in the definition of $H_{q+1}^\delta(\pi)$.

Note that $\bar{H}_q^\delta(\pi) \subset H_q^\delta(\pi)$, $\forall\, q$ in the recursion above. This follows from the particular structure of the problem reflected in Proposition 2.3.1. The proposal decision at any stage π with $1 \in R(\pi)$ is as follows. Each $i \in R(\pi)$

[5] In case the feasible set of maximizers in the following optimization problem is empty, $a_i^\delta(\pi) := 0$.

must belong to some $\bar{H}_q^\delta(\pi)$ and therefore proposes any $M_i \cup \{1\}$ such that $M_i \in H_q^\delta(\pi)$ with $i \in M_i$.

Now, recall that any coalition not containing agent 1 has a zero worth. Therefore, at all other stages π' with $1 \notin R(\pi')$, all proposers propose singleton coalitions of themselves and $a_i^\delta(\pi') = 0$, $\forall\, i \in R(\pi')$.

Claim (a): $\exists\ \bar{\delta} \in (0,1)$ such that $\forall\ \delta \in (\bar{\delta}, 1)$, $H_1^\delta(\pi)$ contains *only* the largest member of

$$\underset{T \in [2^{R(\pi)} \cap S(L)]}{\text{argmax}} AV(1, |T| + 2)$$

for all $\pi \in \mathcal{P}$ with $1 \in R(\pi)$.

Proof: Define $H_1(\pi) := \underset{T \in [2^{R(\pi)} \cap S(L)]}{\text{argmax}} AV(1, |T|+2)$. Note that $\lim\limits_{\delta \to 1} \frac{\delta w^1(\{1\} \cup T)}{1+\delta|T|} = AV(1, |T| + 2)$. Therefore, it trivially follows that for values of δ sufficiently close to 1, $H_1^\delta(\pi) \subseteq H_1(\pi)$. Now suppose $\exists\ T, T' \in H_1(\pi)$. Therefore (A) $\frac{w^1(\{1\} \cup T)}{1+\delta|T|} - \frac{w^1(\{1\} \cup T')}{1+\delta|T'|} = 0$ with δ value fixed at 1. Given the structure of the game, it must be that $|T| \neq |T'|$; say $|T| > |T'|$. From (A), $|T| > |T'|$ implies that $w^1(\{1\} \cup T) > w^1(\{1\} \cup T')$ (because we use non-identical valuations). Also, for a “slight” fall in δ value; in the left hand side of (A), the denominator of the first term decreases by more than the second term (since $|T| > |T'|$). Hence, the ‘equals to’ sign in (A), changes to ‘greater than’ for δ values sufficiently close to 1. Therefore, proposal choice of the largest coalition in $H_1(\pi)$ dominates that of the other members of $H_1(\pi)$, for δ sufficiently close to 1. Hence, proved.

Since $p(N) = 1$, an r-ring is formed only if agent 1 proposes $\{1, 2, \ldots, r\}$ acceptably on the SSPE path. This will happen only if $S_1(r) \cup \{1\}$ is the largest coalition in $H_1^\delta(\emptyset)$, that is, $S_1(r) \cup \{1\}$ is the largest average worth maximizing coalition. This implies that $AV(1, r+1) \geq AV(1, t+1), \forall\, t < r$ and $AV(1, r+1) > AV(1, t+1), \forall\, t > r$. These two conditions imply results (1) and (2) respectively. $\square$

If: Define the following strategy $\mathbf{\Sigma}$ in game G:

- At any stage π with $1 \notin R(\pi)$, all proposers choose to stay alone, and set an acceptance threshold of 0.

- Recall that for any stage π with $1 \in R(\pi)$, $H_1(\pi) := \underset{T \in [2^{R(\pi)} \cap S(L)]}{\text{argmax}} AV(1, |T|+2)$. For all such π, let $\bar{H}_1(\pi)$ be the largest coalition in $H_1(\pi)$. Then, at any stage π with $1 \in R(\pi)$, all $i \in [\bar{H}_1(\pi) \cup \{1\}]$ propose $[\bar{H}_1(\pi) \cup \{1\}]$ and set their acceptance thresholds to be $\frac{\delta w^1(\bar{H}_1(\pi) \cup \{1\})}{1+\delta|\bar{H}_1(\pi)|}$. If the sequence $(\bar{H}_1, \bar{H}_2, \ldots, \bar{H}_q)$ is well defined and $R(\pi) \setminus [\bar{H}_q(\pi) \cup \{1\}] \neq \emptyset$; then

$$H_{q+1}(\pi) := \underset{T \in [2^{R(\pi)} \cap S(L)], \bar{H}_q(\pi) \subset T}{\text{argmax}} \frac{w^1(\{1\} \cup T) - w^1(\{1\} \cup \bar{H}_q(\pi))}{|T| - |\bar{H}_q(\pi)|}$$

  with $\bar{H}_{q+1}(\pi)$ is defined as before to be the largest coalition in $H_{q+1}(\pi)$. Then all $j \in [\bar{H}_{q+1}(\pi) \cup \{1\}]$ propose $[\bar{H}_{q+1}(\pi) \cup \{1\}]$ and set their acceptance thresholds to be

$$\frac{\delta w^1(\bar{H}_{q+1}(\pi) \cup \{1\}) - \delta w^1(\bar{H}_q(\pi) \cup \{1\})}{1 + \delta(|\bar{H}_{q+1}(\pi)| - |\bar{H}_q(\pi)| - 1)}$$

It can easily be seen that the recursion in strategy $\mathbf{\Sigma}$ is simply the limit version of the recursion given by (i) and (ii) in the proof of necessity. Then, arguing as in Claim **(a)**, for each round q of this recursion; we see that for δ values very close to 1, $\mathbf{\Sigma}$ is SSPE. So we can find a $\delta' \in (\bar{\delta}, 1)$ such that $\forall\, \delta \in (\delta', 1)$, $\mathbf{\Sigma}$ is SSPE. Then from conditions (1) and (2) in the statement of the theorem it follows that; when $p(N) = 1$, strategy $\mathbf{\Sigma}$ will lead to formation of an r-ring. Thus, the sufficiency is established. □

In Proposition 2.3.2 we assumed a specific protocol function where $p(N) = 1$. What happens if $p(N) \neq 1$ is explained informally using the following example.

Example 2.3.1. Suppose $N = \{1, 2, 3\}$ and $K = \{1\}$ where $v \equiv (v_1 = 70, v_2 = 65, v_3 = 20)$. Note that $AV(1, 2) < AV(1, 3) > AV(1, 4)$. Invoking the strategy $\mathbf{\Sigma}$ in the sufficiency proof of the Proposition 2.3.2, at the stage $\emptyset$ (that is, at the beginning of the game), we see that $\bar{H}_1(\emptyset) = \bar{H}_2(\emptyset) = \{2\}$ and $\bar{H}_3(\emptyset) = \{2, 3\}$. It can be shown that $\forall \delta \in \left(\frac{2}{3}, 1\right)$, agents 1 and 2 propose $\{1, 2\}$, while agent 3 proposes $\{1, 2, 3\}$ at stage $\emptyset$. Therefore,

- if $p(N) \in \{1, 2\}$ then the outcome coalition structure is $\{\{1, 2\}, \{3\}\}$, that is, the 2-ring forms.
- if $p(N) = 3$ then the outcome coalition structure is $\{\{1, 2, 3\}\}$, that is, the 3-ring forms.

2.3.2 Multiple Goods

Consider the subgames with the set of active agents as T such that $L \subseteq T$. For all such subgames, the substructure formed by the departed agents (who have formed coalitions and left the game) does not affect the worth of any coalitions that remaining agents may form in future. That is, at such a stage with active player set T with $L \subseteq T$; $\bar{w}(S; \pi_{N \setminus T}, S, \hat{\pi}_{T \setminus S}) = \bar{w}(S; \pi'_{N \setminus T}, S, \hat{\pi}_{T \setminus S})$, $\forall\, \pi_{N \setminus T}, \pi'_{N \setminus T} \in \Pi(N \setminus T)$, $\forall\, S \subseteq T$, $\forall\, \hat{\pi}_{T \setminus S} \in \Pi(T \setminus S)$.

At these subgames, we refer to the *stage* in the game by the set of *active* agents, instead of the substructure consisting of coalitions who have (formed and) left the game.

At any such stage T (with $L \subseteq T$), define $C_i^\delta(T)$ to be the set of best acceptable proposals (only the coalitions) that agent i can make at that stage. Also define $T^k := \{k\} \cup L$, $\forall\, k \in K$.

Proposition 2.3.3. For any $i, j \in K$ such that $v_i > v_j$, $\exists\, \delta' \in (0,1)$ such that $\forall\, \delta \in (\delta', 1)$; if $S_k(m) \cup \{i\} \in C_i^\delta(T^i)$ then $\exists\, m' \geq m$ such that $S_k(m') \cup \{j\} \in C_j^\delta(T^j)$.

Proof: Note that at any stage T^i, *the subgame becomes equivalent to a single good auction where the only winner is agent* i. This is because the worth of any subset of T^i, irrespective of the substructure formed amongst the agents who have departed from the game, is given by the $w^i(.)$ function. Hence, we can invoke the Proposition 2.3.1(i) and infer that $\forall\, i \in K$, if $X \in C_i^\delta(T^i)$ and $X \neq \{i\}$ then $X \setminus \{i\} \in S(L)$, $\forall\, \delta \in (0,1)$.

Now, from the *continuity* of the objective functions in the maximization programs of (i) and (ii) in the necessity proof of Proposition 2.3.2, it follows that for δ sufficiently close to 1, any agent $i \in K$ proposes acceptably the average worth maximizing coalition (containing i) at stage T^i. So, for δ sufficiently close to 1, $S_k(m) \cup \{i\} \in C_i^\delta(T^i)$ implies that $S_k(m) \cup \{i\}$ is the average worth maximizing coalition among all subsets of T^i. Therefore, $\frac{v_i - v_{m+1}}{|S_k(m)|+1} \geq \frac{v_i - v_{m-l+1}}{|S_k(m-l)|+1}$ for all $l = 0, 1, \ldots, m-k-1$. It is easy to check that this in turn implies that $\frac{v_j - v_{m+1}}{|S_k(m)|+1} \geq \frac{v_j - v_{m-l+1}}{|S_k(m-l)|+1}$ for all $l = 0, 1, \ldots, m-k-1$ when $v_j < v_i$. Now, for suitably high δ, any $j \in \{i+1, \ldots, k\}$ must also choose the average worth maximizing coalition containing j among the subsets of T^j. Hence, it follows that; for a sufficiently high δ (that is, $\exists$ some $\delta' \in (0,1)$ such that $\forall\, \delta \in (\delta', 1)$), there exists an $m' \geq m$ with $S_k(m') \cup \{j\} \in C_j^{\delta}(T^j)$. $\square$

Proposition 2.3.3 states that when δ is sufficiently high; if winners i and j separately find themselves at a stage where the remaining set of agents are T^i and T^j respectively and if i picks a set $S_k(m) \cup \{i\}$ as a best acceptable proposal then there exists an $m' \in \{m, \ldots, n\}$ such that $S_k(m') \cup \{j\}$ is a best acceptable proposal for j, whenever $v_i > v_j$.

Remark 2.3.2. It also follows from Proposition 2.3.3 that at the stage T^k (for any $k \in K$); the game $G = (N, \bar{w}, p, \delta)$ reduces to a single good/single winner bargaining game $G^k = (T^k, w^k, p^k, \delta)$ where $p^k(.)$ is the restriction of the original protocol function $p(.)$ to the set 2^{T^k}. As mentioned earlier, in the bargaining game with single winner, at any stage, no active agent makes an unacceptable proposal. Therefore, $C_i^{\delta}(T^k)$ is the set of coalitions that agent i proposes in *equilibrium* at stage T^k, in game G; $\forall\, i \in T^k$, $\forall\, k \in K$.

Define $C_i^*(T^k)$ to be the set of coalitions that any $i \in T^k$ proposes in equilibrium at any stage $T^k, k \in K$; *as* δ *goes to* 1 *in limit*. From the arguments in proof of Proposition 2.3.3, we see that at any single winner stage T^k, in limit, the winner k chooses the average worth maximizing coalition containing itself. That is, $C_k^*(T^k) = \underset{S \in S(L)}{\text{argmax}} \frac{w^k(S \cup \{k\})}{1+|S|}, \forall\, k \in K$.

The following proposition states that if the winner 1 finds it optimal to collude with all the losers at stage T^1 in limit (that is, $T^1 \in C_1^*(T^1)$); then, irrespective of the value of δ, the *optimal* proposal of all winners i other than 1, at stage T^i, *can only be* T^i itself (that is, $C_i^{\delta}(T^i) = \{T^i\}$).

Proposition 2.3.4. If $S_k(n) \in \underset{S \in S(L)}{\operatorname{argmax}} \frac{w^1(S\cup\{1\})}{|S|+1}$ then $S_k(n) = \underset{S \in S(L)}{\operatorname{argmax}} \frac{w^i(S\cup\{i\})}{\delta|S|+1}$, for all $\delta \in (0,1)$ and all $i \in K$.

Proof: Since $[S_k(n) \cup \{1\}] = T^1 \in C_1^*(T^1)$ and $v_i < v_1$, $\forall\, i \in K \setminus \{1\}$; as in the previous proposition we can say that $\frac{w^i(S_k(n)\cup\{i\})}{|S_k(n)|+1} \geq \frac{w^i(S_k(m)\cup\{i\})}{|S_k(m)|+1}$ for all $S_k(m) \in S(L)$ and for all $i \in K$. For any $S_k(m) \in S(L) \setminus \{S_k(n)\}$, $\forall\, i \in K$, define the function $d_i(S_k(m), \delta) = \frac{w^i(S_k(n)\cup\{i\})}{\delta|S_k(n)|+1} - \frac{w^i(S_k(m)\cup\{i\})}{\delta|S_k(m)|+1}$. By applying proof by contradiction it is easy to prove that $d_i(S_k(m), \delta) > 0$ for all $\delta \in (0,1)$; $\forall\, m \neq n$ and $\forall\, i \in K$. Hence it follows that $S_k(n) = \underset{S \in S(L)}{\operatorname{argmax}} \frac{w^i(S\cup\{i\})}{\delta|S|+1}$, for all $i \in K$. This proves our result. □

We, now, design a recursion that will be used in the theorem to follow. For this we call the agent with the highest (the lowest) valuation in any set $T \subseteq N$ as $\bar{m}^T$ (as $\underline{m}^T$).[6] This recursion is used to optimize the proposal decision of any loser at any stage $T \cup L$ such that $T \subseteq K$. The recursion, essentially, generates the final coalition structure (for a given protocol function) subject to the choice of a set of winners (one winner from each possible stage $T' \cup L$ where $T' \subseteq T \subseteq K$). This choice is done under the assumption that at each such stage $T' \cup L$; if any winner $j \in T'$ gets to propose, he must propose $\{\bar{m}^{T'}\}$ if $j = \bar{m}^{T'}$ and $\{\bar{m}^{T'}, j\}$ otherwise.

Recursion (*): For any $T \subseteq K$, define $\mathbf{b}(T; p(.)) \equiv \{b(T'; p(.))\}_{T' \subseteq T}$ to be a sequence of members of T such that (i) $b(T'; p(.)) = \underline{m}^{T'}$ if $|T'| = 2$ and (ii) $b(T'; p(.)) \in T'$ if $|T'| \neq 2$. To simplify the notations, henceforth we ignore the argument for the protocol function when writing the $b(.)$ expression. For any such $\mathbf{b}(T; p)$ define the sequence of sets $\{B_t\}_{t=1}^h$ such that

[6] Given the non-identical valuations, for any set T, the agents $\bar{m}^T$ and $\underline{m}^T$ are well defined.

- $\{B_t\}_{t=1}^{h}$ is a partition of $T \cup L$.
- $B_1 = \begin{cases} \{\bar{m}^T\} & \text{if } b(T;p) = \bar{m}^T \\ \{\bar{m}^T, b(T;p))\} & \text{otherwise} \end{cases}$
- Suppose the sequence of sets $(B_1, B_2, \ldots, B_{q-1})$ is well defined. Then, define $D_1 := T \cup L$ and $D_q := [T \cup L] \setminus [\cup_{t=1}^{q-1} B_t]$, $\forall\, q > 1$.

$$B_q = \begin{cases} D_q & \text{if } |D_q \cap K| = 1 \\ \begin{matrix} \{\underline{m}^{D_q}\} & \text{if } p(D_q) \neq \bar{m}^{D_q \cap K} \\ \{\bar{m}^{D_q \cap K}\} & \text{otherwise} \end{matrix} & \text{if } |D_q \cap K| = 2 \\ \begin{matrix} \{\bar{m}^{D_q \cap K}\} & \text{if } p(D_q) = \bar{m}^{D_q \cap K} \\ \{\bar{m}^{D_q \cap K}\} & \text{if } p(D_q) \in L \text{ and } b(D_q \cap K) = \bar{m}^{D_q \cap K} \\ \{\bar{m}^{D_q \cap K}, b(D_q \cap K; p)\} & \text{if } p(D_q) \in L \text{ and } b(D_q \cap K; p) \neq \bar{m}^{D_q \cap K} \\ \{\bar{m}^{D_q \cap K}, p(D_q)\} & \text{otherwise} \end{matrix} & \text{if } |D_q \cap K| > 2 \end{cases}$$

- $B_h = T^{j(\mathbf{b}(T;p))}$ for some $j(\mathbf{b}(T;p)) \in T$.

The last term of the recursion B_h should be a set of all the losers and any one winner j from T. The identity of this winner would depend on the choice of the sequence $\mathbf{b}(T;p)$. That is, for any choice of sequence $\mathbf{b}(T;p)$ we would get a $j^{\mathbf{b}(T;p)} \in T$ such that $B_h = T^{j(\mathbf{b}(T;p)}$. Define $\mathbf{b}^*(T;p)$ to be that sequence of winners that maximizes the (valuation of) agent $j^{\mathbf{b}(T;p)}$ and let $k^*(T;p) := b^*(T;p)$.[7]

Theorem 2.3.3. For any $G = (N, \bar{w}, p, \delta)$ if $T^1 \in C_1^*(T^1)$ then there exists $\delta' \in (0,1)$ such that for all $\delta \in (\delta', 1)$, the SSPE strategies of G are such that

[7]It may so happen that we have multiple $k^*(T;.)$ for a given protocol function. In that case, we choose any one.

$\forall\ T \subseteq K$ we have the following:

1. if $|T| = 1$ then $C_t^\delta(T \cup L) = T \cup L$ for all $t \in T \cup L$,

2. if $|T| = 2$ then $C_t^\delta(T \cup L) = \{t\}$ if $t \in T$ and any loser $t \in L$ proposes unacceptably to $\underline{m}^T$ where $\underline{m}^T = \underset{j \in T}{\operatorname{argmin}}\, v_j$, and

3. if $|T| > 2$ then

$$C_t^\delta(T \cup L) = \begin{cases} \{\overline{m}^T, t\} & \text{if } t \in T \setminus \{\overline{m}^T\} \\ \{t\} & \text{if } t = \overline{m}^T \end{cases}$$

   where $\overline{m}^T = \underset{j \in T}{\operatorname{argmax}}\, v_j$ and any $t \in L$ proposes unacceptably to $k^*(T; p)$ where $k^*(T; p)$ follows from Recursion (*).

Proof: Pick any $i \in K$ and consider the stage T^i. At this stage the only winner i and all the losers are active. From Propositions 2.3.2 and 2.3.4 it follows that if $T^1 \in C_1^*(T^1)$ then $\exists\ \delta(i) \in (0,1)$ such that $\forall\ \delta \in (\delta(i), 1)$; $C_l^\delta(T^i) = \{T^i\}$, $\forall\ l \in T^i$. Define $\delta(1) := \max\{\delta(i)\}_{i \in K}$. Therefore, $\forall\ \delta \in (\delta(1), 1)$, $C_l^\delta(T^i) = \{T^i\}$, $\forall\, l \in T^i$, $\forall\, i \in K$; and thus result (1) follows.

Consider any stage $T' = \{i, j\} \cup L$, for any $i, j \in K$. Pick any $\delta \in (\delta(1), 1)$. Then $C_t^\delta(T^j) = \{T^j\}$, $\forall\ t \in T^j$and $C_t^\delta(T^i) = \{T^i\}$, $\forall\ t \in T^i$. W.l.o.g. assume $v_i > v_j$. If i has the proposal power then the first possibility is that he chooses to stay alone, so that in the next stage with T^j agents, the coalition T^j forms (since $C_l^\delta(T^j) = \{T^j\}$ for all $l \in T^j$) and i gets a payoff of v_i. The remaining possibilities do not give agent i any more than $\frac{v_i+v_j}{1+\delta}$.[8] For all $\delta \in \left(\frac{v_j}{v_i}, 1\right)$, $v_i > \frac{v_i+v_j}{1+\delta}$ and so agent i will find it optimal

[8]Note that this payoff is the outcome of two member bargaining over $v_i + v_j$. Such a

to stay alone. Hence $\forall\, \delta \in \left(\max\left\{\delta(1), \frac{v_j}{v_i}\right\}, 1\right)$, agent i stays alone (that is, $C_i^\delta(T') = \{\{i\}\}$). Pick any $\delta \in \left(\max\left\{\delta(1), \frac{v_j}{v_i}\right\}, 1\right)$. As before, if j has the proposal power and he chooses to stay alone then he gets v_j. Otherwise, knowing that winner i can reject any proposal and get a payoff of δv_i, the best agent j can achieve, by proposing some coalition that includes i, is no more than $\frac{v_j+(1-\delta)v_i}{1+\delta}$. Also any non-singleton coalition excluding i gives j less than $\frac{v_j+(1-\delta)v_i}{1+\delta}$. There is also the possibility that agent j proposes $\{i, j\}$ acceptably to get $(1-\delta)v_i + v_j - 2v_{k+1}$. Note that if $v_j \leq 4v_{k+1}$ then $\exists\, \bar{\bar{\delta}} \in (0, 1)$ such that $\forall\, \delta \in (\bar{\bar{\delta}}, 1)$, $\frac{v_j+(1-\delta)v_i}{1+\delta} > (1-\delta)v_i + v_j - 2v_{k+1}$. If $v_j > 4v_{k+1}$ then $\exists\, \underline{\underline{\delta}} \in (0, 1)$ such that $\forall\, \delta \in (\underline{\underline{\delta}}, 1)$, $\frac{v_j+(1-\delta)v_i}{1+\delta} < (1-\delta)v_i + v_j - 2v_{k+1}$. Let $\tilde{\tilde{\delta}} := \max\left\{\delta(1), \bar{\bar{\delta}}, \underline{\underline{\delta}}, \frac{v_i - 2v_{k+1}}{v_i}, \frac{v_j}{v_i}, \frac{v_i}{v_i+v_j}\right\}$. Therefore $\forall\, \delta \in \left(\tilde{\tilde{\delta}}, 1\right)$, $C_t(T') = \{\{t\}\}$, $\forall\, t = i, j$.

We now consider the possible proposals of any loser for $\delta \in \left(\tilde{\tilde{\delta}}, 1\right)$. If any loser $l \in L$ has the proposal power, then he has two choices, (i) to make an acceptable proposal and (ii) to make an unacceptable proposal. If he chooses the former, then $C_l^\delta(T') \subset \{\{i, j\} \cup S_k(t)\}_{t=l}^n$ because, given $S_k(t)$, it is better to take both winners instead of one. For each $t \in \{l, l+1, \ldots, n\}$, the loser can attain a payoff of $\frac{(1-\delta)(v_i+v_j)-2v_{t+1}}{1+\delta(t-k-1)}$. If $\delta \in \left(1 - \frac{2v_n}{v_i+v_j}, 1\right)$ then the maximum attainable payoff is $\frac{(1-\delta)(v_i+v_j)}{1+\delta(n-k-1)}$, resulting from a proposal $T' = \{i, j\} \cup L$. If agent l makes an unacceptable proposal, it may either be directed at a winner or a loser. If it is directed at a winner, the winner (say i) would get the proposer power in the next period and given our restriction on δ, would stay

---

payoff will never materialize if both winners form a coalition and exit the game (because the losers have not colluded with any winner and so will bid their true valuations leading to a price equal to the third highest valuation).

alone and exit the game. This would drive the game to the stage T^j where, as mentioned above, the coalition T^j would form giving l a payoff $\frac{\delta v_j}{1+\delta(n-k)}$ in *the next period.* Observe that given $v_i > v_j$, the loser will never unacceptably propose to i, because he could do better by unacceptably proposing to j and getting a payoff $\frac{\delta v_i}{1+\delta(n-k)}$ in the next period. If the unacceptable proposal is directed at a loser l', the stage of the game would not change, there would be a period of delay, and in the next period the proposal power would be with loser l' who faces the same options as l with one period delay. Thus unacceptable proposal directed to a loser is suboptimal. Thus given $\delta \in \left(1 - \frac{2v_n}{v_i+v_j}, 1\right)$, loser l has two options. Either propose T' acceptably and get a payoff $\frac{(1-\delta)(v_i+v_j)}{1+\delta(n-k-1)}$ or propose unacceptably to j and get one-period discounted payoff $\frac{\delta^2 v_i}{1+\delta(n-k)}$. Define $F(\delta) := \frac{\delta^2 v_i}{1+\delta(n-k)} - \frac{(1-\delta)(v_i+v_j)}{1+\delta(n-k-1)}$. Note that $F(\delta)$ is strictly increasing and continuous in δ and $\lim_{\delta\to 1} F(\delta) = \frac{v_i}{n-k+1} > 0$. Therefore $\exists\, \bar{\delta} \in (0,1)$ such that $\forall\, \delta \in (\bar{\delta}, 1)$; $F(\delta) > 0$, and so, given the restriction on δ, making unacceptable proposal strictly dominates making acceptable proposal for the loser l. Define $\delta(i,j) := \max\left\{\tilde{\bar{\delta}}, 1 - \frac{2v_n}{v_i+v_j}, \bar{\delta}\right\}$. So $\forall\, \delta \in (\delta(i,j), 1)$, $C_t^\delta(T') = \{\{t\}\}$, $\forall\, t = i, j$ and any loser proposing at stage T' unacceptably proposes to j (the lower valuation winner). Hence, for all $\delta \in (\delta(2), 1)$ result(2) follows where $\delta(2) := \max\{\delta(i,j)\}_{i,j\in N, i\neq j}$.

Suppose that at the stage $T'' \cup L$ with $T'' \subset K$ and $2 \leq |T''| \leq m-1$ result (3) holds $\forall\, \delta \in (\delta^{m-1}, 1)$. Then consider the stage $T \cup L$ where $|T| = m$. Define the winners $\{j_t\}_{t=1}^m$ in T, where $j_1 = \bar{m}^T$ and $j_t = \bar{m}^{T\setminus\{j_1, j_2, \ldots, j_{t-1}\}}$. Fix a $\delta \in (\max\{\delta(2), \delta^{m-1}\}, 1)$. The following STEPS 1 and 2 describe the proposal choice of j_1 and the winners other than j_1, respectively; *when they propose acceptably* at stage $T \cup L$. STEP 3 establishes that no winner in T

proposes unacceptably at stage $T \cup L$. Finally, STEP 4 describes the proposal choice of the losers at stage $T \cup L$.

STEP 1: Pick the agent $j_1 \in T$ ($j_1 = \bar{m}^T$). Strict inequality guarantees that j_1 is well defined. Now, by staying alone j_1 can get at least $\delta^{m-2} v_{j_1}$. This is because, from our hypothesis (and the specified range of δ) it follows that at all the later stages (consequent to j_1 staying alone) other than the single winner stage; *only* the winners make acceptable proposals, and all these acceptable proposals are either directed at themselves (that is, they stay alone) or at exactly one active winner (that is, forming a two agent coalition). This implies that after j_1 has stayed alone, the game must arrive at a single winner stage. From Proposition 2.3.2, 2.3.3 and 2.3.4; it follows that given $T^1 \in C_1^*(T^1)$, all the active agents in this single winner stage collude amongst themselves (irrespective of the identity of that single winner) and the game ends. Therefore, the final coalition structure yields j_1 a payoff of v_{j_1}[9]. Given our hypothesis, delay can occur along this path if (and only if) at some intermediate stage, an active loser gets to propose. There can be at most $m-2$ such stages; and so staying alone yields j_1 at least $\delta^{m-2} v_{j_1}$.

The maximum that j_1 can get by colluding with any other active agent is given by $\max\left\{\frac{v_{j_1}+v_{j_2}}{1+\delta}, \ldots, \frac{\sum_{t=1}^{m-1} v_{j_t}}{1+(m-2)\delta}\right\}$[10]. For any $t' = 2, \ldots, m-1$, the

[9] This follows from our worth of partition function; where any singleton (winner) member of a partition gets his valuation as payoff, if that partition contains another member set where all the losers collude with one or more winners.

[10] Agent j_1 attains the payoff of $\frac{v_{j_1}+\ldots+v_{j_t}}{1+(t-1)\delta}$, for any $t < m$; when j_1 acceptably proposes $\{j_1, j_2, \ldots, j_t\}$ at this stage and the remaining winners (or winner) colludes with all the losers in the next stage.

difference $\left[\delta^{m-2}v_{j_1} - \frac{\sum_{t=1}^{t'} v_{j_t}}{1+(t'-1)\delta}\right]$ is continuous and strictly increasing in δ with the $\delta \to 1$ limit being positive. Therefore, for δ sufficiently close to 1, this difference is positive. Thus, $\exists\, \delta_1 \in (\max\{\delta(2), \delta^{m-1}\}, 1)$ such that $\delta^{m-2}v_{j_1} > \max\left\{\frac{\sum_{t=1}^{t'} v_{j_t}}{1+(t'-1)\delta}\right\}_{t'=2}^{m-1}$ $\forall\, \delta \in (\delta_1, 1)$. Therefore $C_{j_1}^{\delta}(T \cup L) \equiv C_{\bar{m}^T}^{\delta}(T \cup L) = \{\{j_1\}\}, \forall\, \delta \in (\delta_1, 1)$.

STEP 2: Fix a $\delta \in (\delta_1, 1)$ and consider the agent j_2. For such a δ, our hypothesis implies that if j_2 stays alone, the maximum payoff (attained if no delay occurs in the intermediate stages) he can get is v_{j_2} and the minimum payoff (attained if there is delay at each of the intermediate stages) that he can get is $\delta^{m-3}v_{j_2}$. If j_2 acceptably proposes $\{j_1, j_2\}$, he gets *at least*, $\delta^{m-3}\left[v_{j_2} + (1-\delta)v_{j_1}\right]$. Any other collusive venture gives j_2 a maximum possible payoff of

$$\max\left\{\frac{(1-\delta)v_{j_1} + \sum_{t=2}^{t'} v_{j_t}}{1+(t'-2)\delta}\right\}_{t'=3}^{m-1}$$

Like in the previous case, there exists a $\bar{\delta}_2 \in (\delta_1, 1)$ such that $\forall\, \delta \in (\bar{\delta}_2, 1)$, $\delta^{m-3}\left[v_{j_2} + (1-\delta)v_{j_1}\right] > \max\left\{v_{j_2}, \max\left\{\frac{(1-\delta)v_{j_1} + \sum_{t=2}^{t'} v_{j_t}}{1+(t'-2)\delta}\right\}_{t'=3}^{m-1}\right\}$.[11] That is,

[11]For any $t' = 3, \ldots, m-1$, the difference $\delta^{m-3}v_{j_2} - \frac{(1-\delta)v_{j_1} + \sum_{t=2}^{t'} v_{j_t}}{1+(t'-2)\delta}$ is continuous and strictly increasing in δ with a positive value in the limit (tends to 1). Therefore for δ *sufficiently close* to 1, the difference is always positive. Hence for δ sufficiently close to 1, the difference $\left[\delta^{m-3}v_{j_2} - \max\left\{\frac{(1-\delta)v_{j_1} \sum_{t=2}^{t'} v_{j_t}}{1+(t'-2)\delta}\right\}_{t'=3}^{m-1}\right]$ is positive and so staying alone strictly dominates formation of any coalition other than $\{j_1, j_2\}$. However, the difference $\delta^{m-3}[v_{j_2} + (1-\delta)v_{j_1}] - v_{j_2} = (1-\delta)\left[\delta^{m-3}v_{j_1} - \left(1 + \delta + \delta^2 + \ldots + \delta^{m-4}\right)v_{j_2}\right]$ is positive *iff* $H(\delta) > \frac{v_{j_2}}{v_{j_1}}$ where $H(\delta) := \frac{\delta^{m-3}}{1+\delta+\delta^2+\ldots+\delta^{m-4}}$. Since $\frac{v_{j_2}}{v_{j_1}} \in (0,1)$ and $H(\delta)$ is a strictly increasing function of δ, once again the for *sufficiently high* δ, the difference $\delta^{m-3}[v_{j_2} + (1-\delta)v_{j_1}] - v_{j_2}$ is positive. Thus a $\bar{\delta}_2$ can indeed be found.

$C^{\delta}_{j_2}(T \cup L) = \{\{j_1, j_2\}\}$, and $a_{j_2}(T \cup L) \geq \delta^{m-2}[v_{j_2} + (1-\delta)v_{j_1}]$.[12]

Note that we can also find a $\delta_2 \in (\bar{\delta}_2, 1)$ such that $\forall\, \delta \in (\delta_2, 1)$, $\delta^{m-2}[v_{j_2} + (1-\delta)v_{j_1}] > v_{j_2}$. This means that for this range of discount factor; any acceptable proposal directed at j_2 must give him at least a payoff greater than v_{j_2} (which is the maximum possible marginal contribution that j_2 can make to any coalition containing it). This is always suboptimal and therefore, for this range of δ, no acceptable proposal directed at j_2 is ever made in equilibrium. In this manner we can generate a sequence $\{\delta_t\}_{t=3}^{m}$ such that $\delta_3 \in (\delta_2, 1)$ and $\delta_{t+1} \in (\delta_t, 1)$, $\forall\, t$ with the property that (i) $C^{\delta}_{t}(T \cup L) = \{j_1, j_t\}$, $\forall\, \delta \in (\delta_t, 1)$, $\forall\, t \geq 3$; and (ii) *no acceptable proposal* containing any of the members in $\{j_2, \ldots, j_t\}$ will be made in the equilibrium if the discount factor exceeds $\bar{\delta}_2$.[13]

STEP 3: Fix a $\delta \in (\delta_m, 1)$. Consider any $j \in T \setminus \{j_1\}$ and suppose that j makes an unacceptable proposal. This is optimal only if, this leads to transfer of proposal power to some other active player $j' \in T$ who makes an acceptable proposal $S^{j'}$ excluding j.

If $j' \in L$, then from (ii) in STEP 2, $S^{j'} \in \{\{j_1\} \cup S_k(t)\}_{t=l}^{n}$. For any $t \in \{l, \ldots, n\}$, the acceptable proposal $[\{j_1\} \cup S_k(t)]$ gives l a maximum possible payoff $\frac{(1-\delta)v_{j_1} - v_{t+1}}{1+(t-k-1)\delta}$. Observe that if $\delta > \frac{v_{j_1} - v_n}{v_{j_1}}$, then only acceptable proposal giving l a positive (maximum possible) payoff $\frac{(1-\delta)v_{j_1}}{1+(n-k-1)\delta}$ is $[\{j_1\} \cup S_k(n)]$. Then $\forall\, \delta \in \left(\max\left\{\delta_m, \frac{v_{j_1} - v_n}{v_{j_1}}\right\}, 1\right)$, $C^{\delta}_{l}(T \cup L) = [\{j_1\} \cup S_k(n)]$. But then, the game goes to stage $T \setminus \{j_1\}$, where the maximum possible

[12] Recall that j_2 can always reject a proposal, incur a period of delay, and then acceptably propose $\{j_1, j_2\}$.

[13] This is because the expression $\delta^{m-2}[x + (1-\delta)v_{j_1}] - x$ is decreasing in x.

payoff that j can get is $\delta[v_j + (1-\delta)v_{\bar{m}^{T\setminus\{j_1\}}}]$[14] It can be easily seen that $\exists\, \delta'_m \in \left(\max\left\{\delta_m, \frac{v_{j_1}-v_n}{v_{j_1}}\right\}, 1\right)$ such that $\forall \delta \in (\delta'_m, 1)$, $\delta^{m-2}[v_{j_2}+(1-\delta)v_{j_1}] > \delta[v_j + (1-\delta)v_{\bar{m}^{T\setminus\{j_1\}}}]$ (since $v_{j_1} > \bar{m}^{T\setminus\{j_1\}}$) and so making an acceptable proposal dominates making an unacceptable proposal.

Fix any $\delta \in (\delta'_m, 1)$. If $j' \in T \setminus \{j\}$, then either $j' = j_1$ or $j' \neq j_1$. If $j' = j_1$, then from STEP 1, $C_{j_1}(T \cup L) = \{j_1\}$, and so the game goes to the stage $[T \setminus \{j_1\}] \cup L$ with $m-1$ winners. Then from the induction hypothesis, we get that $C_j([T \setminus \{j_1\}] \cup L) = \left\{\bar{m}^{T\setminus\{j_1\}}, j\right\}$. Given the range of δ, from STEP 2, $C_j(T \cup L) = \{j_1, j\}$; and so, payoff to j from proposing $\{j_1, j\}$ acceptably at stage $T \cup L$ exceeds that from proposing $\left\{\bar{m}^{T\setminus\{j_1\}}, j\right\}$ acceptably, at stage $T \cup L$ (which, in turn, is weakly greater than doing the same at stage $[T \setminus \{j_1\}] \cup L$). Therefore, proposing acceptably dominates doing otherwise. When $j' \neq j_1$, from STEP 2, agent j' acceptably proposes $\{j_1, j'\}$ (since $C_{j'}(T \cup L) = \{j_1, j'\}$); and so, the game proceeds to the next stage $[T \setminus \{j_1, j'\}] \cup L$ with $m-2$ winners. Then, from induction hypothesis, $C_j([T \setminus \{j_1, j'\}] \cup L) = \left\{\bar{m}^{T\setminus\{j_1,j'\}]\cup L}, j\right\}$. Given the range of δ, from STEP 2, $C_j(T \cup L) = \{j_1, j\}$; and so, arguing as before, proposing acceptably dominates doing otherwise.

Finally, consider the possibility that $j = j_1$. Then, $j' \neq j_1$. Therefore, $\forall\, \delta \in (\delta'_m, 1)$, as mentioned before, $j' \in L \Rightarrow C_l^\delta(T \cup L) = [\{j_1\} \cup S_k(n)]$ and $j' \in T \setminus \{j_1\} \Rightarrow C_{j'}(T \cup l) = \{j_1, j'\}$. But for both these cases, j_1 could have proposed the same coalition acceptably, in the first place; thereby saving a period of delay (and getting the (higher) proposer's share out of the worth of

[14]As in STEP 2, we can show that it is suboptimal for j to acceptably propose to any other winner in $T \setminus \left\{j_1, \bar{m}^{T\setminus\{j_1\}}\right\}$, at stage $T \setminus \{j_1\}$.

$[\{j_1\} \cup S_k(n)]$, in case of $j' \in L$). Hence, proposing unacceptably turns out to be sub-optimal for j_1 at stage $T \cup L$. Therefore, $\forall\, \delta \in (\delta'_m, 1)$, no winner in T makes an unacceptable proposal at stage $T \cup L$.

STEP 4: If any loser l proposes acceptably at stage $T \cup L$, then, from STEP 3, $\forall\, \delta \in (\delta'_m, 1)$, $C_l^\delta(T \cup L) = [\{j_1\} \cup S_k(n)]$ and l gets a maximum possible payoff of $\frac{(1-\delta)v_{j_1}}{1+(n-k-1)\delta}$ is $[\{j_1\} \cup S_k(n)]$. On the other hand, like in the two winner stage $\{i, j\} \cup L$, given the specified range of δ and our hypothesis, an unacceptable proposal by l to some winner, yields at least $\frac{\delta v_{j_1}}{1+(n-k)\delta}$ in the final single winner stage, at most $m - 1$ periods later. That is, the least l gets by making an unacceptable proposal when $\delta \in (\delta'_m, 1)$ is $\frac{\delta^m v_{j_1}}{1+(n-k)\delta}$. The difference $\frac{\delta^m v_{j_1}}{1+(n-k)\delta} - \frac{(1-\delta)v_{j_1}}{1+(n-k-1)\delta}$ is continuous and strictly increasing in δ and this difference is positive in the limit. Therefore, $\exists\, \underline{\delta} \in (\delta'_m, 1)$ such that for all $\delta \in (\underline{\delta}, 1)$ the difference is positive, that is, making unacceptable proposal is the optimal action. The particular identity of the winner in T to whom any l must unacceptably propose is given Recursion (*).

Define $\delta^m := \max\left\{\delta'_m, \frac{v_{j_1}-v_n}{v_{j_1}}, \underline{\delta}\right\}$. Then, $\forall\, \delta \in (\delta^m, 1)$; at the stage $T \cup L$ such that $|T| = m$, all losers make an unacceptable proposal at some active winner and $C_t^\delta(T \cup L) = \{j_1, t\} = \{\bar{m}^T, t\}$, $\forall\, t \neq j_1 = \bar{m}^T$ with $C_{\bar{m}^T}^\delta(T \cup L) = \{\bar{m}^T\}$. We can continue such a recursion to get a sequence of $\{\delta^m\}_{m=3}^n$ such that result (3) follows by simply choosing $\delta' := \max\{\delta^m\}_{m=3}^n$. □

An obvious consequence of Theorem 2.3.3 is the resulting coalition structure contingent on the protocol function. This is summarized in the next corollary using the δ' obtained in Theorem 2.3.3.

Corollary 2.3.4. For any $G = (N, \bar{w}, p, \delta)$ if $T^1 \in C_1^*(T^1)$ and $\delta \in (\delta', 1)$, then for any given $p(.)$, the SSPE coalition structure is a protocol contingent partition $(E_1, \ldots, E_s)$ of the agent set N such that

$$E_1 = \begin{cases} \{1\} & \text{if } \{p(N) = 1\} \text{ or } \{p(N) \in L \text{ and } k^*(K;p) = 1\} \\ \{p(N), 1\} & \text{if } p(N) \in K \setminus \{1\} \\ \{k^*(K;p), 1\} & \text{if } p(N) \in L \text{ and } k^*(K;p) \neq 1 \end{cases}$$

Suppose the sequence is $\{E_1 \cup \ldots \cup E_q\}$ well defined and $R_q := N \setminus \{E_1 \cup \ldots \cup E_q\} \neq \emptyset$. Then

$$E_{q+1} = \begin{cases} T^i & \text{if } R_q \cap K = \{i\} \\ \{p(R_q)\} & \text{if } \{p(R_q) = \overline{m}^{R_q}\} \text{ or if } \left\{p(R_q) \in K \setminus \{\overline{m}^{R_q}\} \text{ and } |R_q \cap K| = 2\right\} \\ \left\{p(R_q), \overline{m}^{R_q}\right\} & \text{if } p(R_q) \in K \setminus \{\overline{m}^{R_q}\} \text{ and } |R_q \cap K| > 2 \\ \left\{\underline{m}^{[R_q \cap K]}\right\} & \text{if } p(R_q) \in L \text{ and } |R_q \cap K| = 2 \\ \left\{k^*(R_q \cap K; p), \overline{m}^{R_q}\right\} & \text{if } p(R_q) \in L \text{ and } |R_q \cap K| > 2 \end{cases}$$

Example 2.3.5. Suppose $N = \{1, 2, 3, 4, 5\}$, $K = \{1, 2, 3\}$. Let the protocol function $p(S) := \max_{j \in S}\{\succ\}$, $\forall\, S \subseteq N$ for some linear order "$\succ$" defined on the agent set N. For δ sufficiently close to 1, if

1. $1 \succ 2 \succ 3 \succ 4 \succ 5$ then final coalition structure is $\{\{1\}, \{2\}, \{3, 4, 5\}\}$.

2. $3 \succ 2 \succ 5 \succ 4 \succ 1$ then final coalition structure is $\{\{1, 3\}, \{2, 4, 5\}\}$.

3. $4 \succ 1 \succ 5 \succ 2 \succ 3$ then final coalition structure can either be $\{\{1, 3\}, \{2, 4, 5\}\}$ or it can be $\{\{1\}, \{3\}, \{2, 4, 5\}\}$.

Note that two possibilities arise for the coalition structure in the third case. That is because a loser (agent 4) gets to propose at a stage with more than 2 winners. Recall that the **Recursion** (*) did not guarantee a unique $k^*(.)$; which is why $k^*(N;p) \in \{1, 3\}$, thereby leading to two possible coalition structures.

An interesting coalition structure is the one where the lowest valuation winner k colludes with all the losers in L while all other winners stay alone. This is interesting because the coalition $T^k = \{k\} \cup L$ ensures that all the losers bid zero at the auction, thereby reducing the $(k+1)$th price to zero. Thus the other winners $\{1, \ldots, k-1\}$ get their own valuations as the equilibrium payoff in the limit as δ tends to 1. Agent k, however, gets only $\frac{v_k}{n-k+1}$ in the limit. In other words, winner k generates the gains from cooperation while the other winners *free ride*. The following proposition provides the restriction on the protocol function that characterizes formation of this coalition structure in equilibrium.

Proposition 2.3.5. For any $G = (N, p, \bar{w}, \delta)$, if $T^1 \in C_1^*(T^1)$ then $\forall\, \delta \in (\delta', 1)$[15]; the SSPE outcome is $\{\{1\}, \ldots, \{k-1\}, \{k, k+1, \ldots, n\}\}$ if and only if the $p(.)$ satisfies the property

$$p(N) = 1, p\left(N \setminus \{1, \ldots, i\}\right) = i + 1, \forall\, i \in K \setminus \{k-1, k\} \qquad (2.2)$$

Proof: The sufficiency of condition 2.2 follows from Corollary 2.3.4. To establish the necessity, consider the member $T^k = \{k, k+1, \ldots, n\}$. For T^k to have formed; on the equilibrium path, at some stage $\hat{T}$ (such that $T^k \subseteq \hat{T}$), some member $i \in T^k$ must have acceptably proposed T^k. Now if $|\hat{T} \cap K| \geq 2$ then, given the specified range of δ, irrespective of whether $i = k$ or $i \in L$, we get a contradiction to the equilibrium strategies defined in Theorem 2.3.3. Hence $\hat{T} = T^k$.

Now consider the singleton coalition $\{k-1\}$. Since T^k must have formed at the stage T^k itself, $\{k-1\}$ must have formed at a stage $\bar{T}$ such that

[15] The δ' is taken from Theorem 2.3.3.

$\{\{k-1\}\cup T^k\} \subseteq \bar{T}$. Given the range of δ, the only possibility where agent $k-1$ would choose to stay alone without contradicting our findings in Theorem 2.3.3; is when $\bar{T} = \{\{k-1\}\cup T^k\}$. Now if $p(\bar{T}) \in L$, then it must unacceptably propose to the lower value winner k, who would then stay alone. If $p(\bar{T}) = k$ then it is optimal for k to stay alone so that T^{k-1} forms in the next stage. Therefore in either case we have a contradiction. Therefore, $p(\{k-1\}\cup T^k) = k-1 \Rightarrow p(N\setminus\{1,\dots,k-2\}) = k-2+1 = k-1$. Continuing in this manner, for the rest of the singleton coalitions, $\{k-2\}, \{k-3\}, \dots, \{1\}$; the result follows.[16] □

In fact, the strategies in Theorem 2.3.3 generate a class of coalition structures where any one winner colludes with all the losers on the equilibrium path as δ approaches 1 *irrespective of the protocol function*. This is presented formally in the following corollary.

Corollary 2.3.6. For any $G = (N, \bar{w}, p, \delta)$, if $T^1 \in C_1^*(T^1)$ then $\forall\, \delta \in (\delta', 1)$, the SSPE outcome belongs to the class of coalition structures $\bar{\mathcal{P}} \subset \Pi(N)$ such that $\forall\, \pi \in \bar{\mathcal{P}}$,

1. $\exists\, j(\pi) \in K$ such that $T^{j(\pi)} \in \pi$.[17]
2. if $S \in \pi \setminus \left\{T^{j(\pi)}\right\}$ then $|S| \in \{1, 2\}$.
3. $|\{j \in K : \{j\} \in \pi, v_j < v_{j(\pi)}\}| \in \{0, 1\}$.

Proof: (1) and (2) follow from the Theorem 2.3.3. To prove (3), suppose the contrary holds. That is, there exists a $\pi \in \mathcal{P}$ and a pair of winners

[16] Note that $p(T^k) = p(N\setminus\{1,\dots,k-1\})$ is free from any restriction because any agent in T^k proposes T^k optimally.

[17] If $|N| > 2$ then $j(\pi) \in K \setminus \{1\}$, $\forall\, \pi \in \mathcal{P}$.

$j\,, j' \in K$ such that $v_j < v_{j'} < v_{j(\pi)}$ and $\{j\}, \{j'\} \in \pi$.[18] Now, from Theorem 2.3.3 it follows that coalition $T^{j(\pi)}$ forms at stage $T^{j(\pi)}$, that is, the single winner stage (after which the game ends). This means either $\{j\}$ or $\{j'\}$ must have formed at some stage T' such that $\left\{\{j\} \cup \{j'\} \cup T^{j(\pi)}\right\} \subseteq T'$. In either case, this is in contradiction to the equilibrium proposal decisions in Theorem 2.3.3 for the specified range of δ. Hence the result (3) follows. □

Example 2.3.7. Take the simplest multiple goods case where there are two goods, that is, $N = \{1, 2, 3, 4, 5, 6\}$ and $K = \{1, 2\}$. Fix the δ value sufficiently high so that the comparison amongst the average worths gives us the ranking between different coalitions according to their profitability as collusive ventures. Assume that (**a**) $\frac{v_i - v_5}{3} > \max\left\{v_i - v_3, \frac{v_i - v_4}{2}, \frac{v_i - v_6}{4}, \frac{v_i}{5}\right\}$, $\forall\, i = 1, 2$; that is, $C_i^{\delta}(\{i, 3, 4, 5, 6\}) = \{i\} \cup \{3, 4\}$, $\forall\, i = 1, 2$.

Suppose $p(N) = 1$ and $p(N \setminus \{1\}) = 2$. Therefore, if agent 1 stays alone at the stage N, then at the next stage 2 acceptably proposes $\{2, 3, 4\}$ leading to the coalition structure $\{\{1\}, \{2, 3, 4\}, \{5\}, \{6\}\}$ which gives agent 1 a payoff of $v_1 - v_5$. However, if 1 forms $\{1, 6\}$ at stage N, then the payoff is

$$\begin{cases} \frac{v_1 - v_3}{2} & \text{if 2 stays alone at the next stage } \{2,3,4,5\} \\ \frac{v_1 - v_4}{2} & \text{if 2 forms } \{2,3\} \text{ at the next stage } \{2,3,4,5\} \\ \frac{v_1 - v_5}{2} & \text{if 2 forms } \{2,3,4\} \text{ at the next stage } \{2,3,4,5\} \\ \frac{v_1}{2} & \text{if 2 forms } \{2,3,4,5\} \text{ at the next stage } \{2,3,4,5\} \end{cases}$$

Therefore, for 1 to make the optimal proposal choice at stage N (that is, to evaluate the proposal $\{1, 6\}$ at stage N), it needs to know the proposal choice of agent 2 at stage $\{2, 3, 4, 5\}$. Note that our assumption (**a**) puts no

[18] This means $|\{j \in K : \{j\} \in \pi, v_j < v_{j(\pi)}\}| = 2$.

restriction on the ranking of average worths of subsets of $\{2, 3, 4, 5\}$, that agent 2 can propose acceptably (keeping in mind that agent 6 has already colluded with agent 1 and so will bid zero at the auction) at stage $\{2, 3, 4, 5\}$. That is, (**a**) does not impart any ranking of the numbers $v_2 - v_3$, $\frac{v_2 - v_4}{2}$, $\frac{v_2 - v_5}{3}$ (payoffs from forming $\{2\}$, $\{2, 3\}$ and $\{2, 3, 4\}$ respectively) with respect to $\frac{v_2}{4}$ (payoff from forming $\{2, 3, 4, 5\}$). Hence the problem becomes fairly intractable, even with two goods case, once we allow $C_i^{\delta}(T^i)$ to be strict subset of T^i for all (or some) $i \in K$.

Also in such a case the final coalition structure may or may not have one winner colluding with all the losers, depending upon the protocol function. That is, if we use the protocol function $p(N) = 1, p(N \setminus \{1\}) = 6$; then it is optimal for agent 1 to stay alone at stage N since, in the next stage, the loser 6 proposes (who has no choice but to acceptably propose) $\{2, 3, 4, 5, 6\}$ leading to formation of the coalition structure $\{\{1\}, \{2, 3, 4, 5, 6\}\}$ giving 1 a payoff of v_1 (which is the best that agent 1 can get).

2.4 Conclusion

In this chapter, we analyze coalition formation at Vickrey auction with single as well as multiple identical indivisible identical goods; with unit demand and complete information. The assumption of complete information is restrictive but it turns out that this case is already quite rich. We provide, for sufficiently patient bidders, the necessary and sufficient conditions for formation of bidding ring at the single good auction, when the highest valuation agent is the first proposer. In the multiple goods case, we specify the sufficient conditions for formation of the class of coalition structures, where exactly one winner colludes with all the losers *irrespective of the protocol function.* Our work, therefore, turns out to be the complete information benchmark with regard to collusion at such auctions. Of course, further research needs to be done to extend this line of coalition formation to the incomplete information case.

Chapter 3

Group Strategyproof Indivisible Good Allocation

3.1 Introduction

In this chapter, we consider the problem of allocation of a set of indivisible identical goods with monetary transfers. This problem has many practical applications. The set of indivisible identical goods may consist of houses, jobs, locations, frequencies, etc. The agents are assumed to have *unit demand* and quasi-linear preferences over goods and money. In particular, each agent is identified with a non-negative valuation for the good and has utility as a linear function of money.

This valuation is private information, and so, irrespective of the allocation rule employed by the planner; agents have incentive to misreport their valuations. The planner, therefore, needs to design a mechanism to ensure truthful revelation of valuations. A mechanism in this context, is a pair consisting of an assignment function which determines which agents get goods and a vector of monetary transfers.

The popular notion of robustness of mechanisms with respect to misrepresentation of valuations, in the mechanism design literature, has been *strategyproofness*. A mechanism is said to be strategyproof if truth-telling is a weakly dominant strategy for all agents in the direct revelation game induced by it. Holmström's [21] general result implies that in this context the Vickrey-Clarke-Groves mechanisms (Vickrey [47]; Clarke [9]; Groves [16]) are the only decision efficient and strategyproof mechanisms.

It would seem desirable from ethical as well as practical perspective; that such a mechanism, conform to certain *fairness* criteria. In this chapter, we use two notions of fairness; one, the popular concept of *no-envy* introduced by Foley [11] and Varian [46], and two, the concept of *anonymity in welfare* used by Hashimoto and Saitoh [18] and many others. A mechanism is said to satisfy no-envy, if each agent prefers his bundle of good and money, to that of others. A mechanism is said to satisfy anonymity in welfare if utility levels of any two agents get interchanged when their valuations are interchanged. Tadenuma and Thomson [44] mention that no-envy is an appealing concept because of its normative feature as well as compatibility with decision efficiency - in this setting, no-envy implies efficiency. This notion of fairness has been widely used to study the problem of indivisible good allocation with monetary transfers.[1]

We completely characterize the class of fair mechanisms which are im-

[1]Ohseto [36], Tadenuma and Thomson [44],[45], Fujinaka and Sakai [13],[12], Svensson and Larsson [43], Pápai [37].

mune to strategic misrepresentation of valuations by any *group* of agents. For this we extend the notion of strategyproofness to group strategyproofness. In doing this, we could think of coalitional deviations of the kind discussed in Chapter 2 where the deviating group seeks to increase the sum of the utilities of all members. We could also think of coalitional deviations where at least one member of the deviating group is strictly better off with no other members being strictly worse off. We show that there exist no fair mechanisms which are immune to the coalitional deviation of the latter kind. Since, the latter kind of deviations are essentially a subset of the former kind, we get the non-existence of fair mechanisms which are immune to the coalitional deviation of the former kind.

We, therefore, water down our restriction on the mechanisms by considering only those coalitional deviations that make all the participating members strictly better off. These coalitional deviations are the ones, most relevant in present setting of incomplete information, where binding contracts are difficult to write. This is because there is no incentive for the post-deviation redistribution of payoffs necessary to compensate any member who may have become worse off. In other words, colluding members may end up cheating each other, and so, it makes sense for the members of group to agree to a coalitional deviation only if each member is strictly better off. We call mechanisms immune to these kinds of coalitional misreporting; weak group strategyproof mechanisms. In this chapter, we specify the class fair mechanisms that satisfy weak group strategyproofness.[2] We find that the two

[2]This notion of group strategyproofness has also been used in different contexts by Bogomolnaia and Moulin [4], Barbera, Berga and Moreno [1] and Hatsumi and Serizawa [19].

notions of fairness, no-envy and anonymity in welfare, are equivalent in the sense that both criteria yield the same class of weak group strategyproof mechanisms.

The problem of indivisible good allocation with monetary transfers is a well-studied one. Tadenuma and Thomson [44] show that no proper sub-solution of the no-envy solution satisfies consistency. Ohseto [36] shows that there is no strategyproof, decision efficient and budget balanced mechanism on restricted preference domains which are sufficiently large. Svensson and Larsson [43] study the strategyproof and non-bossy allocations, and show that any neutral allocation rule satisfying these properties must be serially dictatorial. Fujinaka and Sakai [12] and Tadenuma and Thomson [45] discuss no-envy multi-valued solutions in this context.

The study closest to the research program in this chapter, is by Fujinaka and Sakai [22]. They focus on no-envy and fully efficient mechanisms immune to unilateral strategic misreporting of preferences in a direct revelation game. In particular, they look for ε-Nash Implementable mechanisms which satisfy full efficiency, no-envy and other normative properties. In this chapter, we relax full efficiency to feasibility and decision efficiency; and extend the notion of truth-telling from Nash equilibrium to group strategyproofness. We find that the Pivotal mechanism adjusted for a (suitably chosen) constant is the only feasible no-envy mechanism satisfying weak group strategyproofness. Using a zero transfer condition, we then obtain a new characterization of Pivotal mechanism.

Section 3.2 states the model and required definitions. Section 3.3 and 3.4 contain the results and conclusion. Section 3.5 is the appendix containing the proofs.

3.2 Model

Consider a set of agents $\{1,\ldots,n\}$ with $n \geq 2$ and a set of indivisible identical goods $\{1,\ldots,k\}$ with $n > k \geq 1$. Each agent has a demand for only one good; $b_i \in \Re_+$ denotes the private valuation of agent i, for the good. A mechanism is a tuple (d,τ) such that at any reported profile of valuations $b \in \Re^n_+$, each agent i is allocated a transfer $\tau_i(b) \in \Re$ and a $d_i(b) \in \{0,1\}$. $d_i(b) = 1$ implies that agent i gets a good, while $d_i(b) = 0$ stands for i not getting the good.

The utility to agent i with a true valuation of b_i at any profile $b' \in \Re^n_+$, from the mechanism (d,τ) is given by $u_i((d_i(b'),\tau_i(b'));b_i) = b_i d_i(b') + \tau_i(b')$. Also define the function $l_i(b,b') := u_i((d_i(b),\tau_i(b));b_i) - u_i((d_i(b'),\tau_i(b'));b_i)$, $\forall\ i \in N$ and $\forall\, b,b' \in \Re^n_+$. Therefore, $l_i(b,b')$ denotes the change in utility to agent i when the (announced) profile changes from b to b' and the true valuation of i is b_i. Let $b_{-i} := (b_1,\ldots,b_{i-1},b_{i+1},\ldots,b_n)$, $\forall\ i \in N$, $b_{-S} := (b_i)_{i\in N\setminus S}$ and $b_S := (b_i)_{i\in S}$, $\forall\ S \subseteq N$, $\forall\, b \in \Re^n_+$. Define $M(x) := \max\{x_t\}$ for all $x \in \Re^p_+$, $\forall\ p \in \mathbb{N}$. Therefore, $M(x)$ is the largest element of any vector x, in the non-negative orthant of Euclidean space of any dimension. Also define $b(r)$ to be the rth ranked valuation in a non-increasing arrangement of components of any $b \in \Re^+_n$ for all $r = 1,2\ldots,n$. In case of ties, w.l.o.g. we use the tie-breaking rule $1 \succ \ldots \succ n$. Therefore $b(1) = M(b)$ for all $b \in \Re^n_+$.

Definition 3.2.1. A mechanism (d^e,t) is *decision efficient* (EFF) if $\forall b \in \Re^n_+$,

$\forall\, i \in N$,

$$[d_i^e(b) = 1] \implies [b_i \geq b(k+1)]$$

Decision efficiency is obtained by solving the following program for any reported valuation profile b,

$$\max_{d_1,d_2,\ldots,d_n} \left\{ \sum_{t=1,\ldots,n} d_t b_t \right\}$$

Using the tie-breaking rule $1 \succ \ldots \succ n$ we obtain a unique efficient decision for each $b \in \Re^n_+$.

Definition 3.2.2. A mechanism (d, τ) satisfies *no envy* (NE) if $\forall\, b \in \Re^n_+$, $\forall\, \{i, j\} \subseteq N$,

$$u_i(d_i(b), \tau_i(b); b_i) \geq u_i(d_j(b), \tau_j(b); b_i)$$

Definition 3.2.3. A mechanism (d, τ) satisfies *equal treatment of equals* (ETE) if $\forall\, b \in \Re^n_+$, $\forall\, \{i, j\} \subseteq N$,

$$[b_i = b_j] \implies [u_i(d_i(b), \tau_i(b); b_i) = u_j(d_j(b), \tau_j(b); b_j)]$$

Definition 3.2.4. A mechanism (d, τ) satisfies *anonymity in welfare* (AN) if $\forall\, i \neq j \in N$, $\forall\, b \in \Re^n_+$,

$$u_i(d_i(b_i, b_j, b_{-i-j}), \tau_i(b_i, b_j, b_{-i-j}); b_i) = u_j(d_j(b_j, b_i, b_{-i-j}), \tau_j(b_j, b_i, b_{-i-j}); b_i)$$

Clearly, AN implies ETE.

Definition 3.2.5. A mechanism (d, τ) is *strategyproof* if $\forall i \in N$, $\forall b_i, b'_i \in \Re_+$, $\forall\, b_{-i} \in \Re^{n-1}_+$,

$$u_i(d_i(b_i, b_{-i}), \tau_i(b_i, b_{-i}); b_i) \geq u_i(d_i(b'_i, b_{-i}), \tau_i(b'_i, b_{-i}); b_i)$$

Strategyproof mechanism guarantees that revealing the true valuation is a weakly dominant strategy for every agent. If a mechanism achieves EFF and strategyproofness, then we say that the efficient decision is implementable in dominant strategies. But there remains the possibility of agents forming coalitions and misreporting together. Ideally a mechanism should also be immune to such coalitional misreporting. Hence, we discuss two new incentive compatibility criteria. First, we introduce the following notation. For any $b, b' \in \Re^n_+$; b' is an S-profile of b if $\forall\, i \notin S$, $b_i = b'_i$, for any non-empty $S \subseteq N$.

Definition 3.2.6. A mechanism (d, τ) is *strong group strategyproof* (SGS) if $\forall\, b \in \Re^n_+$, $\nexists\, S \subseteq N$ such that

$$u_i(d_i(b), \tau_i(b); b_i) \leq u_i(d_i(b'), \tau_i(b'); b_i), \forall\, i \in S$$

$$\text{and } u_j(d_j(b), \tau_j(b); b_j) < u_j(d_j(b'), \tau_j(b'); b_j) \text{ for some } j \in S$$

where b' is an S-profile of b.

Definition 3.2.7. A mechanism (d, τ) is *weak group strategyproof* (WGS) if $\forall\, b \in \Re^n_+$, $\forall\, S \subseteq N$, $\exists\, j \in S$;

$$u_j(d_j(b), \tau_j(b); b_j) \geq u_j(d_j(b'), \tau_j(b'); b_j)$$

where b' is an S-profile of b.

It can easily be seen that SGS implies WGS which in turn implies strategyproofness.

Definition 3.2.8. A mechanism (d, τ) is *feasible* if $\sum_{i \in N} \tau_i(b) \leq 0, \forall\, b \in \Re^n_+$.

Definition 3.2.9. A mechanism (d, τ) satisfies *zero transfer* (ZT) if

$$\tau_i(0, 0, \ldots, 0) = 0, \forall\, i \in N$$

3.3 Results

We first, show that the any fair mechanism must be decision efficient.

Proposition 3.3.1. If a mechanism (d, τ) satisfies NE, then it must satisfy EFF and ETE.

Proof: For any $\{i, j\} \subseteq N$, $\forall\, b \in \Re^n_+$, no-envy implies that $b_i d_i(b) + \tau_i(b) \geq b_i d_j(b) + \tau_j(b)$ and $b_j d_j(b) + \tau_j(b) \geq b_j d_i(b) + \tau_i(b)$ for any profile b. From these two inequalities, it follows that **(i)** $b_i(d_j(b) - d_i(b)) \leq \tau_i(b) - \tau_j(b) \leq b_j(d_j(b) - d_i(b))$ for all b. This implies that $b_i(d_j(b) - d_i(b)) \leq b_j(d_j(b) - d_i(b))$; from which it follows that $d_i(b) > d_j(b) \implies b_i \geq b_j$, $\forall\, i, j \in N$. Hence the mechanism (d, τ) is EFF.

If $b_i = b_j$ then from **(i)** it follows that either $d_i(b) = d_j(b), \tau_i(b) = \tau_j(b)$ or $d_j(b) > d_i(b), \tau_i(b) = b_j + \tau_j(b)$ or $d_i(b) > d_j(b), \tau_j(b) = b_i + \tau_i(b)$. In all three case, $u_i(d_i(b), \tau_i(b); b_i) = u_j(d_j(b), \tau_j(b); b_j)$. Hence, the mechanism (d, τ) satisfies ETE. □

Corollary 3.3.1. If a mechanism (d, τ) satisfies NE and strategyproofness, then $\forall\, i \in N$, $\forall\, b \in \Re^n_+$,

$$\tau_i(b) = \begin{cases} g(b_{-i}) & \text{if } d^e_i(b) = 0 \\ -b(k+1) + g(b_{-i}) & \text{if } d^e_i(b) = 1 \end{cases}$$

Proof: By Proposition 3.3.1, no-envy implies efficiency of decision. Then from Holmstr'́om [21] it follows that the efficient and strategyproof mechanism is VCG mechanism given by the transfer $\tau_i(b) = \sum_{j \neq i} d^e_j(b) b_j + h_i(b_{-i})$, $\forall\, b \in \Re^n_+$. Substituting $h_i(b_{-i}) = -\sum_{j \neq i} d^e_j(b_{-i}) b_j + g_i(b_{-i})$, we can write that $\tau_i(b) = -\sum_{j\neq i}(d^e_j(b_{-i}) - d^e_j(b))b_j + g_i(b_{-i})$. Therefore, if $d_i(b) = 1$, $\tau_i(b) = -b(k+1) + g_i(b_{-i})$, or else $\tau_i(b) = g_i(b_{-i})$.

All we need to show is that for all i, the $g_i(.)$ function is independent of agent label. To prove this, *fix* any vector $\bar{x} \equiv (x^1, x^2, \ldots, x^{n-1}) \in \Re^{n-1}_+$. For all $i = 1, \ldots, n-1$ define the profile $\hat{b}^{i,\bar{x}}$ such that

$$\hat{b}^{i,\bar{x}}_j = \begin{cases} x^j & \text{if } j \leq i \\ x^i & \text{if } j = i+1 \\ x^{j-1} & \text{if } j > i \end{cases}$$

Note that $\forall\, i$, $\hat{b}^{i,\bar{x}}_i = \hat{b}^{i,\bar{x}}_{i+1}$. Therefore, by proposition 3.3.1, NE implies that for all $i = 1\ldots, n-1$,

$$u_i((d^e_i(\hat{b}^{i,\bar{x}}), \tau_i(\hat{b}^{i,\bar{x}})); x^i) = u_{i+1}((d^e_i(\hat{b}^{i+1,\bar{x}}), \tau_i(\hat{b}^{i+1,\bar{x}})); x^i) \tag{3.1}$$

From condition 3.1 it follows that if $d^e_i(\hat{b}^{i,\bar{x}}) = d^e_{i+1}(\hat{b}^{i,\bar{x}})$ then $g_i(\bar{x}) = g_{i+1}(\bar{x})$. Now, if $d^e_i(\hat{b}^{i,\bar{x}}) > d^e_{i+1}(\hat{b}^{i,\bar{x}})$ or $d^e_{i+1}(\hat{b}^{i,\bar{x}}) > d^e_i(\hat{b}^{i,\bar{x}})$ then $\hat{b}^{i,\bar{x}}(k+1) = x^i$. So condition 3.1 implies that $g_i(\bar{x}) = g_{i+1}(\bar{x})$. Therefore, $g_1(\bar{x}) = g_2(\bar{x}) = \ldots = g_{n-1}(\bar{x}) = g_n(\bar{x})$. Since $\bar{x}$ was chosen arbitrarily, for all $i \in N$, the $g_i(.)$ function must be independent of agent label i. □

Remark 3.3.2. Corollary 3.3.1 implies that the $g(.)$ functions are symmetric and independent of agent labels. Pápai [37] (Observation 3) proves the same result in the heterogeneous indivisible good allocation context.

3.3.1 Single Good

We, now, state the results for $k = 1$.

Proposition 3.3.2. There exists no mechanism (d, τ) which satisfies SGS and NE.

Proof: Since SGS implies strategyproofness, by Corollary 3.3.1 $\tau_i(b) = \sum_{j \neq i}[b_j(d_j^e(b) - d_j^e(b_{-i}))] + g(b_{-i})$.

W.l.o.g. fix a $b \in \Re_+^n$ such that $b_1 > b_2 > \ldots > b_n$. Say $\exists\, x \geq b_2$ such that $g(x, b_{-1-2}) \neq g(b_1, b_{-1-2})$. If $g(x, b_{-1-2}) > g(b_1, b_{-1-2})$ then consider a $\{1,2\}$-deviation from the true profile b to (x, b_{-1}). If $g(x, b_{-1-2}) < g(b_1, b_{-1-2})$ then consider a $\{1,2\}$-deviation from the true profile (x, b_{-1}) to b. In either case SGS is violated and so it must be that $g(x, b_{-1-2}) = C(b_{-1-2})$, $\forall\, x \geq b_2$. But, now consider another $\{1,2\}$-deviation from b to $(b_2 - \epsilon, b_{-2})$ with $\epsilon \in (0, b_2)$. Since the efficient decision remains unchanged, the transfers $\tau_2(b) = \tau_2(b_2 - \epsilon, b_{-2})$ and $\tau_1(b) = b_1 - b_2 + C(b_{-1-2}) < b_1 - (b_2 - \epsilon) + C(b_{-1-2})$. Hence, SGS is violated and so contradiction. □

Proposition 3.3.2 rules out existence of mechanisms immune to those group deviations which make no member worse off and at least one member strictly better off. This implies non-existence of mechanisms immune to coalitional deviations where the sum of utilities of all members strictly increases. Hence, we focus on WGS mechanisms.

Proposition 3.3.3. If a mechanism (d, τ) satisfies NE and strategyproofness, then $\forall\, i \neq j \in N$, $\forall\, b \in \Re_+^n$, $\forall\, \{x^u\} \subseteq [M(b_{-i-j}), \infty)$,

$$[\{x^u\} \rightarrow \tilde{x}] \implies [\{g(x^u, b_{-i-j})\} \rightarrow g(\tilde{x}, b_{-i-j})]$$

Proof: W.l.o.g. fix a $b \in \Re_+^n$ such that $b_1 > b_2 \geq M(b_{-1-2})$. Now, NE (for agents 1 and 2) implies that $b_1 - b_2 + g(b_2, b_{-1-2}) \geq g(b_1, b_{-1-2})$ and

$g(b_2, b_{-1-2}) \leq g(b_1, b_{-1-2})$. Therefore, $\forall b_2 \in [M(b_{-1-2}), b_1]$, $0 \leq g(b_1, b_{-1-2}) - g(b_2, b_{-1-2}) \leq b_1 - b_2$. Therefore, for any sequence $\{b_2^u\} \subseteq [M(b_{-1-2}), b_1)$ such that $\{b_2^u\} \to b_1$, the corresponding sequence $\{g(b_2^u, b_{-1-2})\}$ converges to $g(b_1, b_{-1-2})$. Since b was arbitrarily chosen, the result follows. □

Henceforth, for simplicity, whenever we are concerned with the efficient decision, we write $d(.)$ instead of $d^e(.)$.

The following theorem describes the behaviour of the $g(x, b_{-i-j})$ function when $x \geq M(b_{-i-j})$, for any chosen b_{-i-j} vector, and any $\{i, j\} \subseteq N$.

Theorem 3.3.3. If a mechanism (d, τ) satisfies NE and WGS then for any $i \neq j \in N$, $\forall\, b_{-i-j} \in \Re_+^{n-2}$, $\exists\, \eta(b_{-i-j}) \in [0, \infty]$ such that

$$g(x, b_{-i-j}) = \bar{C}(b_{-i-j}) + \min\{x, \eta(b_{-i-j})\}, \forall\, x \geq M(b_{-i-j})$$

Proof: See Appendix. □

Theorem 3.3.4. A mechanism (d, τ) satisfies NE and WGS *if and only if* $\forall\, i \in N$, $\forall\, b \in \Re_+^n$,

$$\tau_i(b) = \begin{cases} K + \min\{\eta, M(b_{-i})\} & \text{if } d_i(b) = 0 \\ K + \min\{\eta, M(b_{-i})\} - M(b_{-i}) & \text{if } d_i(b) = 1 \end{cases}$$

for some $\eta \in [0, \infty]$.

Proof of Only If: Since $k = 1$, $\forall\, i \in N$, $\forall\, b \in \Re_+^n$, $d_i(b) = 1$ implies that $b(2) = M(b_{-i})$. Therefore, we simply need to show that $g(b_{-i}) = K + \min\{\eta, M(b_{-i})\}$ for some $\eta \in [0, \infty]$, in Corollary 3.3.1. We do so by obtaining, for all i, the behaviour of $g(b_{-i})$ function in response to partial change in any component of the vector b_{-i}. So, we pick any $j \in N \setminus \{i\}$ and fix any arbitrary vector $b_{-i-j} \in \Re_+^{n-2}$. Theorem 3.3.3 specifies the impact of b_j on $g(b_{-i})$ when $b_j \geq M(b_{-i-j})$. In the following lemma, we consider the case when $b_j < M(b_{-i-j})$ and show that $g(., b_{-i-j})$ remains constant over the interval $[0, M(b_{-i}))$.

Lemma 3.3.5. If a mechanism (d, τ) satisfies NE and WGS then $\forall\, b_{-i-j} \in \Re_+^{n-2}$,

$$g(x, b_{-i-j}) = \underline{C}(b_{-i-j}), \forall\, x < M(b_{-i-j})$$

Proof of Lemma: Consider a pair of profiles b, b' such that $b_i, b_j, b'_i, b'_j < M(b_{-i-j})$. WGS implies that $[l_i(b, b') \geq 0$ or $l_j(b, b') \geq 0]$ and $[l_i(b', b) \geq 0$ or $l_j(b', b) \geq 0]$. W.l.o.g. suppose $l_i(b, b') > 0$ and $l_j(b', b) > 0$. This would imply that $g(b_j) > g(b'_j)$ and $g(b'_i) > g(b_i)$. But then $l_t((b_i, b'_j, b_{-i-j}), (b'_i, b_j, b_{-i-j})) < 0$, $\forall\, t = i, j$ which would violate WGS. Also note that $l_i(b, b') = 0 \implies g(b_j) = g(b'_j) \implies l_j(b, b') = 0$ (and vice-versa). Hence, $g(x, b_{-i-j}) = g(x', b_{-i-j})$, $\forall\, x, x' < M(b_{-i-j})$, $\forall\, b_{-i-j} \in \Re_+^{n-2}$. □

Therefore, we can write that for some $\eta(b_{-i-j}) \in [0, \infty]$,

$$g(x, b_{-i-j}) = \begin{cases} \bar{C}(b_{-i-j}) + \min\{x, \eta(b_{-i-j})\} & \text{if } x \geq M(b_{-i-j}) \\ \underline{C}(b_{-i-j}) & \text{otherwise} \end{cases}$$

Hence, any discontinuity in $g(., b_{-i-j})$ can occur only at point $M(b_{-i-j})$ and it can only be a jump discontinuity. Define

$$\begin{aligned} G_{ij} := & \lim_{\{x^u\} \to M(b_{-i-j})+} g(x^u, b_{-i-j}) - \lim_{\{x^u\} \to M(b_{-i-j})-} g(x^u, b_{-i-j}) \\ = & \quad \bar{C}(b_{-i-j}) + \min\{M(b_{-i-j}), \eta(b_{-i-j})\} - \underline{C}(b_{-i-j}) \end{aligned}$$

If there exists a jump discontinuity in $g(., b_{-i-j})$ map at the point $M(b_{-i-j})$, then $G_{ij} \neq 0$.

If $G_{ij} > 0$, then there exist b_i, b_j, b'_i, b'_j such that $b_j < b_i < M(b_{-i-j}) < b'_j < b'_i$, $b'_j - b_i < G_{ij}$, $g(b'_i, b_{-i-j}) - g(b_i, b_{-i-j}) \geq g(b'_j, b_{-i-j}) - g(b_j, b_{-i-j}) \geq G_{ij}$. Therefore, $\{i,j\}$-deviation from (b_i, b_j, b_{-i-j}) to (b'_i, b'_j, b_{-i-j}) violates WGS. If $G_{ij} < 0$ then there are two possibilities, (i) $\eta(b_{-i-j}) > M(b_{-i-j})$ and (ii) $\eta(b_{-i-j}) \leq M(b_{-i-})$. If (i) holds then there exist b_i, b_j, b'_i, b'_j such that $b'_j < M(b_{-i-j}) < b_j < b_i < b'_i < \eta(b_{-i-j})$, $g(b'_j, b_{-i-j}) > g(b_j, b_{-i-j})$ and $g(b'_i, b_{-i-j}) > g(b_i, b_{-i-j})$. Clearly the $\{i,j\}$-deviation from (b_i, b_j, b_{-i-j}) to (b'_i, b'_j, b_{-i-j}) violates WGS. If (ii) holds then there exist b_i, b_j, b'_i, b'_j such that $b'_j < b'_i < M(b_{-i-j}) < b_j < b_i$, $b_i - b_j < |G_{ij}|$ and $g(b'_i, b_{-i-j}) - g(b_i, b_{-i-j}) = g(b'_j, b_{-i-j}) - g(b_j, b_{-i-j}) = |G_{ij}|$.[3] Once again WGS is violated in $\{i,j\}$-deviation from (b_i, b_j, b_{-i-j}) to (b'_i, b'_j, b_{-i-j}).

Therefore, it must be that $G_{ij} = 0$, and so there can be no discontinuity in the $g(., b_{-i-j})$ map. This allows us to obtain the functional form of the $g(.)$ function by aggregating the results in Theorem 3.3.4 and Lemma 3.3.5 in the following manner. W.l.o.g. consider any $b \in \Re^n_+$ such that $b_i > b_j \geq M(b_{-i-j})$. Then, we can write for any $k \in N \setminus \{i,j\}$,

$$g(b_{-i}) = \underline{C}(b_{-i-k}) \Rightarrow g(b_{-i}) = \bar{C}(b_{-i-j-k}) + \min\{b_j, \eta(b_{-i-j-k})\}$$

Arguing similarly for all $k' \notin \{i,j,k\}$, we get that

$$g(b_{-i}) = K + \min\{b_j, \eta\}$$

where K is a constant and $\eta \in [0, \infty]$. □

Proof of If: W.l.o.g. pick a profile b such that $b_1 \geq b_2 \geq \ldots \geq b_n$. Therefore $d_1(b) = 1$ and $d_j(b) = 0$, $\forall\, j \neq 1$ and $\tau_j(b) = \tau_{j'}(b)$, $\forall\, j, j' \neq 1$. Hence,

[3] Note that for case (ii), $G_{ij} = M(b_{-i-j}) - \underline{C}(b_{-i-j})$.

NE is trivially satisfied for all such pair of agents. Now, choose any $i \neq 1$ and consider the pair of agents 1 and i. Define $D_i^1 := b_i - b_2 + \min\{b_2, \eta\} - \min\{b_1, \eta\}$ and $D_1^i := \min\{b_1, \eta\} - [b_1 - b_2 + \min\{b_2, \eta\}]$. Now, since $b_i < b_1$, $D_i^1 \leq 0$ and $D_1^i \leq 0$ irrespective of the ordering amongst b_1, b_2, η. Hence NE holds.

To check for WGS, define $\bar{M}_i(C) := \max\left\{\max_{t \in C \setminus \{i\}} x_t, \max_{t \notin C} b_t\right\}$, $\forall\, i \in C$, $\forall C \subseteq N$. Pick any $C \subseteq N \setminus \{1\}$ with $|C| \geq 2$ and any $x \in \Re_+^{|C|}$. Now, if $\exists\, i \in C$ such that $d_i(b) = 0$ and $d_i(x, b_{-C}) = 1$; then $l_i(b, (x, b_{-C})) = \min\{b_1, \eta\} - \min\{\bar{M}_i(C), \eta\} - b_i + \bar{M}_i(C)$. By construction $\bar{M}_i(C) \geq b_1 \geq b_i$ and so; $\eta < b_1 \Rightarrow l_i(b, (x, b_{-C})) = \bar{M}_i(C) - b_i$, $\eta \in [b_1, \bar{M}_i(C)) \Rightarrow l_i(b, (x, b_{-C})) = (b_1 - b_i) + (\bar{M}_i(C) - \eta)$, and $\eta \geq \bar{M}_i(C) \Rightarrow l_i(b, (x, b_{-C})) = b_1 - b_i$. Therefore, irrespective of the value of η, $l_i(b, (x, b_{-C})) \geq 0$ and so, WGS is not violated. If $d_i(b) = d_i(x, b_{-C}) = 0$ for all $i \in C$, then $l_i(b, (x, b_{-C})) = 0$ (since $1 \notin C$) and again, WGS is not violated.

Now, consider any $C \subseteq N$ with $1 \in C$. Suppose that $\exists\, x \in \Re_+^{|C|}$ such that WGS is violated a C-deviation from b to (x, b_{-C}). Therefore, $l_j(b, (x, b_{-C})) < 0$, for all $j \in C$. If $d_1(x, b_{C}) = 1$, then all members in $C \setminus \{1\}$ are strictly better off in this deviation, only if $\min\{x_1, \eta\} > \min\{b_1, \eta\}$. This implies that $b_2 \leq b_1 < \eta$, and so, $l_1(b, (x, b_{-C})) = \min\{b_2, \eta\} - b_2 + \bar{M}_1(C) - \min\{\bar{M}_1(C), \eta\}$ which is non-negative irrespective of the value of η. Hence, contradiction. If $\exists\, j \in C \setminus \{1\}$ such that $d_j(x, b_{-C}) = 1$, then $l_1(b, (x, b_{-C})) < 0$ implies that $\min\{b_2, \eta\} + b_1 < b_2 + \min\{x_j, \eta\}$. It can easily be checked that this condition holds only if $\eta > b_2$. In that case, the condition reduces to $b_1 < \min\{x_j, \eta\}$; which can be true only if $b_1 < \eta$. But then $l_j(b, (x, b_{-C})) = b_1 - b_j + \bar{M}_j(C) - \min\{\bar{M}_j(C), \eta\} \geq 0$. Hence, contradiction. Finally, if $d_j(x, b_{C}) = 0$, $\forall\, j \in C$, then $l_1(b,(x,b_{-C})) \geq 0$ and once again, contradiction.

Therefore, we can say that there does not exist a coalitional deviation which violates WGS. □

An obvious question that arises out of Theorem 3.3.4: what are the mechanisms satisfying AN and WGS? This question is relevant because AN, too, is a popular notion of fairness (Hashimoto and Saitoh [18]). In the following proposition we show that any mechanism in the single good allocation problem satisfying AN and strategyproofness, must be EFF.

Theorem 3.3.6. If a mechanism $(\hat{d},\tau)$ satisfies AN and strategyproofness, then it must satisfy EFF.

We, first, prove the following lemma;

Lemma 3.3.7. If a mechanism $(\hat{d},\tau)$ satisfies strategyproofness, then for any $i \in N, b \in \Re^n_+$,

$$t_i(b) = \begin{cases} K_i(b_{-i}) - T_i(b_{-i}) & \text{if } \hat{d}_i(b) = 1 \\ K_i(b_{-i}) & \text{if } \hat{d}_i(b) = 0 \end{cases}$$

Proof: Fix any i and any $b'_i \in \Re_+$. Strategyproofness for profiles b and (b'_i, b_{-i}) implies that

$$b_i(\hat{d}_i(b'_i,b_{-i}) - \hat{d}_i(b)) \leq \tau_i(b) - \tau_i(b'_i,b_{-i}) \leq b'_i(\hat{d}_i(b'_i,b_{-i}) - \hat{d}_i(b)) \tag{3.2}$$

Therefore, (**i**) $\hat{d}_i(b'_i,b_{-i}) = \hat{d}_i(b) \implies \tau_i(b) = \tau_i(b'_i,b_{-i})$ and (**ii**) $\hat{d}_i(b'_i,b_{-i}) > \hat{d}_i(b) \implies b'_i \geq b_i$. The non-decreasingness of the $\hat{d}_i(.,b_{-i})$ function in (**ii**) implies that there exists a $T_i(b_{-i}) \in \Re_+$ such that

$$\hat{d}_i(x,b_{-i}) = \begin{cases} 1 & \text{if } x \geq T_i(b_{-i}) \\ 0 & \text{if } x < T_i(b_{-i}) \end{cases} \tag{3.3}$$

In case, $T_i(b_{-i}) = T_j(b_{-j})$ for some $b \in \Re^n_+$ and $i \neq j \in N$, the good is allocated according to any arbitrarily chosen linear order $\succ$ defined on N. Given b_{-i} profile of other agents' reports, $T_i(b_{-i}) = 0$ implies that i always gets the good and $T_i(b_{-i}) = \infty$ implies that i never gets the good, *irrespective* of what i reports.

From (**i**) it follows that $\forall\, x \in \Re_+, \forall\, b_{-i} \in \Re^{n-1}_+$,

$$t_i(x, b_{-i}) \in \{t_i(\hat{d}_i = 0; b_{-i}), t_i(\hat{d}_i = 1; b_{-i})\}$$

where given b_{-i}, $t_i(\hat{d}_i = 1; b_{-i})$ denotes the transfer to i if he/she gets the good and $t_i(\hat{d}_i = 0; b_{-i})$ denotes the transfer if he/she does not get the good. Then, for any two sequences $\{\underline{x}^u\} \subset [0, T_i(b_{-i}))$ and $\{\bar{x}^u\} \subset [T_i(b_{-i}), \infty)$ with $\{\underline{x}^u\} \to T_i(b_{-i})$ and $\{\bar{x}^u\} \to T_i(b_{-i})$ respectively, (3.2) implies that

$$\forall\, u, \;\; \underline{x}^u \leq [t_i(\hat{d}_i = 0; b_{-i}) - t_i(\hat{d}_i = 1; b_{-i})] \leq \bar{x}^u$$

As $u \to \infty$, we get $[t_i(\hat{d}_i = 0, b_{-i}) - t_i(\hat{d}_i = 1, b_{-i})] = T_i(b_{-i})$. Defining $t_i(\hat{d}_i = 0, b_{-i}) := K_i(b_{-i})$ the result follows. □

Proof of Theorem 3.3.6: It can be easily seen that we simply need to prove that $T_i(b_{-i}) = M(b_{-i})$ for all i and all b in (3.3). For this, we define $\forall b \in \mathbb{R}^n_+$, $\forall x \in \mathbb{R}_+, \forall k = 1, \ldots, n$; $c^b_x := |\{i \in N : b_i = x\}|$ and $\bar{x}^k := (x, x \ldots, x) \in \mathbb{R}^k_+$. Also define the sequence of sets $\{S^x_k\}^{n-1}_{k=1}$ where $\forall\, k$,

$$S^x_k := \left\{b \in \mathbb{R}^n_+ | b(1) = x \text{ and } c^b_x = n - k + 1\right\}$$

Therefore, for all $x \geq 0$, $S^x_1 = \{\bar{x}^n\}$. We now prove that strategy-proofness and AN implies decision efficiency in the following three steps.

STEP 1. For all i and all $x \geq 0$, $T_i(\bar{x}^{n-1}) = x$.

Proof of Step: For any $x \geq 0$, consider the profile $\bar{x}^n$. W.l.o.g. suppose that $\hat{d}_1(\bar{x}^n) = 1$. Therefore, AN implies that (**i**) $K_j(\bar{x}^{n-1}) = x + K_1(\bar{x}^{n-1}) - T_1(\bar{x}^{n-1}), \forall j \neq 1$. From Lemma 3.3.7, strategy-proofness implies that $\forall \epsilon > 0$, $\hat{d}_1(x + \epsilon, \bar{x}^{n-1}) = 1$. Again, AN implies that

$$u_1(\hat{d}_1(x+\epsilon, \bar{x}^{n-1}), \tau_1(x+\epsilon, \bar{x}^{n-1}); x+\epsilon) = u_2(\hat{d}_2(x, x+\epsilon, \bar{x}^{n-2}), \tau_2(x, x+\epsilon, \bar{x}^{n-2}); x+\epsilon)$$

Given (**i**) this implies that $\hat{d}_2(x, x + \epsilon, \bar{x}^{n-2}) = 1$. From Lemma 3.3.7, it follows that $\forall \epsilon > 0$, $x + \epsilon \geq T_2(\bar{x}^{n-1})$ and so $x \geq T_2(\bar{x}^{n-1})$. Our supposition $\hat{d}_1(\bar{x}^n) = 1 \Rightarrow \hat{d}_2(\bar{x}^n) = 0$ implies that $x_2 \leq T_2(\bar{x}^{n-1})$ and so $T_2(\bar{x}^{n-1}) = x$. Arguing similarly for all $j \neq 1, 2$, we get that $T_j(\bar{x}^{n-1}) = x$

To prove the result for agent 1, consider the profile $(x, x - \epsilon, \bar{x}^{n-2})$. By Lemma 3.3.7 and our supposition, $\hat{d}_2(\bar{x}^n) = 0 \Rightarrow \hat{d}_2(x, x - \epsilon, \bar{x}^{n-2}) = 0$. AN implies that

$$u_1(\hat{d}_1(x-\epsilon, \bar{x}^{n-1}), \tau_1(x-\epsilon, \bar{x}^{n-1}); x-\epsilon) = u_2(\hat{d}_2(x, x-\epsilon, \bar{x}^{n-2}), \tau_2(x, x-\epsilon, \bar{x}^{n-2}); x-\epsilon)$$

As before, (**i**) implies that $\hat{d}_1(x - \epsilon, x, \bar{x}^{n-2}) = 0$ and so, $\forall \epsilon \in (0, x]$, $x - \epsilon \leq T_1(\bar{x}^{n-1}) \Rightarrow x \leq T_1(\bar{x}^{n-1})$. Therefore, $\hat{d}_1(\bar{x}^n) = 1$ implies that $x = T_1(\bar{x}^{n-1})$.

STEP 2. (INDUCTION HYPOTHESIS) For any $x \geq 0$, if for all $1 \leq k \leq n - 2$, $\forall b \in S_k^x$, $\forall i$, $T_i(b_{-i}) = x$; then

$$T_i(b_{-i}) = x, \forall b \in S_{k+1}^x, \forall i \in N$$

Proof of Step: W.l.o.g. consider the profile

$$b = (x, x, \ldots, x, b_{n-k+1}, b_{n-k+2}, \ldots, b_n) \in S_{k+1}^x$$

For any $r = n - k + 1, \ldots, n$, design the profile $\tilde{b}^r$ such that $\forall t \neq r$, $\tilde{b}_t^r = b_t$ and $\tilde{b}_r^r = x$. Clearly, $\tilde{b}^r \in S_k^x$ and so by induction hypothesis, $T_i(\tilde{b}_{-i}^r) = x$

for all $i \in N$. Therefore, by construction, $T_r(b_{-r}) = T_r(\tilde{b}^r_{-r}) = x$ and so, the result is established for all $r = n-k+1, \ldots, n$.

By construction $\forall\, r = n-k+1, \ldots, n$, $x > b_r$ and so $T_r(b_{-r}) > b_r$. Therefore, for all such r, $\hat{d}_r(b) = 0$ and so, any *one* of the agents with valuation equal to x, must get the good. Also, irrespective of the value of $k+1$, there will always be two agents that announce valuation x in b. W.l.o.g. suppose that $\hat{d}_1(b) = 1$ and $\hat{d}_2(b) = 0$. As in Step 1, $\forall\, \epsilon > 0$, $\hat{d}_1(x+\epsilon, b_{-1}) = 1$. AN implies that (**ii**) $K_2(b_{-2}) = x + K_1(b_{-1}) - T_1(b_{-1})$ and $u_1(\hat{d}_1(x+\epsilon, x, b_{-1-2}), \tau_1(x+\epsilon, x, b_{-1-2}); x+\epsilon) = u_2(\hat{d}_2(x+\epsilon, x, b_{-1-2}), \tau_2(x+\epsilon, x, b_{-1-2}); x+\epsilon)$. Arguing as in Step 1, $\hat{d}_2(x, x+\epsilon, b_{-1-2}) = 1$ and so, $\forall\, \epsilon > 0$, $x+\epsilon \geq T_2(b_{-2}) \Rightarrow x \geq T_2(b_{-2})$. Therefore, from our supposition $\hat{d}_2(b) = 0 \Rightarrow x \leq T_2(b_{-2})$ we get $T_2(b_{-2}) = x$. Arguing in this manner, we can show that for all $i \neq 1$ with $b_i = x$, $T_i(b_{-i}) = x$.

As in Step 1, $\forall\, \epsilon > 0$, $\hat{d}_2(x, x-\epsilon, b_{-1-2}) = 0$. AN implies that $u_1(\hat{d}_1(x-\epsilon, x, b_{-1-2}), \tau_1(x-\epsilon, x, b_{-1-2}); x-\epsilon) = u_2(\hat{d}_2(x-\epsilon, x, b_{-1-2}), \tau_2(x-\epsilon, x, b_{-1-2}); x-\epsilon)$. Hence, by (**ii**) it follows that $\hat{d}_1(x-\epsilon, x, b_{-1-2}) = 0 \Rightarrow x-\epsilon \leq T_1(b_{-1})$ for all $\epsilon > 0$. Therefore, $x \leq T_1(b_{-1})$ and so, $\hat{d}_1(b) = 1 \Rightarrow T_1(b_{-1}) = x$. Thus, the result holds for all i with $b_i = x$.

STEP 3. Fix any $b \in \mathbb{R}^n_+$, any $i \in N$ and consider the real number $T_i(b_{-i})$. Construct the profile $\hat{b}$ such that $\hat{b}_i = M(b_{-i})$ and $\hat{b}_{-i} = b_{-i}$. Clearly for some $k = 1, \ldots, n-1$, $\hat{b} \in S_k^{M(b_{-i})}$. Therefore, from the induction argument Step 1 (which serves as the induction base of $k = 1$) and Step 2, it follows that $T_i(b_{-i}) = M(b_{-i})$. □

Proposition 3.3.4. A mechanism $(\hat{d}, \tau)$ satisfies WGS and AN *if and only* *if* $\forall\, i \in N$ and $\forall\, b \in \Re^n_+$,

- $\hat{d}_i(b) = d_i(b)$
- $\tau_i(b) = \begin{cases} K + \min\{M(b_{-i}), \eta\} - M(b_{-i}) & \text{if } d_i(b) = 1 \\ K + \min\{M(b_{-i}), \eta\} & \text{if } d_i(b) = 0 \end{cases}$

Proof: The sufficiency is easy to check. So we prove the necessity only. Since WGS implies strategyproofness, Theorem 3.3.6 implies EFF and so, for all i and b, $\hat{d}_i(b) = d_i(b)$. This implies that any mechanism satisfying WGS and AN must belong to the class of VCG mechanisms that satisfy ETE. Therefore, arguing as in Theorem 3.3.3, we obtain the same result, but with a possibility of discontinuity at point $\eta(b_{-i-j})$. However, arguing as in Claim 4.6.8 (in the appendix 4.6 of the next chapter) we can rule out this possibility. Thus, we can argue in lines of Theorem 3.3.4 and get the desired result. □

Remark 3.3.8. Pápai [37] discusses the class of VCG mechanisms satisfying NE in a heterogeneous indivisible good allocation (with *non*-unit demand) setting. When adapted to the present single good context, her result (Theorem 1) requires that for any VCG mechanism satisfying NE, $\forall\, i \neq j$, $\forall\, b$,

$$W(b_{-i}) > W(b_{-j}) \Rightarrow 0 \leq \frac{h(W(b_{-i})) - h(W(b_{-j}))}{W(b_{-i}) - W(b_{-j})} \leq 1$$

where $h(W(b_{-i})) = W(b_{-i}) - g(b_{-i})$ and $W(b_{-i}) = \max\limits_{\{d_j\}} \sum_{j \neq i} d_j(b) b_j$. Clearly, for our setting, $W(b_{-i}) = M(b_{-i})$.

Consider an EFF mechanism with transfers as defined in Corollary 3.3.1 with $g(b_{-i}) = \frac{M(b_{-i})}{2}$. This mechanism does not belong to the class specified by Theorem 3.3.4. However, this mechanism belongs to the class specified by

Theorem 1 in Pápai [37]. To see this, consider w.l.o.g. the profile b such that $b_1 > b_2 > M(b_{-1-2})$. Then $W(b_{-2}) - W(b_{-1}) = b_1 - b_2 > 0$, $h(W(b_{-2})) - h(W(b_{-1})) = \frac{b_1 - b_2}{2}$ and so the result follows.

Definition 3.3.9. A mechanism $(\hat{d}, \tau)$ is *Pivotal* if $\forall\, b \in \Re^n_+$, $\forall\, i \in N$,

- $\hat{d}_i(b) = d_i(b)$
- $\tau_i(b) = -\sum_{j \neq i}(d_j(b_{-i}) - d_j(b))b_j$

Using Theorem 3.3.4 above, we provide a complete characterization of Pivotal mechanism in the indivisible good allocation problem.

Proposition 3.3.5. The following statements are equivalent;

1. (d, τ) is a feasible WGS mechanism satisfying NE and ZT.
2. (d, τ) is the Pivotal mechanism.

Proof: It is easy to check that 2 implies 1. To prove that 1 implies 2, note that whenever $\eta > 0$, from Theorem 3.3.4 it follows that $\sum\limits_{i \in N} \tau_i(b) = nK - \max\limits_{t' \in \{j : d_j(b) = 0\}} b_{t'} + \sum\limits_{i \in N} \min\{M(b_{-i}), \eta\}$ for all $b \in \Re^n_+$. Then imposing ZT and checking for profile $b^\eta \in \Re^n_{++}$ such that $b^\eta_t < \eta$ for all $t \in N$; we get that $\sum\limits_{i \in N} \tau_i(b^\eta) = (n-1)\min\{M(b^\eta), \eta\} > 0$ and hence, not feasible. This implies that any feasible NE and WGS mechanism satisfying ZT must have $\eta = 0$. Hence, 1 implies 2. □

Remark 3.3.10. From Proposition 3.3.4 it follows that the result in Proposition 3.3.5 continues to hold even if NE is replaced with AN.

3.3.2 Multiple Goods

We now discuss the case $n > k > 1$. Recall that for any vector of valuations $b \in \Re^n_+$ and for all $r = 1, 2 \ldots, n$, $b(r)$ is the rth ranked valuation in a non-increasing arrangement of the components of b (where ties are broken according to the rule $1 \succ \ldots \succ n$). Therefore, for any announced valuation profile b, any mechanism satisfying NE (as discussed in Proposition 3.3.1) requires that all agents in $\{j \in N | b_j = b(r), 1 \leq r \leq k\}$ get a good. In this section, for simplicity of notation we denote $b(r)$ as $b^{(r)}$.

Like in the single good case, we can easily show (by designing $\{k, k+1\}$-deviations as done in Proposition 3.3.2) that there does not exist any mechanism satisfying NE and SGS. Also, as in Proposition 3.3.3, we can show that any mechanism satisfying NE and strategyproofness must have the $g(.)$ function continuous over $\left[b^{(k)}_{-i-j}, b^{(k-1)}_{-i-j}\right)$.[4]

The structure of externality, however, is a little different for multiple goods. To see this, consider a profile of non-identical valuations. The externality imposed by the winners of the goods is not on the agent with second highest valuation, but on the agent with the $(k+1)$th highest valuation. Therefore, in this case, we get qualitatively similar class of mechanisms satisfying NE and WGS; albeit, with a different formulation of the $g(.)$ function, where $M(b_{-i})$ is replaced by $b_{-i}(k)$. This is given by the following theorem.

Theorem 3.3.11. If a mechanism (d, τ) satisfies NE and WGS then $\forall b \in \Re^n_+$,

[4] And so the $g(.)$ function is continuous over $\left[b^{(k)}_{-i-j}, \infty\right)$. In case $b^{(k)}_{-i-j}$ is undefined, we take it to be 0.

$\forall\, i \in N$,

$$\tau_i(b) = \begin{cases} K + \min\{\eta, b_{-i}(k)\} & \text{if } \ d_i(b) = 0 \\ K + \min\{\eta, b_{-i}(k)\} - b(k+1) & \text{if } \ d_i(b) = 1 \end{cases}$$

Proof: The proof is similar to that of Theorem 3.3.4. Pick any $\{i, j\} \subseteq N$ and any $b \in \Re^n_+$. Then as in lemma 3.3.5, we can show that (i) $g(x, b_{-i-j}) = \underline{C}'(b_{-i-j}), \forall\, x \in \left[0, b^{(k)}_{-i-j}\right)$, and (ii) $g(x, b_{-i-j}) = \tilde{C}'(b_{-i-j}), \forall\, x \geq b^{(k-1)}_{-i-j}$. Finally, as done in Theorem 3.3.3, we can show that (iii) $g(x, b_{-i-j}) = \bar{C}'(b_{-i-j}) + \min\{x, \eta(b_{-i-j})\}, \forall\, x \in \left[b^{(k)}_{-i-j}, b^{(k-1)}_{-i-j}\right)$ where $\eta(b_{-i-j}) \in [0, \infty]$.

Note that in this case, there can be jump discontinuity at two points $b^{(k)}_{-i-j}$ (when it is positive) and $b^{(k-1)}_{-i-j}$. As in Theorem 3.3.4, WGS rules out jump discontinuity at $b^{(k)}_{-i-j}$. To obtain same result for the other point $b^{(k-1)}_{-i-j}$, define

$$\begin{aligned} \bar{G}_{ij} := & \lim_{\{x^u\} \to b^{(k-1)}_{-i-j}+} g(x^u, b_{-i-j}) - \lim_{\{x^u\} \to b^{(k-1)}_{-i-j}-} g(x^u, b_{-i-j}) \\ = & \quad \tilde{C}'(b_{-i-j}) - \bar{C}'(b_{-i-j}) - \min\{b^{(k-1)}_{-i-j}, \eta(b_{-i-j})\} \end{aligned}$$

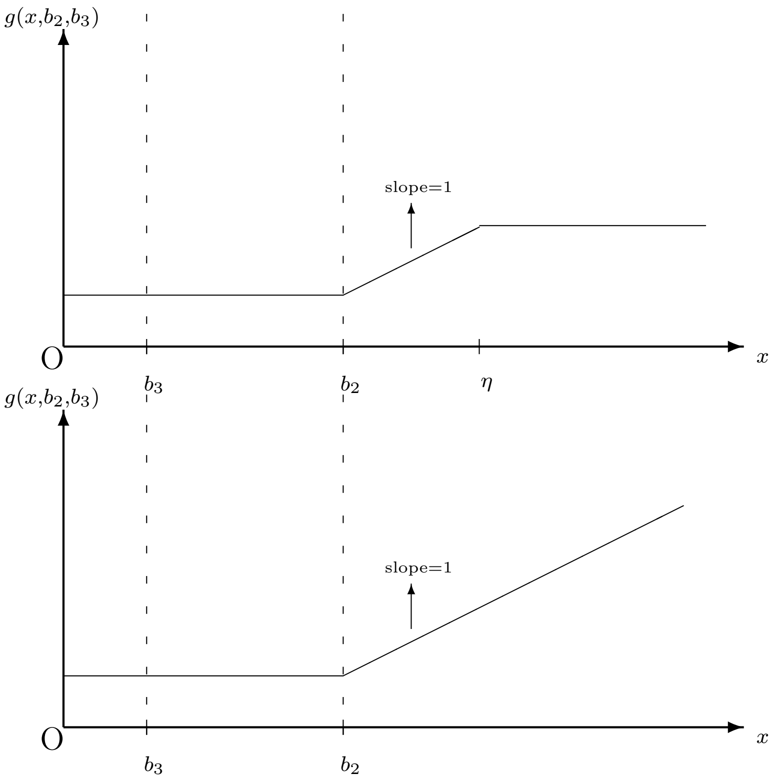
As before, jump discontinuity at $b^{(k-1)}_{-i-j}$ implies that $\bar{G}_{ij} \neq 0$.

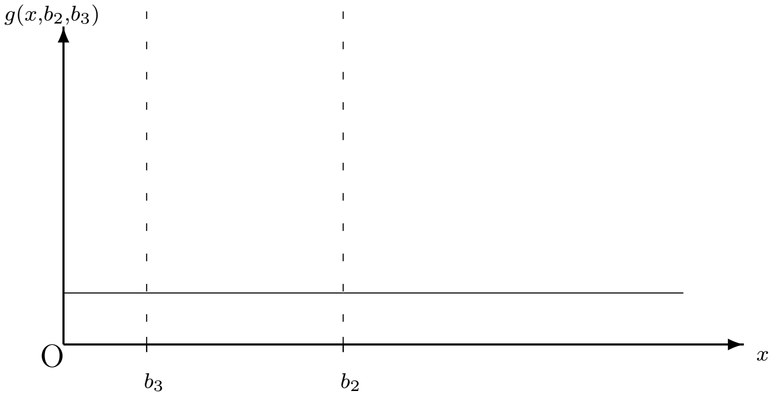
If $\bar{G}_{ij} < 0$ then either (a) $\eta(b_{-i-j}) < b^{(k-1)}_{-i-j}$ or (b) $\eta(b_{-i-j}) \geq b^{(k-1)}_{-i-j}$. If (a) holds then there exist b_i, b_j, b'_i, b'_j such that $\max\{\eta(b_{-i-j}), b^{(k)}_{-i-j}\} < b'_i < b'_i < b^{(k-1)}_{-i-j} < b_j < b_i$, $b_j - b^{(k-1)}_{-i-j} < |\bar{G}_{ij}|$ and $g_i(b'_j, b_{-i-j}) - g_i(b_j, b_{-i-j}) = g_j(b'_i, b_{-i-j}) - g_j(b_i, b_{-i-j}) = |\bar{G}_{ij}|$. Clearly, WGS is violated in the $\{i, j\}$-deviation from (b_i, b_j, b_{-i-j}) to (b'_i, b'_j, b_{-i-j}). If (b) holds then there exist b_i, b_j, b'_i, b'_j such that $b^{(k)}_{-i-j} < b'_j < b'_i < b^{(k-1)}_{-i-j} < b_j < b_i$, $g(b'_i, b_{-i-j}) - g(b_i, b_{-i-j}) > 0$, $g(b'_j, b_{-i-j}) - g(b_j, b_{-i-j}) > b_j - b^{(k-1)}_{-i-j}$. Again WGS is violated in $\{i, j\}$-deviation from (b_i, b_j, b_{-i-j}) to (b'_i, b'_j, b_{-i-j}). Finally, if $\bar{G}_{ij} > 0$ then there exist b_i, b_j, b'_i, b'_j such that $b^{(k)}_{-i-j} < b_j < b_i < b^{(k-1)}_{-i-j} < b'_j < b'_i$,

$b_{-i-j}^{(k-1)} - b_j < \bar{G}_{ij}$. The the $\{i, j\}$-deviation from (b_i, b_j, b_{-i-j}) to (b'_i, b'_j, b_{-i-j}) violates WGS.

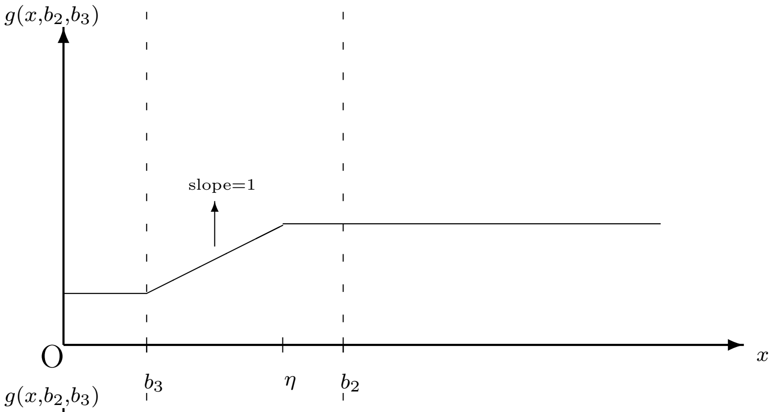
Therefore, WGS implies that $\bar{G}_{ij} = 0$ and so, there can be no discontinuity in $g(., b_{-i-j})$ map. Then, aggregating the functions (i), (ii) and (iii), as in Theorem 3.3.4, we arrive at the result. □

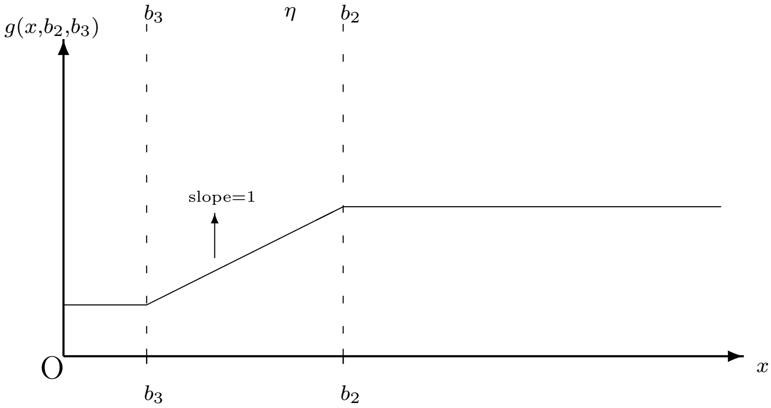
As in the single good case, the result in Theorem 3.3.11 would continue to hold if NE were replaced by AN. Also, Theorems 3.3.4 and 3.3.11 are qualitatively similar. This becomes clear on making a graphical comparison between the two results. Take for example the case where $n = 4$. If $k = 1$, the graph of Theorem 3.3.4 will look like one of the following maps (when $b_3 < b_2$);

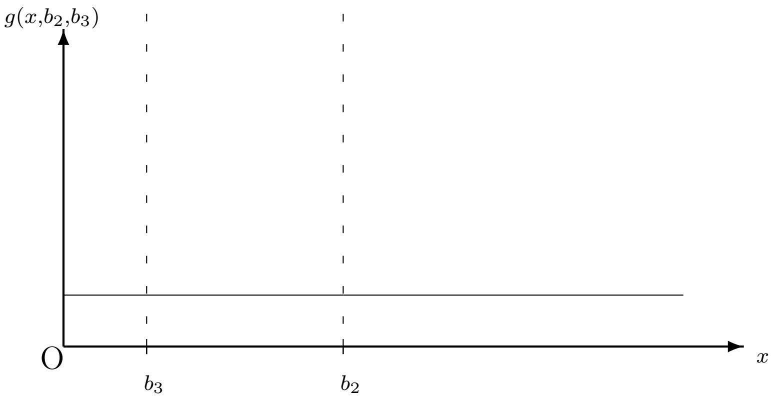




If $k = 2$, the graph of Theorem 3.3.11 is one of the following maps;







Note that for $k = 2$ case, there is no $g(., b_2, b_3)$ map similar to the second map in the panel of graphs for the case $k = 1$. This particular feature of multiple goods case, follows from the different externality structure in this setting.

3.4 Conclusion

We completely characterize the class of fair mechanisms satisfying weak group strategyproofness for the indivisible good allocation problem. We show that for both fairness criteria of no-envy and anonymity in welfare, we get the same class of mechanisms. Pivotal mechanism (with a constant term) turns out to be a member of this class. We use the zero transfer axiom and the feasibility requirement to prune this class down to Pivotal mechanism.

3.5 Appendix

W.l.o.g. consider a pair of profiles b, b' such that b' is an $\{1, 2\}$-profile of b with $b_1 \geq b_2 \geq M(b_{-1-2})$ and $b'_1 \geq b'_2 \geq M(b_{-1-2})$. Therefore $d_1(b) = d_1(b') = 1$ while $d_2(b) = d_2(b') = 0$.

Now, WGS with respect to $\{1,2\}$-deviation between the profiles b, b'; requires that [either $l_1(b,b') \geq 0$ or $l_2(b,b') \geq 0$] *and* [either $l_1(b',b) \geq 0$ or $l_2(b',b) \geq 0$]. Note that $l_t(b,b') = 0 \Rightarrow l_t(b',b) = 0$ for any $t = 1,2$. Therefore, if there exists a $t \in \{1,2\}$ such that $l_t(b,b') = 0$, then WGS among b, b' is trivially satisfied. We refer to this situation as WGS *holding with equality* for b and b'. There is also the possibility of WGS *holding with a strict inequality.* That is, *either* $[l_1(b,b') > 0$ and $l_2(b',b) > 0]$ *or* $[l_2(b,b') > 0$ and $l_1(b',b) > 0]$.

W.l.o.g. we analyze the first case where $l_1(b,b') > 0$ and $l_2(b',b) > 0$.

Proposition 3.5.1. If $l_1(b,b') > 0$ and $l_2(b',b) > 0$, then $\forall\, x \geq M(b_{-1-2})$,

$$g(x, b_{-1-2}) = \bar{C}(b_{-1-2}) + \min\{x, \eta(b_{-1-2})\}$$

where $\eta(b_{-1-2}) = g(b'_1, b_{-1-2}) - g(b_1, b_{-1-2}) + b_1$.

Proof: The conditions $l_1(b,b') > 0$ and $l_2(b',b) > 0$ imply that the following restrictions hold simultaneously

$$-b'_2 + g(b'_2, b_{-1-2}) < -b_2 + g(b_2, b_{-1-2}) \tag{3.4}$$

$$g(b_1, b_{-1-2}) < g(b'_1, b_{-1-2}) \tag{3.5}$$

The first step of the proof is to show that if we can find a pair of profiles b and b' such that (3.4) and (3.5) hold; then $M(b_{-1-2}) \leq b_2 \leq b_1 < b'_2 \leq b'_1$.[5] Henceforth, whenever obvious, we suppress the b_{-1-2} argument.

[5]If the other combination of inequalities $[l_2(b,b') > 0, l_1(b',b) > 0]$ were to be chosen, the direction of the inequalities in (3.4) and (3.5) would get reversed. However, even with this different combination of inequalities; using the same techniques, we could show that $M(b_{-1-2}) \leq b'_2 \leq b'_1 < b_2 \leq b_1$ and derive similar results to arrive at the same $g(.)$ maps.

Lemma 3.5.1. If equations (3.4) and (3.5) hold; then $M \leq b_2 \leq b_1 < b_2' \leq b_1'$.

Proof: As shown in Proposition 3.3.3, NE implies that $g(y) \leq g(x)$, $\forall\, x > y \geq M$. Hence, $g(., b_{-1-2})$ is a non-decreasing function for all values greater than or equal to M. Then (3.5) implies that $M \leq b_1 < b_1'$. By construction $b_2 \leq b_1$ and $b_2' \leq b_1'$. Now if $M \leq \max\{b_2, b_2'\} \leq b_1 < b_1'$, then (3.4) and (3.5) imply that $l_t(\alpha, \beta) < 0$, $\forall\, t = 1, 2$ where $\alpha \equiv (b_1, b_2', b_{-1-2})$ to $\beta \equiv (b_1', b_2, b_{-1-2})$, which violates WGS. So the only other possibility that remains is $M \leq b_2 \leq b_1 < b_2' \leq b_1'$. □

W.l.o.g. assume that $M < b_2 < b_1 < b_2' < b_1'$ (the same functional from would follow, no matter what combination of inequalities we take). We, now, show that if (3.4) and (3.5) hold, then $g(., b_{-1-2})$ function is kinked over the domain of values greater than or equal to M. Such a kink occurs at the point η in the interval (b_1, b_2'). Over this domain, the function turns out to be a straight line with slope of 1 for all values less than η and flat line after η.

Lemma 3.5.2. If equations (3.4) and (3.5) hold; then

1. $g(x) - g(x') = x - x', \forall\, x, x' \in [M, b_1]$

2. $g(x) = g(b_1'), \forall\, x \geq b_2'$

Proof: For any $x, x' \in [M, b_1]$; if $g(x) - x > g(x') - x'$, then from equation (3.5), WGS is violated in a deviation from true profile (b_1, x') to (b_1', x). Similarly, WGS is violated in a deviation from (b_1, x) to (b_1', x') if $g(x) - x < g(x') - x'$. Hence, statement 1 is proved. Now, equation (3.4) implies that for any $x, x' \geq b_2'$, if $g(x) < g(x')$ then $l_t((x, b_2'), (x', b_2)) < 0$, for both $t = 1, 2$.

Similarly, if $g(x') < g(x)$ then $l_t((x', b_2'), (x, b_2)) < 0, \forall\, t = 1, 2$. In both cases WGS is violated and so, statement 2 is proved. □

Taking the results in lemma 3.5.2 above, along with (3.4) and (3.5); we see that

$$0 < g(b_2', b_{-1-2}) - g(b_1, b_{-1-2}) < b_2' - b_1 \tag{3.6}$$

This (3.6) implies that $\exists\, \eta \in (b_1, b_2')$ such that $g(b_2') - g(b_1) = \eta - b_1$.[6] We now show that in the region (b_1, b_2'), the $g(., b_{-1-2})$ graph is upward sloping on the left of η and flat on the right.

Lemma 3.5.3. If equations (3.4) and (3.5) hold; then

1. $g(x) - g(x') = x - x', \forall\, x, x' \in (b_1, \eta)$
2. $g(x) = g(b_1'), \forall\, x \in (\eta, b_2')$

Proof: If $\exists\, x \in (\eta, b_2')$ such that $g(x) > g(b_1')$, then equation (3.4) implies that $l_t((b_1', b_2'), (x, b_2)) < 0$, $\forall\, t = 1, 2$. Therefore, (i) $g(x) \leq g(b_1')$, $\forall\, x \in (\eta, b_2')$. Now pick any $x, x' \in (\eta, b_2')$ with $x > x'$. From construction of η, statement 2 of lemma 3.5.2 and (i); we get that $g(b_1) - b_1 > g(x') - x'$ which in turn implies that $l_1((x, x'), (b_2', b_1)) < 0$. WGS, therefore, requires that $l_2((x, x'), (b_1', b_1)) \geq 0$, that is, $g(x) \geq g(b_1')$. The statement 2, then, follows from (i).

If $\exists\, x \in (b_1, \eta)$ such that $g(x) - g(b_2) > x - b_2$, then equation (3.5) implies that $l_t((b_1, b_2), (b_1', x)) < 0$, $\forall\, t = 1, 2$. Therefore, it must be that (ii) $g(x) - g(b_2) \leq x - b_2$, $\forall\, x \in (b_1, \eta)$. Statement 1 of lemma 3.5.2, then,

[6]In (3.6), the left hand inequality implies that $\eta > b_1$ while the right hand inequality implies that $\eta < b_2'$.

implies that $g(x) < g(b_1) + \eta - b_1$. Therefore, by construction of η and statement 2 of lemma 3.5.2, $g(x) < g(b'_2) = g(b'_1)$, $\forall\ x \in (b_1, \eta)$. Now, if (ii) holds with strict inequality then $l_t((x, x), (b'_1, b_2)) < 0$, $\forall\, t = 1, 2$ and so, contradiction to WGS. Hence, statement 1 must be true. $\square$

Proof of Proposition 3.5.1: Lemmas 3.5.1, 3.5.2 and 3.5.3, have been proved keeping all other agents' announcements fixed at b_{-1-2}. Hence the kink point η obtained may be a function of the b_{-1-2} vector.[7] Therefore, the continuity of $g(., b_{-1-2})$ map over the interval $[M(b_{-1-2}), \infty)$ implied by Proposition 3.3.3, completes the proof. $\square$

Proposition 3.5.1 shows that for any vector $b_{-1-2} \in \Re_+^{n-2}$; if there exist $b_1, b_2, b'_1, b'_2 \in \Re_+$ with **(a)** $b_1 \geq b_2 \geq M(.)$, $b'_1 \geq b'_2 \geq M(.)$ and **(b)** WGS holds with strict inequality between profiles $(b_1, b_2), (b'_1, b'_2)$; then $g(x, b_{-1-2}) = \bar{C}(b_{-1-2}) + \min\{x, \eta(b_{-1-2})\}$, $\forall\ x \geq M(.)$ where $\eta(b_{-1-2}) = g(b'_1, b_{-1-2}) - g(b_1, b_{-1-2}) + b_1$. But there remains the possibility that WGS never holds with strict inequalities. In other words, there may not exist any four real numbers satisfying the conditions **(a)** and **(b)**. That is, either $l_1((b_1, b_2, b_{-1-2}), (b'_1, b'_2, b_{-1-2})) = 0$ or $l_2((b_1, b_2, b_{-1-2}), (b'_1, b'_2, b_{-1-2})) = 0$ for all choices of non-negative b_1, b_2, b'_1, b'_2 such that **(a)** holds. The following proposition gives the $g(., b_{-1-2})$ map in this case, by analyzing those $\{1, 2\}$-deviations where both agents misreport (that is, $b_1 \neq b'_1$ and $b_2 \neq b'_2$).

Proposition 3.5.2. For all $\forall\ b, b' \in \Re_+^n$ such that b' is an $\{1, 2\}$-profile of b with $b_1 > b_2 > M(.)$ and $b'_1 > b'_2 > M(.)$; *if* $\exists\ t \in \{1, 2\}$ such that

[7] From Corollary 3.3.1 it follows that $g(.)$ must be independent of agent labels. Hence, η cannot depend on agent labels.

$l_t(b, b') = 0$, *then,*

$$\begin{aligned} &\text{either} \quad g(x, b_{-1-2}) = \bar{C}(b_{-1-2}), \forall\, x \geq M(.) \\ &\text{or} \qquad g(x, b_{-1-2}) = \bar{C}(b_{-1-2}) + x, \forall\, x \geq M(.) \end{aligned}$$

Proof: We claim that for any such pair of profiles b, b'; if $\exists\, t \in \{1, 2\}$ with $l_t(b, b') = 0$ then $l_t(b, b'') = 0$, $\forall\, b'' \neq b'$ with $M(.) \leq b_2'' \leq b_1''$. Suppose not. That is, suppose w.l.o.g. that there exists $b'' \equiv (b_1'', b_2'', b_{-1-2})$ such that $l_1(b, b') = 0$ but $l_1(b, b'') > 0$. Then, by assumption, (i) $l_2(b, b'') = 0$. Now there are three possibilities; either $l_2(b, b') > 0$ or $l_2(b, b') = 0$ or $l_2(b, b') < 0$.

If $l_2(b, b') > 0$ then (i) implies that $l_2(b'', b') > 0$. Also note that $l_1(b, b') = 0$ and $l_1(b, b'') > 0$ taken together imply that $l_1(b', b'') > 0$. Thus, we find a pair of profiles b', b'' such that WGS among them holds with a strict inequalities and hence, contradiction. Again, if $l_2(b, b') < 0$ then (i) implies that $l_2(b'', b') < 0$. As before, combining $l_1(b, b') = 0$ and $l_1(b, b'') > 0$, we get that $l_1(b'', b') < 0$ and hence, WGS is violated leading to contradiction.

Therefore, the only possible implication of $[l_1(b, b') = 0$ and $l_1(b, b'') > 0]$ that remains; is $l_2(b, b') = 0$. By assumption $b_1 \neq b_1'$ and $b_2 \neq b_2'$, and so, $l_1(b, b') = 0 \Rightarrow g(b_2, b_{-1-2}) \neq g(b_2', b_{-1-2})$, from which, as before, we can show that WGS requires that $g(x, b_{-1-2}) - g(x', b_{-1-2}) = x - x'$, $\forall\, x, x' \in (M(.), \min\{b_2, b_2'\})$. Similarly, $l_2(b, b') = 0 \Rightarrow g(b_1, b_{-1-2}) - g(b_1', b_{-1-2}) \neq b_1 - b_1'$, from which, as before, we can show that WGS requires that $g(x, b_{-1-2}) - g(x', b_{-1-2}) = 0$, $\forall\, x, x' > \max\{b_1, b_1'\}$. By assumption $M(.) < \min\{b_2, b_2'\} < \max\{b_1, b_1'\}$, and so, the $g(., b_{-1-2})$ graph is rising with a slope of 1 in the region $(M(.), \min\{b_2, b_2'\})$ while it is flat for all values greater than $\max\{b_1, b_1'\}$. From lemma 3.5.1, NE requires that the $g(., b_{-1-2})$ function be non-decreasing. Hence, $g(x, b_{-1-2}) > g(x', b_{-1-2})$, for all x, x' such that $M(.) < x' < \min\{b_2, b_2'\} \leq \max\{b_1, b_1'\} < x$. Define $\underline{m} := \min\{b_2, b_2'\}$ and $\bar{m} := \max\{b_1, b_1'\}$; and consider a deviation from $\alpha \equiv (\underline{m}-\epsilon, \underline{m}-2\epsilon, b_{-1-2})$ to $\beta \equiv (\bar{m}+2\epsilon, \bar{m}+\epsilon, b_{-1-2})$ where $\epsilon \in \left(0, \frac{\underline{m}-M(.)}{2}\right)$. It is easy to check that $l_t(\alpha, \beta) \neq 0$ for both $t = 1, 2$, and hence, contradiction.

Therefore, if $l_t(b, b') = 0$ then $l_t(b, b'') = 0, \forall\, b'' \neq b'$ with $M(.) \leq b_2'' \leq b_1''$. Hence, the result follows. □

Proof of Theorem 3.3.3: Proposition 3.5.1 establishes the result in the theorem for the finite positive values of the term $\eta(b_{-1-2})$. The possibility of $\eta(b_{-1-2}) = 0$ is captured by the result $g(x, b_{-1-2}) = \bar{C}(b_{-1-2}), \forall\, x \geq M(.)$ in Proposition 3.5.2. The possibility of $\eta(b_{-1-2}) = \infty$[8] is captured by the result $g(x, b_{-1-2}) = \bar{C}(b_{-1-2}) + x, \forall x \geq M(.)$ in Proposition 3.5.2. Since the pair of agents $\{1, 2\}$ and the b_{-1-2} vector were chosen arbitrarily, Propositions 3.5.1 and 3.5.2 prove the theorem. □

[8]With a slight abuse of notation

Chapter 4

Group Strategyproof Queueing

4.1 Introduction

A queueing problem involves a set of agents wanting to consume a service provided by one or many machines, and a set of machines who can only serve the agents sequentially (one by one). Such a problem with n agents and m machines has the following features: (i) each agent has exactly one job to complete using any one of these machines, (ii) each machine can process only one job at a time, (iii) the jobs are identical across agents so that for a given machine, they take the same time to get processed, (iv) the machines are non-identical so that for a given job, different machines may take different times to process.

This model captures a multitude of real life situations; a typical example would be the problem of provision of the quickest possible service to n customers waiting at m cashier windows. Similar situations arise in a printing press, truckload transportation, people waiting on ATM machines, amateur astronomers waiting to use public telescopes and whole host of other possibilities. Maniquet [28] discusses many other interesting applications of this problem in the single machine context. Apart from the aforementioned practical relevance, queueing models are also important from a theoretical point of view. Mitra and Sen [34] show that for any multiple heterogenous good allocation problem; if there exists a mechanism satisfying full efficiency and strategyproofness, then the underlying structure of the problem must be like that of a queueing problem. Such wide applicability of the queueing model has led to an extensive literature[1].

The server wants to ensure the efficiency in decision, that is, minimize the aggregate waiting cost of provision of the service to the agents. In case the waiting costs of agents is private information, this requires the agents to reveal their waiting costs to the server. But in doing so, they have the incentive to misreport so as to ensure a favorable outcome (distinct from the socially optimal one). Thus, the server runs into a problem of information extraction; and so needs to apply a cost revelation mechanism. Information extraction problems of this kind have been analyzed by Vickerey [47], Clarke [9] & Groves [16] leading to the formulation of VCG mechanisms, which are sufficient for decision efficiency and strategyproofness. For smoothly connected domains, Holmström [21] established the uniqueness of VCG mechanisms in this regard.

As in the previous chapter, we are interested in decision efficient mecha-

[1]Dolan [10], Suijs [42], Mitra [32], Moulin [35], Hashimoto and Saitoh [18], Kayi and Ramaekers [26], Maniquet [20], Chun [7], Katta and Sethuraman [25], Kar, Mitra and Mutuswami [24], Chun and Heo [8], Mitra and Mutuswami [33]

nisms which are immune to not only unilateral misreporting but also group misreporting. Our goal is to identify the class of continuous mechanisms that satisfy decision efficiency and group strategyproofness in the incomplete information queueing problem with multiple machines. Continuity, in this problem too[2], turns out to have a fairness interpretation as it is implied by equal treatment of equals. As before, strategyproofness is implied by group strategyproofness; and therefore, we identify the class of VCG mechanisms that satisfy group strategyproofness.

Mitra and Mutuswami [33] discuss the two variants of group strategyproofness, *weak* and *strong* in the setting of single machine queueing. As in Proposition 3.3.2 of Chapter 3; they show that there does not exist any strong group strategyproof mechanism that satisfies efficiency of decision in a single machine queueing context. They also argue that the notion of strong group strategyproofness presumes the ability of agents to arrange credible side payments, which is not reasonable as reporting honestly is a weakly dominant strategy for VCG mechanisms. Therefore, as in Chapter 3, we focus on weak group strategyproof mechanisms.

Mitra and Mutuswami [33] identify the class of linear mechanisms necessary for efficiency (in decision) and pairwise strategyproofness; and remark that for two or four agents, there is no budget balancing mechanism which satisfies the aforementioned properties. They also characterize the efficient, pairwise strategyproof, linear class of mechanisms with the fairness property of equal treatment of equals in a single machine queueing problem.

By restricting their attention to only linear mechanisms, Mitra and Mu-

[2]As in Proposition 3.3.3 of Chapter 3.

tuswami [33] implicitly assume linearity (implying absence of any kind of *kink*) and hence, continuity of the transfer maps. They restrict their characterization result to only pairwise misreports, that too, in a single machine setting. This chapter generalizes their results, by (i) allowing for multiple non-identical machines (ii) allowing groups to contain more than two agents while deviating, (iii) assuming no more than continuity, (iv) showing that the transfer maps may contain a kink[3], (v) showing that for two agents, decision efficiency and weak group strategyproofness rule out the possibility of discontinuity at more than one point on the transfer map and (vi) providing a general class of efficient and weak group strategyproof mechanisms (of which the k-Pivotal mechanisms are a special case of). A particular feature of the general multiple machine case is that more than one agents may have to wait same amount of time to get the service. In contrast, in the single machine special case, no two agents ever wait the same time to get the service. The generalizations (ii)-(vi), however, continue to hold in both cases.

When number of agents is two, efficiency and weak group strategyproofness are found to imply lower semi-continuity of the transfer schedule. By considering all possible deviations three possible cases emerge; (a) flat straight line, (b) positively sloped straight line, and (c) initially positively sloped but later flat straight line (a kink occurs in the map). We see that discontinuity in graph can only occur for case (c), that too at the kink point with the only possibility of a sudden fall in value at that point. Thus there can be at most one point of discontinuity in the transfer map, that too with both side limits being equal (at that point). Further, such a point of discontinuity, if present,

[3]This possibility is ruled out by the weak linearity axiom in Mitra and Mutuswami [33].

can only occur for one agent.

We provide a necessary condition for continuous mechanisms that satisfy decision efficiency and weak group strategyproofness. We then provide the class of kinked mechanisms which satisfy continuity, decision efficiency and weak group strategyproofness.

Section 4.2 states the model. The section 4.3.1 states the two agent results while the section 4.3.2 states the $n > 2$ results. Section 4.4 discusses the possible extensions. Section 4.5 states the conclusion and section 4.6 is the appendix.

4.2 Model

Let $N = \{1, \ldots, n\}$, $n \geq 2$ be the set of agents with identical jobs[4] and $M = \{1, \ldots, m\}$ be the set of machines. Each machine j is identified with a speed of $s_j \in (0, 1]$ which is the time taken by the machine to process one job. W.l.o.g. we assume $s_1 \leq s_2 \leq \ldots \leq s_m$. Similarly each agent i is identified with $\theta_i \in \Re_+$ which denotes the disutility incurred by i per unit of waiting time. Let $\theta = (\theta_i)_{i \in N}$ denote the profile of waiting costs and θ_{-i} denote the cost vector $(\theta_1, \ldots, \theta_{i-1}, \theta_{i+1}, \ldots, \theta_n)$. The cost of waiting on the machine

[4]W.l.o.g. the processing time of the job is assumed to be 1. If we assume private processing times then an agent's utility from the service depends *directly* on the announcements of processing times of other agents. Then strategyproofness is not possible and is typically replaced by "*implementation in ex-post equilibrium*", which is Bayesian incentive compatibility under all priors (Hain and Mitra [17]).

j, to agent i, in the position k is given by $ks_j\theta_i$.[5] Agents have quasilinear preferences over queue positions and money. So an agent i waiting on the machine j in the kth position with money $t_i \in \Re$ gets a utility $t_i - ks_j\theta_i$.

The objective of the planner is to achieve decision efficiency, which means that the n jobs need to be scheduled through m machines in such a way that the aggregate waiting cost is minimized. To attain decision efficiency, the planner needs to pick the smallest n numbers from the set of all possible waiting times $\{\{ks_j\}_{k\in N}\}_{j\in M}$, arrange them in a non-decreasing order and then assign these waiting times to the agents in such a way that decision efficiency is achieved. Let $(z(1), \ldots, z(n))$ denote the smallest n waiting times arranged in such an order.

Now, any ranking of n agents can be represented by a bijection $\hat{\sigma} : N \to N$ so that the agent i is ranked $\hat{\sigma}_i$. Let Σ be the set of all such bijections. Then, for any profile of waiting costs θ, the planner simply picks a efficient ranking of agents[6], $\sigma(\theta) = (\sigma_i(\theta))_{i\in N}$; and then assigns to each agent i a waiting time $z(\sigma_i(\theta))$. This efficient ranking is unique if and only if no two agents have the same waiting cost per unit time. To ensure that the efficient ranking be a single valued selection, a *tie-breaking* rule is required. A *strict order* $\succ$ is defined on N, for this purpose. This relation is used to break ties in

[5] Note that an agent is supposed to incur a cost of waiting until a machine ends processing its job; unlike in Mitra and Mutuswami [33]) where an agent incurs a cost of waiting *until* a machine starts processing it. This allows for the situations where an agent prefers the kth (some $k > 1$) position on the queue of a faster machine then the first position on the queue of a slower machine.

[6] For any $\theta \in \Re^n_+$, an efficient ranking satisfies the following; $\sigma(\theta) \in \text{argmin}_{\hat{\sigma}\in\Sigma} \sum_{i=1}^n \hat{\sigma}_i\theta_i$.

the following manner; if any two agents i and j have same waiting cost per unit time, then $\sigma_i(\theta) < \sigma_j(\theta)$ iff $i \succ j$. Also define for any cost profile θ, $P_i'(\theta) := \{k \in N | \sigma_k(\theta) > \sigma_i(\theta)\}$ and $P_i(\theta) := \{k \in N | \sigma_k(\theta) < \sigma_i(\theta)\}$. Therefore, $P_i'(\theta)$ and $P_i(\theta)$ denote the set of agents ranked after agent i and before agent i, respectively, in the efficient order.

Now, if the waiting costs are private information, agents will have the incentive to misreport. In such an incomplete information situation the planner has to design a mechanism to extract information. A mechanism associates to any profile of waiting costs $\theta \in \Re_+^n$, a tuple $(\hat{\sigma}(\theta), \tau(\theta)) \in N \times \Re^n$ where $\tau(\theta) = (\tau_i(\theta))_{i \in N}$ and $\hat{\sigma}(.) \in \Sigma$. Under this mechanism, any agent i gets the rank $\hat{\sigma}_i(\theta)$ and a transfer $\tau_i(\theta)$. The utility to agent i for any reported profile of costs θ is $u_i(\hat{\sigma}_i(\theta), \tau_i(\theta); \theta_i') = -z(\hat{\sigma}_i(\theta))\theta_i' + \tau_i(\theta)$, where θ_i' is the true waiting cost of the agent i. We assume that $\tau_i(0, 0, \ldots, 0) = 0$ *irrespective* of the tie-breaking rule chosen, $\forall\, i \in N$. This essentially means that transfers are independent of agent specific constants no matter what the tie-breaking rule.

Definition 4.2.1. A mechanism $(\hat{\sigma}, \tau)$ is *queue-effcient* (EFF) if $\forall\, \theta \in \Re_+^n$,

$$\hat{\sigma}(\theta) \in \text{ argmin}_{\tilde{\sigma} \in \Sigma} \sum_{i=1}^{n} z(\tilde{\sigma}_i)\theta_i$$

In other words, a mechanism $(\hat{\sigma}, \tau)$ is EFF if $\hat{\sigma}(\theta) = \sigma(\theta)$, $\forall\, \theta \in \Re_+^n$.

Definition 4.2.2. A mechanism $(\hat{\sigma}, \tau)$ is *strategyproof* if $\forall\, i \in N$, $\forall\, \theta_i, \theta_i' \in \Re_+$ and $\forall\, \theta_{-i} \in \Re_+^{n-1}$,

$$u_i(\hat{\sigma}_i(\theta_i, \theta_{-i}), \tau_i(\theta_i, \theta_{-i}); \theta_i) \geq u_i(\hat{\sigma}_i(\theta_i', \theta_{-i}), \tau_i(\theta_i', \theta_{-i}); \theta_i)$$

As in Chapter 3, we introduce the following notation to define weak group strategyproofness in this setting. For any $\theta, \theta' \in \Re^n_+$; θ' is an S-profile of θ if $\forall\, i \notin S$, $\theta_i = \theta'_i$, for any non-empty $S \subseteq N$. Cost profile θ' is said to be an order preserving S-profile of θ if $\forall\, i \in S$, $\hat{\sigma}_i(\theta) = \hat{\sigma}_i(\theta')$.

Definition 4.2.3. A mechanism $(\hat{\sigma}, \tau)$ is *weak group strategyproof* (WGS) if, $\forall\, \theta \in \Re^n_+$, $\forall\, S \subseteq N$, there exists an $i \in S$ such that

$$u_i(\hat{\sigma}_i(\theta), \tau_i(\theta); \theta_i) \geq u_i(\hat{\sigma}_i(\theta'), \tau_i(\theta'); \theta_i)$$

where θ' is an S-profile of θ.

Like in Chapter 3, WGS implies strategyproofness.

Result 4.2.1. An EFF mechanism (σ, τ) is strategyproof *if and only if* for all $\theta \in \Re^n_+$,

$$\tau_i(\theta) = -\sum_{j \neq i} z(\sigma_j(\theta))\theta_j + h_i(\theta_{-i}), \forall\, i \in N$$

Proof: Since the domain of cost profiles $\Re^n_+$ is convex, the result follows from Theorem 2 of Holmström [15]. □

Any EFF mechanism with transfers given by the above equation is known as a Vickrey-Clarke-Groves (VCG) mechanism.

Result 4.2.2. An EFF mechanism (σ, τ) is strategyproof *if and only if*

$$\tau_i(\theta) = -\sum_{j \in P'_i(\theta)} \left[z(\sigma_j(\theta)) - z(\sigma_j(\theta_{-i}))\right]\theta_j + g_i(\theta_{-i}), \forall\, i \in N$$

Proof: This result follows from Result 4.2.1 by substituting $h_i(\theta_{-i}) = \sum_{j \neq i} z(\sigma_j(\theta_{-i}))\theta_j + g_i(\theta_{-i})$. □

From the definition of WGS, it follows that WGS mechanisms are necessarily strategyproof. Thus we need to search the class of transfers given by Result 4.2.2 for a WGS transfer map. Effectively, the additional restriction of WGS imposes a structure on the $g_i(\theta_{-i})$ function.

Definition 4.2.4. An EFF and WGS mechanism (σ, τ) is *upper semi continuous* (USC) if $\forall\, i \in N$ the set $\{x \in \Re_+^{n-1} : g_i(x) \geq \alpha\}$ is closed in $\Re_+$, $\forall\, \alpha \in \Re_+$.

Definition 4.2.5. An EFF and WGS mechanism (σ, τ) is *lower semi continuous* (LSC) if $\forall\, i \in N$ the set $\{x \in \Re_+^{n-1} : g_i(x) \leq \alpha\}$ is closed in $\Re_+$, $\forall\, \alpha \in \Re_+$.

Definition 4.2.6. An EFF and WGS mechanism (σ, τ) is *continuous* if it is USC as well as LSC.

Definition 4.2.7. A mechanism $(\hat{\sigma}, \tau)$ satisfies *equal treatment of equals* if $\forall\, \theta \in \Re_+^n$, $\forall\, \{i, j\} \subseteq N$,

$$\theta_i = \theta_j \implies u_i(\hat{\sigma}_i(\theta), \tau_i(\theta); \theta_i) = u_j(\hat{\sigma}_j(\theta), \tau_j(\theta); \theta_j)$$

Definition 4.2.8. A mechanism $(\hat{\sigma}, \tau)$ is *feasible* if $\forall\, \theta \in \Re_+^n$,

$$\sum_{i \in N} \tau_i(\theta) \leq 0$$

4.3 Results

4.3.1 Two Players

Suppose $N = \{1, 2\}$. We discuss the two agent case, first, because it is the building block of the n agent result.

Theorem 4.3.1. In a 2 agent multiple machine queueing problem, an EFF mechanism (σ, τ) is WGS *if and only if* $\forall\, i, j \in \{1, 2\}, i \neq j$

- For some $\eta \in [0, \infty]$,

$$g_i(\theta_j) = \begin{cases} (z(2) - z(1)) \min\{\theta_j, \eta\} & \text{if } \theta_j \neq \eta \\ \alpha_i[(z(2) - z(1))\eta\,] & \text{if } \theta_j = \eta \end{cases}$$

- $\max\{\alpha_1, \alpha_2\} = 1$

Sketch of the Proof: Suppose $N = \{1, 2\}$. Pick any $\theta, \theta' \in \Re^2_+$ such that θ' is an order-preserving $\{1, 2\}$-profile of θ. W.l.o.g. assume[7] that $\sigma_1(\theta) = \sigma_1(\theta') = 1$, $\sigma_2(\theta) = \sigma_2(\theta') = 2$ (which implies that $\theta_1 \geq \theta_2$ and $\theta'_1 \geq \theta'_2$). Consider the possibility of $\{1, 2\}$-deviation from true profile θ to misreport θ'. WGS, then, requires that either

$$g_1(\theta_2) - g_1(\theta'_2) \geq [z(2) - z(1)][\theta_2 - \theta'_2] \tag{4.1}$$

or

$$g_2(\theta_1) - g_2(\theta'_1) \geq 0 \tag{4.2}$$

must be true.

Now, consider the possibility of $\{1, 2\}$-deviation from true profile θ' to misreport θ. WGS, then, requires that either

$$g_1(\theta_2) - g_1(\theta'_2) \leq [z(2) - z(1)][\theta_2 - \theta'_2] \tag{4.3}$$

or

$$g_2(\theta_1) - g_2(\theta'_1) \leq 0 \tag{4.4}$$

[7] In case of equality of reports, 1 is ranked before 2.

must be true.

If any of the equations above holds with equality, WGS is ensured by the mechanism irrespective of whether $\{1, 2\}$ deviates from θ to θ' or from θ' to θ. Such a situation will be referred to as *WGS holding with equality.* There is also the possibility that from each of two pairs of equations, one holds with strict inequality. WGS requires that the two equations thus chosen must have their inequalities in the opposite direction. Such a situation will be referred to as *WGS holding with strict inequality.* W.l.o.g. assume that equations (4.1) & (4.4) hold with inequality for some particular vectors θ and θ'. Thus rewriting them;

$$g_1(\theta_2) - g_1(\theta_2') > [z(2) - z(1)][\theta_2 - \theta_2'] \tag{4.5}$$

$$g_2(\theta_1) - g_2(\theta_1') < 0 \tag{4.6}$$

Now, either $z(2) - z(1) \neq 0$ or $z(2) - z(1) = 0$. We first prove the result for the former possibility, and then discuss the latter possibility in Remark 4.6.12 in Appendix.

The first step of the proof is to show that if we can find a pair of profiles θ and θ' such that WGS holds with strict inequality then $\theta_2 \leq \theta_1 < \theta_2' \leq \theta_1'$[8]. This is done in Claim 4.6.1 of Appendix. Then, in Claims 4.6.2-4.6.10, we show that the $g(.)$ maps for both agents have a kink in their graphs at some point $\eta \in [\theta_1, \theta_2']$. Only at this point there can be a discontinuity for at most one agent, that too with both side limits being equal. Further, in

[8]Had we taken the different combination of inequalities in equations (4.5) and (4.6); arguing similarly we could show that $\theta_2' \leq \theta_1' < \theta_2 \leq \theta_1$ and then the similar results would follow leading to the same $g(.)$ maps.

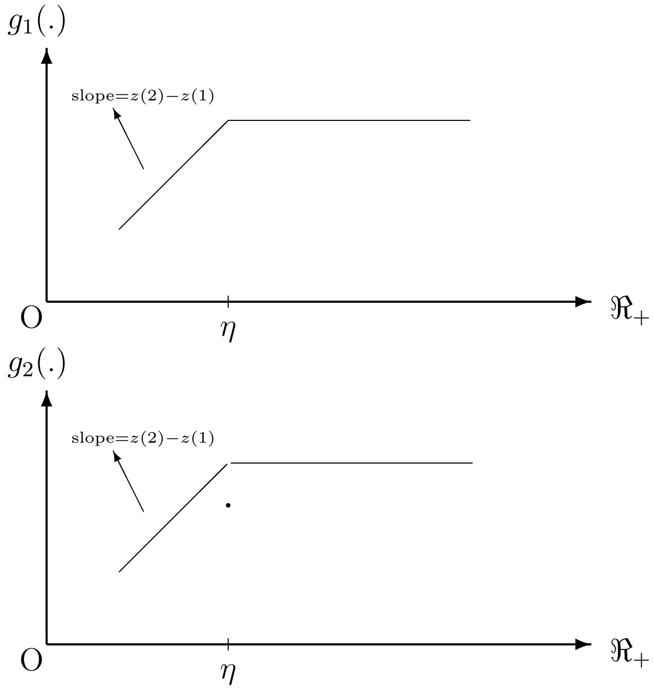


Figure 4.1:

Proposition 4.6.2, we show that if there is no pair of profile exhibiting WGS with strict inequality then either $\eta = \infty$ or $\eta = 0$. Finally the sufficiency is established. □

Figure 4.1 shows the graphical representation of Theorem 4.3.1 when $\eta \in (0, +\infty)$ and $\alpha_2 < 1$. The η value decides the position of the *kink* point while α value decides whether there is any discontinuity at the kink point or not. Also note that Theorem (4.3.1) implies that there cannot be discontinuity in both the $g(.)$ maps. We interpret the values of α and η in the following way;

1. If $\alpha_t = 1, \forall\, t = 1, 2$, then there is no discontinuity in either of the $g(.)$, no matter what the value of η.

2. If $\eta = 0$ then the $g(.)$ maps are horizontal straight lines along the x-axis, irrespective of the values of α.

3. If $\eta = \infty$ then the values of α become immaterial since then $g(.)$ maps will simply be upward sloping straight lines with the slope $z(2) - z(1)$.

Remark 4.3.2. In the single machine setting formalized by Mitra and Mutuswami [33] $z(2) - z(1) = 1$ and hence, the result in Theorem 4.3.1 result reduces to

$$g_i(\theta_j) = \begin{cases} \min\{\theta_j, \eta\} & \text{if } \theta_j \neq \eta \\ \alpha_i \eta & \text{if } \theta_j = \eta \end{cases}$$

with $\max\{\alpha_1, \alpha_2\} = 1$, for any $\theta_j \geq 0$, for some $\eta \in [0, \infty]$.

Remark 4.3.3. From Theorem 4.3.1 it follows that when the number of agents is 2; any mechanism will satisfy EFF and WGS only if the number of discontinuities in the mechanism does not exceed one. Moreover, any discontinuity, if present, can occur for only one of the two agents. Such discontinuity must occur at the kink point of the transfer map and must have equal both side limits.

Corollary 4.3.4. In a 2 agent multiple machine queueing problem, an EFF and WGS mechanism is *Lower Semi-Continuous.*

From the Corollary 4.3.4, it is obvious that in a two agent case, imposition of the USC property, leads to *Continuity* of the mechanism. Hence the following result;

Result 4.3.1. In a 2 agent multiple machine queueing problem, an EFF and USC mechanism (σ, τ) is WGS *if and only if*, $\forall\, \theta \in \Re^2_+$, $\forall\, i \neq j$,

$$g_i(\theta_j) = (z(2) - z(1)) \min\{\eta, \theta_j\}$$

where $\eta \in [0, \infty]$.

Note that the discontinuity in the two agent result is 'mild', in the sense that the both hand limits at the only possible point of discontinuity (the kink point), are equal. Also, if we impose the fairness requirement of equal treatment of equals on Theorem 4.3.1, the possibility of discontinuity gets eliminated (as $\alpha_1 = \alpha_2$) and we get the continuous mechanism specified by Result 4.3.1. Hence, the axiom of continuity has technical as well as fairness justifications in the multiple machine queueing problem.

4.3.2 n Players

Let $\Delta z(i) \overset{def}{=} z(i+1) - z(i), \forall\, i = 1, 2, \ldots, n-1$.

Theorem 4.3.5. In a multiple machine queueing problem, a continuous mechanism (σ, τ) is EFF and WGS only if $\forall\, i \in N$, $\forall\, \theta_{-i} \in \Re^{n-1}_+$,

$$g_i(\theta_{-i}) = \sum_{j \neq i} \Delta z(\sigma_j(\theta_{-i})) \min\{\theta_j, \eta_{ij}(\sigma(\theta_{-i}))\}$$

Proof: Suppose $N = \{1, 2, \ldots, n\}$. Pick a $\theta \in \Re^n_+$ such that $\sigma_1(\theta_{-2}) = \sigma_2(\theta_{-1})$, that is, agents 1 and 2 are adjacently ranked in the efficient order for profile θ. Note that while analyzing the impact of change in 2's report on the $g_1(.)$ function; we assumed that the announcements of other agents, that is, the vector θ_{-1-2} was constant. The impact of this θ_{-1-2} can be deemed to enter the $g_1(.)$ function through a constant intercept term $F_{12}(\theta_{-1-2})$. Also upfront, we cannot rule out the possibility of this $F_{12}(.)$ depending on $\sigma_2(\theta_{-1})$, that is, the rank of agent 2 when 1 is not around. We can then invoke the Result 4.3.1 to write that

$$\begin{aligned} g_1(\theta_{-1}) &= \Delta z(\sigma_2(\theta_{-1})) \min\{\theta_2, \eta_{12}(\sigma_2(\theta_{-1}); \theta_{-1-2})\} \\ &+ F_{12}(\sigma_2(\theta_{-1}); \theta_{-1-2}) \end{aligned} \tag{4.7}$$

Consider a profile $\theta' = (\theta'_1, \theta_{-1})$ such that $\sigma_1(\theta'_{-3}) = \sigma_3(\theta'_{-1})$, that is, agents 1 and 3 are adjacently ranked in the efficient order. Again invoking the Result 4.3.1, now for agents $\{1, 3\}$, we can write

$$\begin{aligned} g_1(\theta_{-1}) &= \Delta z(\sigma_3(\theta_{-1})) \min\{\theta_3, \eta_{13}(\sigma_3(\theta_{-1}); \theta_{-1-3})\} \\ &+ F_{13}(\sigma_3(\theta_{-1}); \theta_{-1-3}) \end{aligned} \tag{4.8}$$

The left hand side of both the above equations are the same. This means that the $F_{13}(.)$ must contain $\Delta z(\sigma_2(\theta_{-1})) \min\{\theta_2, \eta_{12}(.)\}$. This in turn implies that (i) the $\eta_{12}(.)$ function may contain $\sigma_3(\theta_{-1})$ as its argument, (ii) the $\eta_{12}(.)$ does not depend on θ_3, (iii) the $F_{13}(.)$ term may contain $\sigma_2(\theta_{-1})$ as its argument and (iv) the $F_{13}(.)$ does not depend θ_2 as argument.

Arguing similarly for the terms $F_{12}(.)$ and $\Delta z(\sigma_3(\theta_{-1})) \min\{\theta_3, \eta_{13}(.)\}$ in the right hand side of equations (4.7) and (4.8), respectively; we can write that

$$\begin{aligned} g_1(\theta_{-1}) &= \Delta z(\sigma_2(\theta_{-1})) \min\{\theta_2, \eta_{12}(\sigma_2(\theta_{-1}), \sigma_3(\theta_{-1}); \theta_{-1-2-3})\} \\ &+ \Delta z(\sigma_3(\theta_{-1})) \min\{\theta_3, \eta_{13}(\sigma_2(\theta_{-1}), \sigma_3(\theta_{-1}); \theta_{-1-2-3})\} \\ &+ F_{123}(\sigma_2(\theta_{-1}), \sigma_3(\theta_{-1}); \theta_{-1-2-3}) \end{aligned}$$

Continuing this recursion for all $j \neq 1$, we get that

$$g_1(\theta_{-1}) = \sum_{j \neq 1} \Delta z(\sigma_j(\theta_{-1})) \min\{\theta_j, \eta_{1j}(\sigma(\theta_{-1}))\} + C(\sigma(\theta_{-1}))$$

Note that $\tau_1(0, 0, \ldots, 0) = 0$ *irrespective* of what tie-breaking rule we use. Hence, it must be that $C(\sigma(\theta_{-1})) = 0$ and $\eta_{1j}(\sigma(\theta_{-1})) \in [0, \infty], \forall j \in N - \{1\}$ and $\forall\, \theta_{-1} \in \Re_+^{n-1}$. Arguing similarly, we can establish the result for all $i \in N$. $\square$

Remark 4.3.6. For the single machine setting formalized in Mitra and Mutuswami [33], $\Delta z(i) = 1, \forall\, i = 1, 2, \ldots, n-1$. Therefore the expression in Theorem 4.3.5 reduces to

$$g_i(\theta_{-i}) = \sum_{j \neq i} \min\{\theta_j, \eta_{ij}(\sigma(\theta_{-i}))\}$$

for any $\theta_{-i} \in \Re_+^{n-1}, \forall\, i \in N$.

Remark 4.3.7. If we focus only on weakly linear mechanisms, then $\eta_{ij}(s) \in \{0, \infty\}, \forall \{i, j\} \subseteq N$, for all possible ranking s among the players other than i (or j). It follows from Theorem 4.3.1 that $\eta_{ij}(s) = \eta_{ji}(s)$ for all i, j and s. We can also show that $\forall \{i, k\} \subseteq N$, $\theta_i > \theta_k \implies \eta_{ik}(\sigma(\theta_{-k})) \geq \eta_{ki}(\sigma(\theta_{-i}))$ for all $\theta \in \Re_+^n$ with $\theta_1 > \theta_2 > \ldots > \theta_n$.[9] Thus the result in Theorem 4.3.5 reduces to that of Theorem 3.7 in Mitra and Mutuswami [33]), when we consider weakly linear mechanisms *only*. This occurs because, in proving Theorem 4.3.5, we check only for 2 player coalitional deviations which is equivalent to "pairwise strategyproofness" in Mitra and Mutuswami [33].

The following is an example of a continuous transfer map which satisfies WGS and belongs to the class specified by Theorem 4.3.5.

[9]Suppose $\exists\, \theta$ such that $\theta_1 > \ldots > \theta_i > \theta_j > \theta_k > \ldots > \theta_n$ and $\eta_{ik}(\sigma(\theta_{-i})) = 0 < \eta_{ki}(\sigma(\theta_{-k})) = \infty$. Then WGS is violated in an order preserving $\{i, k\}$ deviation from θ to $(\theta'_i, \theta'_k, \theta_{-i-k})$ where $\theta'_i > \theta_i$ and $\theta'_k < \theta_k$.

Example 4.3.8. Let $N = \{1, 2, 3\}$ and $\Delta z(1) = \Delta z(2) = 1$. Therefore, the $\sigma(\theta_{-i})$ term in Theorem 4.3.5 can be either $(1, 2)$ or $(2, 1)$. Let $\eta_{12}(1, 2) = 0$ and $\eta_{ij}(1, 2) = \eta_{ij}(2, 1) = \infty$ for all $ij \neq 12$. Therefore, for any $\theta \in \Re^3_+$,

- if $\theta_1 > \theta_2 > \theta_3$ then $\tau_1(\theta) = -\theta_2, \tau_2(\theta) = 0, \tau_3(\theta) = \theta_1 + \theta_2$
- if $\theta_1 > \theta_3 > \theta_2$ then $\tau_1(\theta) = 0, \tau_2(\theta) = \theta_3, \tau_3(\theta) = \theta_1$
- if $\theta_2 > \theta_1 > \theta_3$ then $\tau_1(\theta) = 0, \tau_2(\theta) = -\theta_1, \tau_3(\theta) = \theta_1 + \theta_2$
- if $\theta_2 > \theta_3 > \theta_1$ then $\tau_1(\theta) = \theta_3, \tau_2(\theta) = 0, \tau_3(\theta) = \theta_2$
- if $\theta_3 > \theta_1 > \theta_2$ then $\tau_1(\theta) = \theta_3, \tau_2(\theta) = \theta_1 + \theta_3, \tau_3(\theta) = 0$
- if $\theta_3 > \theta_2 > \theta_1$ then $\tau_1(\theta) = \theta_2 + \theta_3, \tau_2(\theta) = \theta_3, \tau_3(\theta) = 0$

It can easily be checked that these continuous transfers along with decision efficiency satisfy weak group strategyproofness.

Complete characterization of the class of mechanisms that satisfy continuity, EFF and WGS; would require proving that the transfers in Theorem 4.3.5, when coupled with decision efficiency, satisfy weak group strategyproofness. This turns out to be difficult since the η terms depend on player labels as well as the $\sigma(\theta_{-i})$ vector. Instead, in the following theorem, we generate a class of continuous mechanisms that satisfy EFF and WGS, by allowing the η terms to depend only on the rank $\sigma_j(\theta_{-i})$ (instead of the vector of ranks $\sigma(\theta_{-i})$).

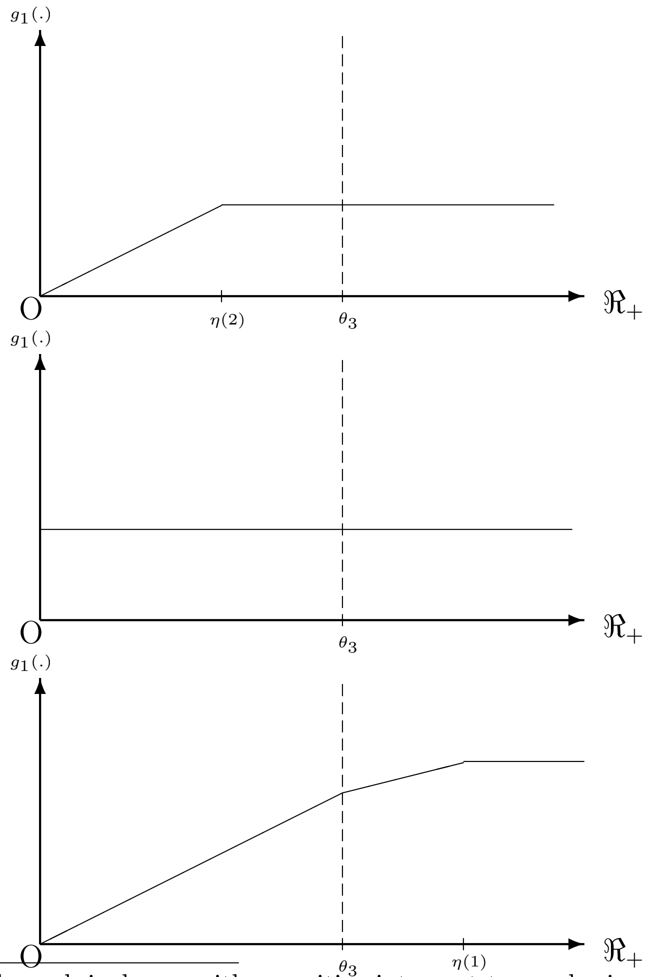
Theorem 4.3.9. In a multiple machine queueing problem, a continuous mechanism (σ, τ) satisfies EFF and WGS if $\forall\, i \in N$, $\forall\, \theta_{-i} \in \Re_+^{n-1}$,

$$g_i(\theta_{-i}) = \sum_{j \neq i} \Delta z(\sigma_j(\theta_{-i})) \min\{\theta_j, \eta(\sigma_j(\theta_{-i}))\} \quad (4.9)$$

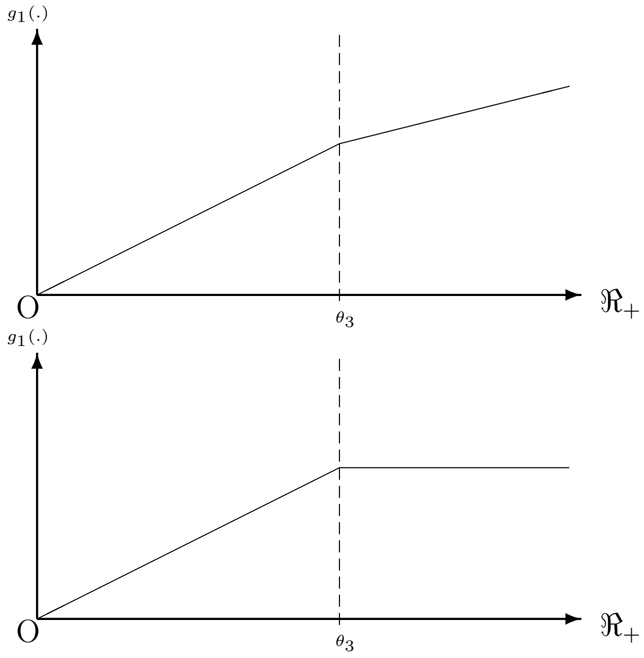
$$\eta(r+1) \geq \eta(r), \forall\, r = 1, 2, \ldots, n-2 \quad (4.10)$$

Proof: See Appendix. □

The following set of graphs capture the implication of the Theorem 4.3.9 when $n = 3$. Equations (4.9) and (4.10) imply that the three agent $g(.)$ maps must look like one of the following five figures[10].



[10]The second graph is drawn with a positive intercept to emphasize the flat curvature.



Remark 4.3.10. As before, in the single machine setting of Mitra and Mutuswami [33], equation (4.9) reduces to $g_i(\theta_{-i}) = \sum_{j \neq i} \min\{\theta_j, \eta(\sigma_j(\theta_{-i}))\}$. We can, then, easily show that the k-pivotal mechanism mentioned Mitra and Mutuswami [33]; is a special case of the mechanism given by Theorem 4.3.9. When

$$\eta(s) = \begin{cases} \infty & \text{if } s \geq k \\ 0 & \text{if } s < k \end{cases}$$

and $\Delta z(i) = 1, \forall i = 1, 2, \ldots, n-1$ (that is, in the single machine setting); the equation (4.9) reduces to the k-pivotal mechanism in for any $k = 1, 2, \ldots, n$. In fact, this arrangement η values gives us the k-pivotal mechanism for the general multiple machine case where $\forall\, \theta \in \Re^n_+$,

$$\tau_i^k(\theta) = \begin{cases} \sum_{s=\sigma_i(\theta)+1,\ldots,k} \Delta z(s-1)\theta(s) & \text{if } \sigma_i(\theta) < k \\ 0 & \text{if } \sigma_i(\theta) = k \\ \sum_{s=k,\ldots,\sigma_i(\theta)-1} \Delta z(s)\theta(s) & \text{if } \sigma_i(\theta) > k \end{cases}$$

where $\theta(s) := \{\theta_j | \sigma_j(\theta) = s\}$, $\forall\, s = 1, \ldots, n$.

Remark 4.3.11. An interesting subclass of the mechanisms described by Theorem 4.3.9 are the feasible mechanisms. Note that for all such mechanisms,

$$\begin{aligned} \sum_{i \in N} \tau_i(\theta) = &- \sum_{s=2,3,\ldots,n} (s-1)\Delta z(s-1)\theta(s) \\ &+ \sum_{s=1,2\ldots,n} [(n-s)\Delta z(s)\min\{\theta(s), \eta(s)\} + (s-1)\Delta z(s-1)\min\{\theta(s), \eta(s-1)\}] \end{aligned}$$

for all $\theta \in \Re^n_+$. It can easily be seen that feasible mechanisms in the class provided by Theorem 4.3.9 must have $\eta(1) = 0$. This means that; in the panel of graphs above, the third and fourth possibilities are ruled out.

4.4 Discussion

The formulation of the queueing problem in this chapter assumes that (i) machine speeds are constant across jobs, and (ii) per unit time waiting cost of each agent is constant over time. Relaxing assumption (i) would mean that each machine j is associated with a sequence $\left\{s_j^t\right\}_{t=1}^{\infty}$ where $s_j^t > 0$, $\forall\, t$. Any agent placed on the kth position in the queue for machine j would have to wait $\sum_{t=1}^{k} s_j^t$ to get his job completed. Since these speeds are known to the planner with certainty; the planner can choose the n smallest numbers out of the set $\left\{\left\{\sum_{t=1}^{k} s_j^t\right\}_{k=1}^{n}\right\}_{j=1}^{m}$ and arrange them in a non-decreasing order to get the $z \equiv (z(1), \ldots, z(n))$ vector. The rest of the analysis would remain same as above.

Relaxing the assumption (ii) is more difficult. If we measure time as a discrete variable and assume the waiting cost to vary with time t; we get that the cost to agent i upon being assigned the rank k is

$$\sum_{t=1}^{[z(k)]} \theta_i(t) + \{z(k) - [z(k)]\} \, \theta_i([z(k)] + 1)$$

where $[x]$ is the integer nearest to x but smaller than x, $\forall \, x > 0$. This leads to a different definition of decision efficiency and may well lead to different results. Identifying the necessary and sufficient conditions for mechanisms to satisfy EFF and WGS in this setting, would be an interesting but very difficult problem.

Similar kinked mechanisms may also be obtained in analysis of group strategyproofness in related fields like the indivisible good (single and multiple) allocation problem and the public good provision problem.

4.5 Conclusion

In this chapter, we analyze queueing problem with multiple machines and identical jobs. The crucial aspect here is how the $g_i(\theta_{-i})$ function (which can be any arbitrary function if we require only strategyproofness) behaves when we require weak group strategyproofness. We assume continuity, which is weaker than the *weak linearity* (unlike in Mitra and Mutuswami [33]) and this results in transfer maps having kinks. The continuity is not demanding in this structure from technical as well as fairness perspective. This is because, for two agents, any decision efficient and weak group strategyproof mechanism is *lower semi-continuous*; and any such mechanism satisfying equal treatment of equals, is continuous.

Our results show that if we restrict the $g_i(.)$ function to be continuous then it must be that (i) it is piecewise linear and (ii) if θ_j $(j \neq i)$ changes without changing the queue order then it cannot have a flat stretch followed by an increasing stretch. This feature prevails even in the single machine case. We also provide a class of mechanisms satisfying continuity, decision efficiency and weak group strategyproofness. The k-pivotal mechanisms in Mitra and Mutuswami [33], are a special case of this class in the single as well as multiple machine setting (Remark 4.3.10).

4.6 Appendix

Proof of Theorem 4.3.1: Define $l_i(\theta, \theta') := u_i(\hat{\sigma}_i(\theta), \tau_i(\theta); \theta_i) - u_i(\hat{\sigma}_i(\theta'), \tau_i(\theta'); \theta_i)$, $\forall\, i \in S$, $\forall\, S \subseteq N$, $\forall\, \theta, \theta' \in \Re^n_+$ such that θ' is an S-profile of θ. Therefore $l_i(\theta, \theta')$ captures the change in utility to member i of the misreporting coalition S as they deviate from (truth) profile θ to (misreport) profile θ'. We first prove the following Claims 2-10 and Propositions 1-2. Using these results, we prove the required necessity and sufficiency.

Claim 4.6.1. Consider an EFF & WGS mechanism (σ, τ). If WGS holds with strict inequality for θ, θ' (equations (4.5) and (4.6) hold); then $\theta_2 \leq \theta_1 < \theta'_2 \leq \theta'_1$.

Proof: Let us eliminate the other possibilities, namely the following five cases;

Case 1: $\theta_1' \geq \theta_2$, $\theta_1 \geq \theta_2'$

Design a $\{1,2\}$ deviation from $\alpha \equiv (\theta_1, \theta_2')$ to $\beta \equiv (\theta_1', \theta_2)$. By equations (4.5) and (4.6), $l_t(\alpha, \beta) < 0$, $\forall\, t = 1, 2$ and thus WGS is violated.

Case 2: $\theta_2' = \theta_1' < \theta_2 < \theta_1$

Design a $\{1,2\}$ deviation from $\beta \equiv (\theta_2, \theta_2')$ to $\alpha \equiv (\theta_1', \theta_2)$. By (4.5), $-z(2)\theta_2 + g_1(\theta_2) > -z(1)\theta_2 - [z(2) - z(1)]\theta_2' + g_1(\theta_2') \implies l_1(\beta, \alpha) < 0$. Thus $WGS \implies l_2(\beta, \alpha) \geq 0$ which means $-z(1)\theta_2' - [z(2) - z(1)]\theta_1' + g_2(\theta_1') \leq -z(2)\theta_2' + g_2(\theta_2) \implies$ (**a**) $g_2(\theta_1') \leq g_2(\theta_2)$. Now in a $\{1,2\}$ deviation from $\tilde{\beta} \equiv (\theta_1, \theta_2')$ to $\tilde{\alpha} \equiv (\theta_2, \theta_2)$; from (4.5), $l_1(\tilde{\beta}, \tilde{\alpha}) < 0$ and so $WGS \implies l_2(\tilde{\beta}, \tilde{\alpha}) \geq 0 \implies$ (**b**) $g_2(\theta_1) \geq g_2(\theta_2)$. Combining conditions (**a**) and (**b**) we get that $g_2(\theta_1) \geq g_2(\theta_1')$ which violates (4.6). Hence, contradiction.

Case 3: $\theta_2' = \theta_1' < \theta_2 = \theta_1$

In a $\{1,2\}$ deviation from $\alpha \equiv (\theta_1, \theta_2')$ to $\beta \equiv (\theta_1', \theta_2)$; (4.5) implies that $-z(2)\theta_2 + g_1(\theta_2) > -z(1)\theta_2 - [z(2) - z(1)]\theta_2' + g_1(\theta_2') \Rightarrow l_1(\alpha, \beta) < 0$. Similarly (4.6) implies that $g_2(\theta_1') - [z(2) - z(1)]\theta_1' - z(1)\theta_2' > g_2(\theta_1) - z(2)\theta_2'$ $\Rightarrow l_2(\alpha, \beta) < 0$ which violates WGS.

Case 4: $\theta_2' < \theta_1' < \theta_2 = \theta_1$

Design a $\{1,2\}$ deviation from $\beta \equiv (\theta_1, \theta_1')$ to $\alpha \equiv (\theta_1', \theta_2)$. (4.6) implies that $g_2(\theta_1') - z(1)\theta_1' - [z(2) - z(1)]\theta_1' > g_2(\theta_1) - z(2)\theta_1' \Rightarrow l_2(\beta, \alpha) < 0$. So $WGS \implies l_1(\beta, \alpha) \geq 0 \implies$ (**c**) $g_1(\theta_2) - g_1(\theta_1') \leq [z(2) - z(1)][\theta_2 - \theta_1']$. Now for an $\{1,2\}$ deviation from $\tilde{\beta} \equiv (\theta_1, \theta_2')$ to $\tilde{\alpha} \equiv (\theta_1', \theta_1')$; (4.6) implies that $l_2(\tilde{\beta}, \tilde{\alpha}) < 0$, and so $WGS \implies l_1(\tilde{\beta}, \tilde{\alpha}) \geq 0$ which implies that (**d**) $g_1(\theta_1') - g_1(\theta_2') \leq [z(2) - z(1)][\theta_1' - \theta_2']$. Then (4.5) minus (**d**) we get that $g_1(\theta_2) - g_1(\theta_1') > [z(2) - z(1)][\theta_2 - \theta_1']$ which contradicts (**c**). Hence, contradiction.

Case 5: $\theta_2' < \theta_1' < \theta_2 < \theta_1$

Consider four profiles (θ_2, θ_2'), (θ_1, θ_2), (θ_1, θ_2'), and (θ_2, θ_2). Given (4.5), if $g_2(\theta_2) < g_2(\theta_1)$ then a $\{1, 2\}$ coalition deviation from the first profile to the second makes both agents strictly better off; and if $g_2(\theta_2) > g_2(\theta_1)$ then a $\{1, 2\}$ coalition deviation from the third profile to the fourth leads to both agents being strictly better off. Therefore $WGS \implies$ (**e**) $g_2(\theta_2) = g_2(\theta_1)$. Now, for a pair of profiles $\alpha \equiv (\theta_2, \theta_1')$ and $\beta \equiv (\theta_1', \theta_2)$; from (4.6) and (**e**) it follows that $g_2(\theta_1') > g_2(\theta_2) \implies g_2(\theta_1') - [z(2) - z(1)]\theta_1' - z(1)\theta_1' > g_2(\theta_2) - z(2)\theta_1' \implies l_2(\alpha, \beta) < 0$. So $WGS \implies l_1(\alpha, \beta) \geq 0 \implies$ (**f**) $g_1(\theta_2) - g_1(\theta_1') \leq [z(2) - z(1)][\theta_2 - \theta_1']$.

Consider four profiles (θ_1, θ_2'), (θ_1', θ_1'), (θ_1, θ_1'), and (θ_1', θ_2'). If $g_1(\theta_1') - g_1(\theta_2') > [z(2) - z(1)](\theta_1' - \theta_2')$, then in a $\{1, 2\}$ deviation from the first profile to the second; from (4.6) it follows that both agents are strictly better off. Again if, $g_1(\theta_1') - g_1(\theta_2') < [z(2) - z(1)](\theta_1' - \theta_2')$, then in a deviation from the third profile to the fourth; from (4.6) it follows that both agents are strictly better off. Thus $WGS \implies g_1(\theta_1') - g_1(\theta_2') = [z(2) - z(1)](\theta_1' - \theta_2')$. Using (**f**), we can then say that $g_1(\theta_2) - g_1(\theta_2') \leq [z(2) - z(1)][\theta_2 - \theta_2']$, which then, contradicts (4.5). □

By Claim 1 we know that $\theta_2 \leq \theta_1 < \theta_2' \leq \theta_1'$. W.l.o.g., assume $\theta_2 < \theta_1 < \theta_2' < \theta_1'$ and continue the proof[11].

[11]This of course rules out the possibility that $\theta_1 = 0$. We will discuss the implications

Claim 4.6.2. Consider an EFF & WGS mechanism (σ, τ). If WGS holds with strict inequality for θ, θ' (equations (4.5) and (4.6) hold); then

A $\forall\, x, y \leq \theta_1,\; g_1(x) - g_1(y) = [z(2) - z(1)][x - y]$

B $\forall\, x, y < \theta_1,\; g_2(x) - g_2(y) = [z(2) - z(1)][x - y]$

C $\forall\, x, y \geq \theta_2',\; g_2(x) - g_2(y) = 0$

D $\forall\, x, y > \theta_2',\; g_1(x) - g_1(y) = 0$

E $\forall\, x \in (\theta_1, \theta_2'),\; g_2(x) \leq g_2(\theta_1')$

F If $\exists$ an $x \in (\theta_1, \theta_2')$ such that $g_2(x) < g_2(\theta_1')$, then

$$g_1(\theta_1) - g_1(y) = [z(2) - z(1)][\theta_2 - y], \forall\, y \in (\theta_1, x]$$

Proof:

A: For any $x, y \leq \theta_1, x \neq y$; consider the profiles $(\theta_1, x), (\theta_1', y)$ (θ_1, y), (θ_1', x). If $g_1(x) - g_1(y) > [z(2) - z(1)][x - y]$ then consider a $\{1, 2\}$ deviation from the third profile to the fourth; and if $g_1(x) - g_1(y) < [z(2) - z(1)][x - y]$ then consider a deviation from the first profile to the second. In both cases, by (4.6), WGS is violated.

B: Pick any x, y, ρ, x' such that $x < y < \rho < x' \leq \theta_1$. If $g_2(x) - g_2(y) > [z(2) - z(1)][x - y]$ then in a $\{1, 2\}$ deviation from $\beta \equiv (y, \rho)$ to $\alpha \equiv (x, x')$; $l_2(\beta, \alpha) < 0$. From the previous case **A**, $g_1(x') - g_1(\rho) > 0 \implies l_1(\beta, \alpha) < 0$ which violates WGS. If $g_2(x) - g_2(y) < [z(2) - z(1)][x - y]$, then in a $\{1, 2\}$ deviation from (x, ρ) to (y, x'), as before, WGS is violated.

of that possibility in Remark 4.6.11.

C: Pick any $x, y \geq \theta_2'$. If $g_2(x) > g_2(y)$ then in a $\{1,2\}$ deviation from $\beta \equiv (y, \theta_2')$ to $\alpha \equiv (x, \theta_2)$, given (4.5); $l_t(\beta, \alpha) < 0, \forall\, t = 1, 2$. If $g_2(x) < g_2(y)$ then in a deviation from (x, θ_2') to (y, θ_2), using (4.5), $l_t(\beta, \alpha) < 0, \forall\, t = 1, 2$. In both cases WGS is violated.

D: Pick any x, y, x', ρ such that $x > y > x' > \rho \geq \theta_2'$. From the case **C**, we get that $g_2(\rho) - g_2(x') = 0 > [z(2) - z(1)][\rho - x']$. Now, if $g_1(x) > g_1(y)$, then $l_t((x', y), (\rho, x)) < 0, \forall\, t = 1, 2$; and if $g_1(x) < g_1(y)$, then $l_t((x', x), (\rho, y)) < 0, \forall\, t$. In both cases WGS is violated.

E: Say $\exists\, x \in (\theta_1, \theta_2')$ such that $g_2(x) > g_2(\theta_1')$. Then in a deviation from profile $\alpha \equiv (\theta_1', \theta_2')$ to $\beta \equiv (x, \theta_2)$, $l_2(\alpha, \beta) < 0$; while by (4.5), $l_1(\alpha, \beta) < 0$. Thus WGS is violated.

F: For any $y \in (\theta_1, x]$, if $g_1(\theta_1) - g_1(y) > [z(2) - z(1)][\theta_1 - y]$ then $l_1(\alpha, \beta) < 0$ where $\alpha \equiv (x, y)$ to $\beta \equiv (\theta_1', \theta_1)$ while $g_2(x) < g_2(\theta_1') \implies l_2(\alpha, \beta) < 0$. If $g_1(\theta_1) - g_1(y) < [z(2) - z(1)][\theta_1 - y]$, then similarly, it can be shown that $l_t((x, \theta_1), (\theta_1', y)) < 0, \forall\, t = 1, 2$. □

The implications of all the subcases of the Claim 4.6.2 is depicted in the following set of Figures 4.2 - 4.7;

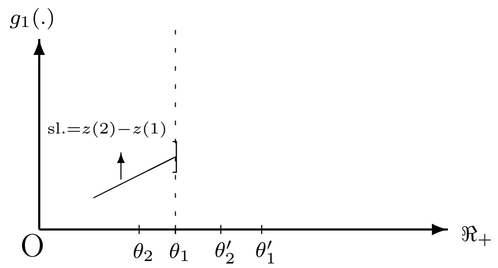


Figure 4.2: Claim 4.6.2A

Claim 4.6.3. Consider an EFF & WGS mechanism (σ, τ). If WGS holds

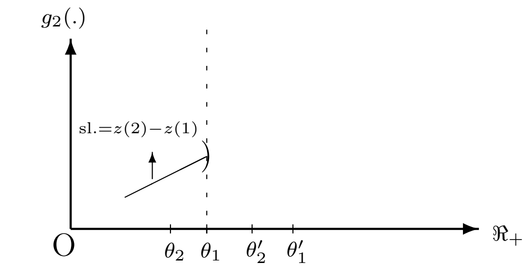


Figure 4.3: Claim 4.6.2B

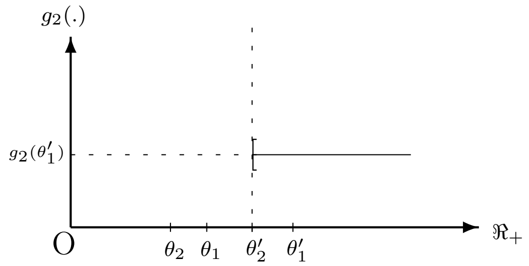


Figure 4.4: Claim 4.6.2C

with strict inequality for θ, θ' (equations (4.5) and (4.6) hold); then *either* of the following must be true,

i $g_1(y) = g_1(\theta_2'), \forall\, y \geq \theta_2'$

ii $g_1(y) = K > g_1(\theta_2'), \forall\, y > \theta_2'$ where K is some constant

Proof: If $\exists\; y > \theta_2'$ such that $g_1(y) < g_1(\theta_2')$ then $l_1(\alpha, \beta) < 0$ where $\alpha \equiv (x', y)$ and $\beta \equiv (\theta_1, \theta_2')$ with $x' \in (\theta_2', y)$. Then $WGS \implies l_2(\alpha, \beta) \geq 0 \implies g_2(\theta_1) \leq g_2(x') - [z(2) - z(1)][x' - \theta_1]$. Now, by Claim 4.6.2C; $g_2(x')$ is constant, $\forall\, x' \geq \theta_2'$. Since there is no upper bound on y and so, on x'; there can be no lower bound on the value $g_2(\theta_1)$. However, (4.6) implies that $g_2(\theta_1) > -g_2(\theta_1')$ and so, contradiction. Claim 4.6.2D completes the proof. $\square$

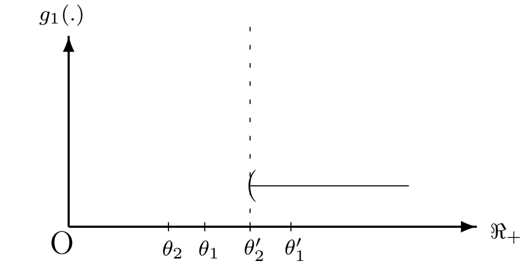


Figure 4.5: Claim 4.6.2D

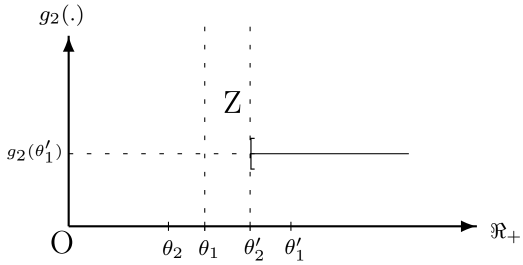


Figure 4.6: Claim 4.6.2E $\implies$ interior of zone Z is vacant

The graphical implication of the above Claim 4.6.3 is given by Figure 4.8. In the following Claim 4.6.4 we analyze the implication of Claim 4.6.3i.

Claim 4.6.4. Consider an EFF & WGS mechanism (σ, τ). If WGS holds with strict inequality for θ, θ' (equations (4.5) and (4.6) hold) and $g_1(y) = g_1(\theta'_2), \forall\, y \geq \theta'_2$ then

$$g_1(x) \leq g_1(\theta'_2), \forall\, x \in [\theta_1, \theta'_2]$$

Proof: If $\exists\ x \in [\theta_1, \theta'_2)$ such that $g_1(x) > g_1(\theta'_2)$ then consider a $\{1,2\}$ deviation from profile $\alpha \equiv (\delta, y)$ to $\beta \equiv (\theta_2, x)$ where $\theta'_2 \leq \delta < y$. Therefore $g_1(y) = g_1(\theta'_2) < g_1(x)$, which implies that $l_1(\alpha, \beta) < 0$. Then $WGS \implies l_2(\alpha, \beta) \geq 0 \implies g_2(\delta) - g_2(\theta_2) \geq [z(2) - z(1)][\delta - \theta_2]$. Note that there is no upper bound on y, and so, on δ; while from Claim 4.6.2C we know that the

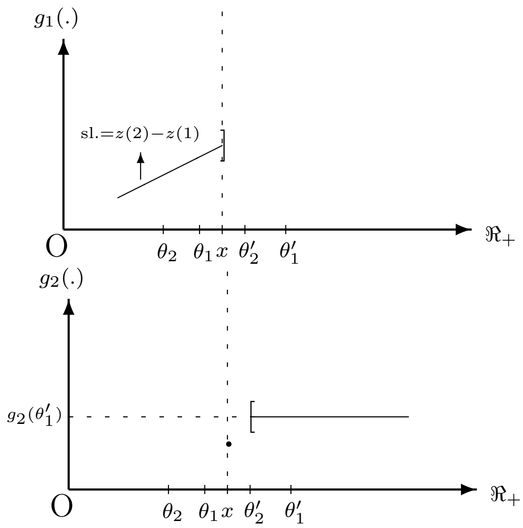


Figure 4.7: Claim 4.6.2F

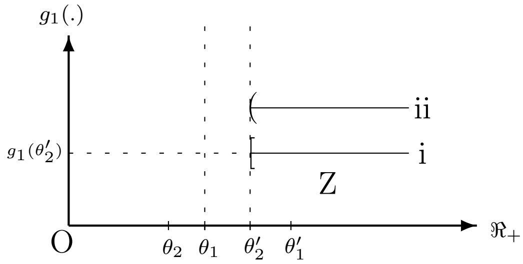


Figure 4.8: Claim 4.6.3 $\implies$ interior of zone Z is vacant $\implies$ either (i) or (ii) must hold

value $g_2(\delta)$ stays constant for all $\delta > \theta'_2$. Then, as in previous Claim 4.6.3, we cannot get a lower bound on the value $g_2(\theta_2)$ and so, contradiction. $\square$

The graphical implication of the above Claim 4.6.4 is given by Figure 4.9. Claim 4.6.4 states that if Claim 4.6.3i holds, then $g_1(\theta_1) \leq g_1(\theta'_2)$. Claims 4.6.5 and 4.6.6 state the consequences of $g_1(\theta_1) = g_1(\theta'_2)$; while Claims 4.6.7 and 4.6.8 state the consequences of $g_1(\theta_1) < g_1(\theta'_2)$.

Claim 4.6.5. Consider an EFF & WGS mechanism (σ, τ). If WGS holds

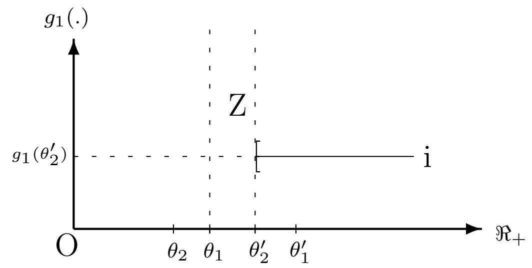


Figure 4.9: Claim 4.6.4 $\implies$ [Claim 4.6.3i $\implies$ interior of zone Z is vacant]

with strict inequality for θ, θ' (equations (4.5) and (4.6) hold) and

- $g_1(y) = g_1(\theta_2'), \forall\, y \geq \theta_2'$
- $g_1(\theta_1) = g_1(\theta_2')$

then $\forall\, t \in \{1, 2\}$,

$$g_t(x) = g_t(\theta_2'), \forall\, x \in (\theta_1, \theta_2')$$

Proof: If $\exists\, x \in (\theta_1, \theta_2')$ such that $g_2(x) < g_2(\theta_2')$, then $l_2(\alpha, \beta) < 0$ when $\alpha \equiv (x, x')$ and $\beta \equiv (\theta_2', \theta_1)$ with $x' \in (\theta_1, x)$. Since $x' \in (\theta_1, \theta_2')$, by Claim 4.6.4, $g_1(x') \leq g_1(\theta_2') = g_1(\theta_1) \implies g_1(\theta_1) - g_1(x') > [z(2) - z(1)][\theta_1 - x'] \implies l_1(\alpha, \beta) < 0$. Therefore $WGS \implies g_2(x) \geq g_2(\theta_2')$. Then from Claim 4.6.2E, it follows that (**a**) $g_2(x) = g_2(\theta_2'), \forall\, x \in (\theta_1, \theta_2')$.

Now, if $\exists\, x \in (\theta_1, \theta_2')$ such that $g_1(x) < g_1(\theta_2')$ then $l_1(\alpha', \beta') < 0$ where $\alpha' \equiv (\delta, x)$ and $\beta' \equiv (\epsilon, \theta_2')$ with $\theta_1 < \epsilon < \delta < x$. Since $\delta, \epsilon \in (\theta_1, \theta_2')$, by condition (**a**), we get $g_2(\epsilon) - g_2(\delta) = 0 > [z(2) - z(1)][\epsilon - \delta] \implies l_2(\alpha', \beta') < 0$. Thus $WGS \implies g_1(x) \geq g_1(\theta_2')$ which coupled with Claim 4.6.4 implies that $g_1(x) = g_1(\theta_2')$. □

The graphical implication of the above Claim 4.6.5 is given by Figure 4.10.

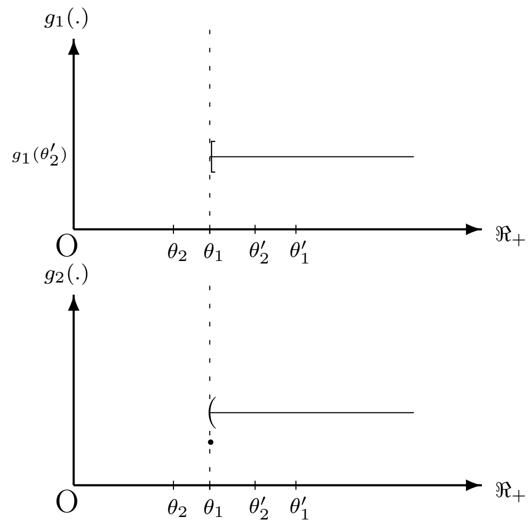


Figure 4.10: Claim 4.6.5

Claim 4.6.6. Consider an EFF & WGS mechanism (σ, τ). If WGS holds with strict inequality for θ, θ' (equations (4.5) and (4.6) hold) and

- $g_1(y) = g_1(\theta_2'), \forall\, y \geq \theta_2'$
- $g_1(\theta_1) = g_1(\theta_2')$

then $\forall\, t \in \{1, 2\}$,

$$\lim_{x \to \theta_1 -} g_t(x) = g_t(\theta_2')$$

Proof: If $\exists\, \nu \in [\theta_2, \theta_1)$ such that $g_2(\nu) > g_2(\theta_2')$, then $l_2(\alpha, \beta) < 0$ where $\alpha \equiv (\theta_2', \theta_2')$ and $\beta \equiv (\nu, \theta_2)$. From (4.5) it follows that $l_1(\alpha, \beta) < 0$.[12] Therefore $WGS \implies g_2(\nu) \leq g_2(\theta_2')$. Now by Claim 4.6.2B; $\forall\, \nu_2 < \nu_1 < \theta_1$, $g_2(\nu_1) > g_2(\nu_2)$. Thus $g_2(\nu) \leq g_2(\theta_2')$, $\forall\, \nu < \theta_1$ which in turn implies that $\lim_{x \to \theta_1 -} g_2(x) \stackrel{def}{=} T \leq g_2(\theta_2')$. Given Claim 4.6.2B, if $T < g_2(\theta_2')$ then $\exists\, \epsilon > 0$

[12]In case $\theta_2 = \theta_1$, we choose any $\nu, \nu' < \theta_1$ such that $\nu < \nu'$. Claim 4.6.2A and (4.5) imply that $l_1((\theta_2', \theta_2'), (\nu, \nu')) < 0$. The rest of the proof remains same.

such that $g_2(\theta_2') - g_2(x) > \epsilon$, $\forall\, x < \theta_1$. Then $\exists\, \delta < \theta_1$ and $\rho \in (\theta_1, \theta_2')$ such that $\rho - \delta < \frac{\epsilon}{(z(2)-z(1))}$. Therefore by Claim 4.6.5 $g_2(\rho) = g_2(\theta_2') \implies g_2(\rho) - g_2(\delta) > \epsilon > [z(2) - z(1)][\rho - \delta]$. This implies that $l_2(\alpha', \beta') < 0$ when $\alpha' \equiv (\delta, \xi)$ and $\beta' \equiv (\rho, \theta_2')$ with $\xi \in (\delta, \theta_1)$. Now, by Claim 4.6.2A, $g_1(\xi) < g_1(\theta_1) = g_1(\theta_2')$, since $\xi < \theta_1$. Therefore $l_1(\alpha', \beta') < 0$ and so $WGS \implies T = g_2(\theta_2')$.

For $t = 1$; from Claim 4.6.2A and the condition $g_1(\theta_1) = g_1(\theta_2')$ we get that $g_1(\theta_2') - g_1(x) = [z(2) - z(1)][\theta_2' - x]$. Therefore as x tends to θ_2', the result is established. □

Claims 4.6.2A, 4.6.2B, 4.6.2C, 4.6.5 & 4.6.6 imply that if Claim 4.6.3i holds true and $g_1(\theta_1) = g_1(\theta_2')$ then $g(.)$ map for each agent will look like in the Figure 4.11 where the kink point $\eta = \theta_1$. Note that there is only one point of discontinuity and that too for a single agent, here agent 2.

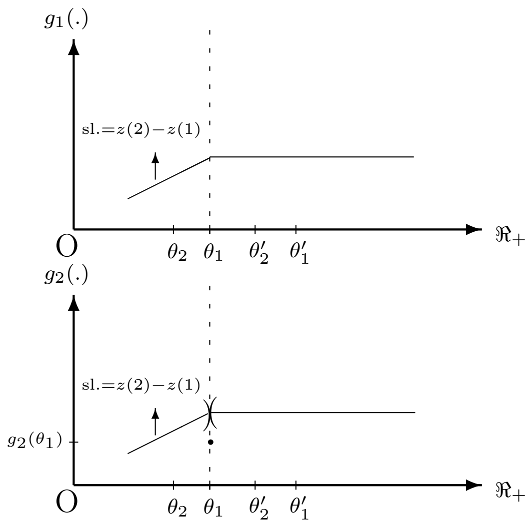


Figure 4.11: Claim 4.6.6

Let us now consider the other possible implication of the Claim 4.6.3i, that is, $g_1(\theta_1) < g_1(\theta_2')$. Given Claim 4.6.2A; (4.5) implies that $g_1(\theta_1) - g_1(\theta_2') > [z(2) - z(1)][\theta_1 - \theta_2']$. Since $g_1(\theta_1) < g_1(\theta_2')$; we can extend the straight line with slope $z(2) - z(1)$ passing through the $(\theta_2, g_1(\theta_2))$ point in the $g_1(.)$ map; on the right side of θ_1, to find a number $\omega \in (\theta_1, \theta_2')$ such that[13]

$$g_1(\theta_2) + [z(2) - z(1)][\omega - \theta_2] = g_1(\theta_2') \tag{4.11}$$

By Claim 4.6.4 and equation (4.11) above;

$$g_1(\theta_2) - g_1(x) > [z(2) - z(1)][\theta_2 - x], \forall\, x > \omega \tag{4.12}$$

Claim 4.6.7. Consider an EFF & WGS mechanism (σ, τ). If WGS holds with strict inequality for θ, θ' (equations (4.5) and (4.6) hold) and

- $g_1(y) = g_1(\theta_2'), \forall\, y \geq \theta_2'$
- $g_1(\theta_1) < g_1(\theta_2')$

then,

A $g_t(x) = g_t(\theta_2'), \forall\, x \in (\omega, \theta_2'], \forall\, t \in \{1, 2\}$

B $g_t(\theta_2) - g_t(x) = [z(2) - z(1)][\theta_2 - x], \forall\, x < \omega, \forall\, t \in \{1, 2\}$

C $g_t(\omega) = g_t(\theta_2')$ for some $t = 1, 2$

Proof:

A: For $t = 2$; pick any x, x' such that $\omega < x' < x < \theta_2'$. Then by (4.12), $l_1(\alpha, \beta) < 0$ when $\alpha \equiv (x, x')$ and $\beta \equiv (\theta_2', \theta_2)$. Therefore, by Claim 4.6.2E , $WGS \implies$ (**a**) $g_2(x) = g_2(\theta_2'), \forall\, x \in (\omega, \theta_2')$.

[13] Note that we are not assuming continuity; but simply extending the line continuously to locate the value ω.

For $t = 1$; if $\exists\, x \in (\omega, \theta_2')$ such that $g_1(x) < g_1(\theta_2')$ then consider a deviation from $\alpha' \equiv (\rho, x)$ and $\beta' \equiv (\delta, \theta_2')$ where $\omega < \delta < \rho < x$. By condition (**a**); $g_2(\delta) - g_2(\rho) = 0 > [z(2) - z(1)][\delta - \rho] \implies l_2(\alpha', \beta') < 0$. Therefore by Claim 4.6.4, $WGS \implies g_1(x) = g_1(\theta_2'), \forall\, x \in (\omega, \theta_2')$.

B: For $t = 2$, from Claim 4.6.2B it follows that the statement is satisfied for any $x < \theta_1$. Now, if $g_2(\theta_2) - g_2(\theta_1) > [z(2) - z(1)][\theta_2 - \theta_1]$ then $l_2(\alpha, \beta) < 0$ when $\alpha \equiv (\theta_1, \psi)$ and $\beta \equiv (\theta_2, \theta_2')$, where $\psi \in (\theta_1, \omega)$. Hence, by Claim 4.6.4, $WGS \implies g_1(\psi) = g_1(\theta_2'), \forall\, \psi \in (\theta_1, \omega)$. This coupled with (4.11), implies that $g_1(\psi) - g_1(\theta_2) > [z(2) - z(1)][\psi - \theta_2] \implies l_1(\beta', \alpha') < 0$ where $\alpha' \equiv (\theta_1', \psi)$ and $\beta' \equiv (\theta_1, \theta_2)$. Then from (4.6), it follows that $l_2(\beta', \alpha') < 0$ and so WGS is violated. Thus $WGS \implies g_2(\theta_2) - g_2(\theta_1) \leq [z(2) - z(1)][\theta_2 - \theta_1]$. If this equation holds with strict inequality then $l_2(\alpha'', \beta'') < 0$ when $\alpha'' \equiv (\theta_2, \theta_1)$ and $\beta'' \equiv (\theta_1, \theta_2')$ while $g_1(\theta_1) < g_1(\theta_2') \implies l_1(\alpha'', \beta'') < 0$. Therefore $WGS \implies$ (**b**) $g_2(\theta_2) - g_2(\theta_1) = [z(2) - z(1)][\theta_2 - \theta_1]$. □

We now show that (**b**) holds even if θ_1 is replaced by any real number lying in the open interval (θ_1, ω). If $\exists$ a $\psi \in (\theta_1, \omega)$ such that $g_1(\psi) = g_1(\theta_2')$ then (4.11) implies that $g_1(\psi) - g_1(\theta_1) > [z(2) - z(1)](\psi - \theta_1)$ and so, from (4.6); $l_t((\theta_1, \theta_1), (\theta_1', \psi)) < 0, \forall\, t = 1, 2$. Hence, by Claim 4.6.4 , $WGS \implies$ (**c**) $g_1(\psi) < g_1(\theta_2')$. But from (**c**) it follows that $l_1((\theta_2, \psi), (x, \theta_2')) < 0$ where $x \in (\theta_1, \psi)$; and so $WGS \implies$ (**d**) $g_2(\theta_2) - g_2(x) \geq [z(2) - z(1)][\theta_2 - x]$. Now, if (**d**) holds with strict inequality then equation (**c**) $\implies l_t((x, \psi), (\theta_2, \theta_2')) < 0, \forall\, t = 1, 2$ which violates WGS. Therefore (**d**) must hold with equality. Using (**b**) and Claim 4.6.2B, then, we complete the proof for $t = 2$.

For $t = 1$, pick any ψ, ϵ such that $\theta_1 < \psi < \epsilon < \omega$. As proved in the paragraph above, we can say that $g_2(\epsilon) - g_2(\psi) = [z(2) - z(1)][\epsilon - \psi] > 0$

which implies that $l_2((\psi, x), (\epsilon, \theta_1)) < 0$ where $x \in (\theta_1, \psi)$. Then $WGS \implies$ $(\mathbf{e}) g_1(x) - g_1(\theta_1) \geq [z(2) - z(1)][x - \theta_1]$. Now, if $(\mathbf{e})$ holds with strict inequality then $l_1((x, \theta_1), (\epsilon, x)) < 0$. Again, from the statement proved in the previous paragraph $\epsilon > x \implies g_2(\epsilon) > g_2(x) \implies l_2((x, \theta_1), (\epsilon, x)) < 0$ and so WGS is violated. Therefore $(\mathbf{e})$ must hold with equality. Claim 4.6.2A, then, completes the proof for $t = 1$. □

C: If $g_1(\omega) < g_1(\theta_2')$ then from (4.11) it follows that $g_1(\omega) < g_1(\theta_2') \implies g_1(\theta_2) - g_1(\omega) > [z(2) - z(1)][\theta_2 - \omega] \implies l_1((\omega, \omega), (\theta_2', \theta_2)) < 0$ and so by Claim 4.6.2E, $WGS \implies g_2(\omega) = g_2(\theta_2')$. Therefore, given Claim 4.6.4, we can say that there always exists a $t' \in \{1, 2\}$ such that $g_{t'}(\omega) = g_{t'}(\theta_2')$. □

The graphical implications of Claim 4.6.7 are given by Figure 4.12.

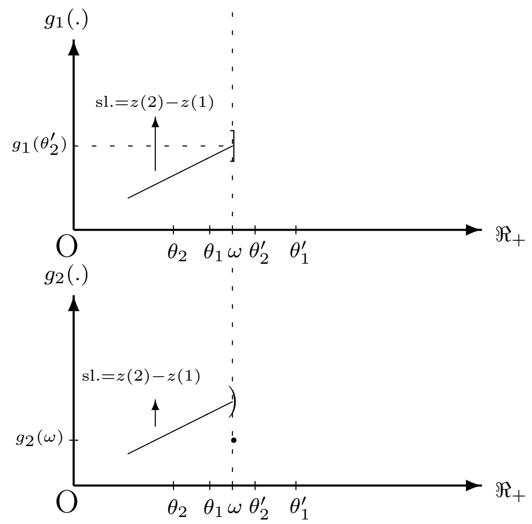


Figure 4.12: Claim 4.6.7 with $g_1(\omega) = g_1(\theta_2')$

Claim 4.6.8. Consider an EFF & WGS mechanism (σ, τ). If WGS holds with strict inequality for θ, θ' (equations (4.5) and (4.6) hold) and

- $g_1(y) = g_1(\theta_2'), \forall\, y > \theta_2'$

- $g_1(\theta_1) < g_1(\theta_2')$

then $\forall\, t = 1, 2$,

$$\lim_{x \to \omega-} g_t(x) = g_t(\theta_2')$$

Proof: From Claim 4.6.7C, w.l.o.g. assume that $g_1(\omega) = g_1(\theta_2')$. Then from (4.11) and Claim 4.6.7B we get that $g_1(\theta_2') - g_1(x) = [z(2) - z(1)][\omega - x]$, $\forall\, x \leq \omega$. Therefore as x tends to ω, the result is established for $t = 1$.

For $t = 2$; notice that Claim 4.6.2E and Claim 4.6.7B imply that $g_2(x) \leq g_2(\theta_2')$, $\forall\, x < \theta_2'$. Thus $\lim_{x \to \omega-} g_2(x) \stackrel{def}{=} T' \leq g_2(\theta_2')$. As in Claim 4.6.6, the possibility of $T' < g_2(\theta_2')$ can be ruled out. □

Claims 4.6.2C, 4.6.7 and 4.6.8 imply that if Claim 4.6.3i holds true with $g_1(\theta_1) < g_1(\theta_2')$ then discontinuity in the $g(.)$ maps, if present, shall occur only at a single point (at the kink point) and for at most one agent out of the two. The $g(.)$ map for each agent will look like in the Figure 4.13 where the kink point $\eta = \omega \in (\theta_1, \theta_2')$.

Now that the consequences of two possible implications of Claim 4.6.3i have been analyzed, let us move to the implications of Claim 4.6.3ii.

Claim 4.6.9. Consider an EFF & WGS mechanism (σ, τ). If WGS holds with strict inequality for θ, θ' (equations (4.5) and (4.6) hold); and $g_1(y) = K > g_1(\theta_2')$, $\forall\, y > \theta_2'$, then

A $g_t(x) - g_t(x') = [z(2) - z(1)][x - x']$, $\forall\, x, x' < \theta_2'$, $\forall\, t \in \{1, 2\}$

B $g_2(\theta_2') - g_2(x) = [z(2) - z(1)][\theta_2' - x]$, $\forall\, x \leq \theta_2'$

Proof:

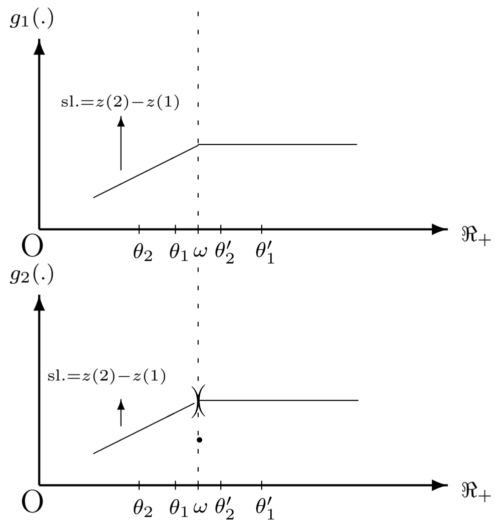


Figure 4.13: Claim 4.6.8 with $g_1(\omega) = g_1(\theta_2')$

A: For $t = 2$, pick any $x, x' < \theta_2'$. If $g_2(x) - g_2(x') > [z(2) - z(1)][x - x']$, then consider the deviation from $l_2((x', \theta_2'), (x, y)) < 0$, where $y > \theta_2'$. Also $g_1(y) = K > g_1(\theta_2') \implies l_1((x', \theta_2'), (x, y)) < 0$. Hence $WGS \implies g_2(x) - g_2(x') \leq [z(2) - z(1)][x - x']$. But if this holds with strict inequality then again $l_t((x, \theta_2'), (x', y)) < 0$, $\forall\, t = 1, 2$. Thus $WGS \implies$ (**a**) $g_2(x) - g_2(x') = [z(2) - z(1)][x - x']$.

For $t = 1$, pick any ν, ϵ such that $\epsilon < \nu < \theta_2'$ and any $x, x' < \epsilon$. By (**a**), $g_2(\nu) - g_2(\epsilon) = [z(2) - z(1)][\nu - \epsilon] > 0$. By checking the deviation from (ϵ, x') to (ν, x) and then the deviation from (ϵ, x) to (ν, x'), we see that WGS is violated unless $g_1(x') - g_1(x) = [z(2) - z(1)][x' - x]$. □

B: Pick any ε, y such that $\theta_2' < \varepsilon < y$. By assumption, $g_1(y) = K > g_1(\theta_2') \implies l_1((\theta_2, \theta_2'), (\varepsilon, y)) < 0$. Then $WGS \implies$ (**a**) $g_2(\theta_2) - g_2(\varepsilon) \geq [z(2) - z(1)][\theta_2 - \varepsilon]$, $\forall\, \varepsilon > \theta_2'$. Also by Claim 4.6.2C, $g_2(\theta_2') = g_2(\varepsilon)$, which coupled with (**a**) implies that $g_2(\theta_2) - g_2(\theta_2') \geq [z(2) - z(1)][\theta_2 - \varepsilon]$. Since ε

was chosen arbitrarily, this equation must hold for all $\varepsilon > \theta_2'$. This implies that (**b**) $g_2(\theta_2) - g_2(\theta_2') \geq [z(2) - z(1)][\theta_2 - \theta_2']$.[14]

Now, if (**b**) holds with strict inequality, then there exists a $\zeta < \theta_2'$ such that $g_2(\zeta) > g_2(\theta_2')$[15]. Therefore $l_2((\theta_2', \theta_2'), (\zeta, x')) < 0$ where $x' < \zeta$. From the previous case **A** (for $t = 1$) and (4.5), $l_1((\theta_2', \theta_2'), (\zeta, x')) < 0$. Thus WGS requires that condition (**b**) hold with equality. This along with case **A** (for $t = 2$) completes the proof. □

The graphical exposition of this Claim 4.6.9 is given by Figure 4.14.

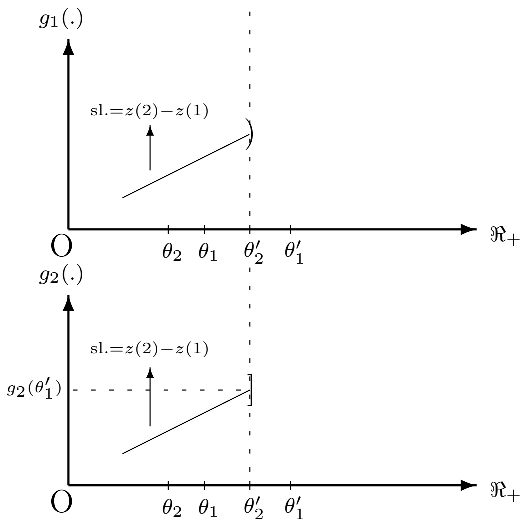


Figure 4.14: Claim 4.6.9

Claim 4.6.10. Consider an EFF & WGS mechanism (σ, τ). If WGS holds with strict inequality for θ, θ' (equations (4.5) and (4.6) hold); and $g_1(y) =$

[14]Suppose not, that is, $g_2(\theta_2) - g_2(\theta_2') = [z(2) - z(1)][\theta_2 - \theta_2'] - \nu$ for some $\nu > 0$; then we can find some $\varepsilon' \in \left(\theta_2', \theta_2' + \frac{\nu}{z(2)-z(1)}\right)$ such that **(a)** is violated.

[15]If $g_2(\zeta) \leq g_2(\theta_2'), \forall \zeta < \theta_2'$, then by invoking the previous case **A** for $t = 2$, we get that $g_2(\theta_2) - g_2(\zeta) = (\theta_2 - \zeta) \implies g_2(\theta_2) - g_2(\theta_2') \leq (\theta_2 - \zeta), \forall \zeta < \theta_2'$. Then as ζ tends to θ_2', in limit this violates *condition* (**b**) *holding with strict inequality.*

$K > g_1(\theta_2'), \forall\, y > \theta_2'$, then $\forall\, t = 1, 2$,

$$\lim_{x \to \theta_2'-} g_t(x) = K$$

Proof: Given Claim 4.6.9B, as $\{x\} \to \theta_2'$, the result is established for $t = 2$. For $t = 1$; if $\exists\, x \in (\theta_1, \theta_2')$ such that $g_1(x) > K$ then as in Claim 4.6.4, WGS is violated. Again from Claim 4.6.9A, $\forall\, \zeta < \theta_2'$, $g_1(.)$ is an increasing in ζ. Therefore $g_1(x) \leq K, \forall\, x < \theta_2' \implies \lim_{x \to \theta_2'-} g_1(x) \stackrel{def}{=} T'' \leq K$. If $T'' < K$ then as in Claim 4.6.6, we can design a violation of WGS. □

Claims 4.6.2C, 4.6.9 & 4.6.10 imply that if $g_1(y) = K > g_1(\theta_2'), \forall\, y > \theta_2'$, then there is only one point of discontinuity (at the kink point $\eta = \theta_2'$) and that too for a single agent, here agent 1 (as shown in Figure 4.15).

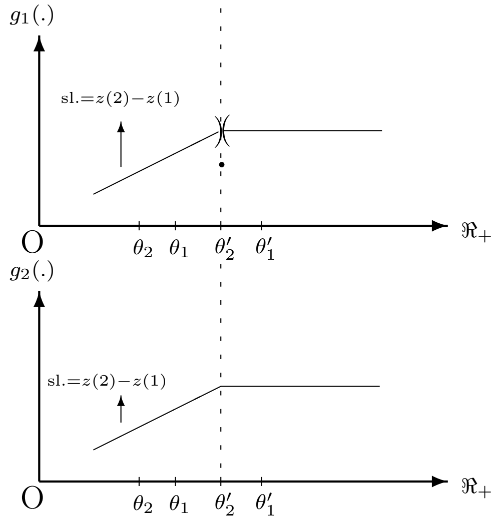


Figure 4.15: Claim 4.6.10

Thus we come to our first proposition.

Proposition 4.6.1. Consider an EFF & WGS mechanism (σ, τ). If WGS holds with strict inequality for θ, θ' (equations (4.5) and (4.6) hold), then

- $\forall\, i \neq t \in \{1, 2\}$,

$$g_i(\theta_t) = \begin{cases} (z(2) - z(1)) \min\{\theta_t, \eta\} & \text{if } \theta_t \neq \eta \\ \alpha_i[(z(2) - z(1))\eta] & \text{if } \theta_t = \eta \end{cases}$$

  for some $\eta \in [\theta_1, \theta_2']$

- $\max\{\alpha_1, \alpha_2\} = 1$.

Proof: Claims 4.6.1-4.6.10 prove the proposition. □

Proposition 4.6.2. Consider an EFF & WGS mechanism (σ, τ). If $\nexists\, \theta, \theta' \in \Re^2_+$ such that θ' is an order preserving $\{1, 2\}$-profile of θ and WGS holds with strict inequality for θ, θ' (equations (4.5) and (4.6) hold); then either

$$g_t(x) = 0, \forall\, x \in \Re_+, \forall\, t \in \{1, 2\}$$

or

$$g_t(x) = [z(2) - z(1)]x, \forall\, x \in \Re_+, \forall\, t \in \{1, 2\}$$

Proof: Pick any $\theta \in \Re^2_+$ and any two order preserving $\{1, 2\}$-profiles of θ; θ' and θ''. W.l.o.g. assume that 1 precedes 2 in the efficient order for all three profiles. From the statement of the proposition; WGS *holds with equality* for both pairs θ, θ' and θ, θ''. W.l.o.g. suppose that for the pair θ, θ', (**ai**) $g_1(\theta_2) - g_1(\theta_2') = [z(2) - z(1)](\theta_2 - \theta_2')$ and (**aii**) $g_2(\theta_1) - g_2(\theta_1') > 0$; while for the pair θ, θ'', (**bi**) $g_1(\theta_2) - g_1(\theta_2'') > [z(2) - z(1)](\theta_2 - \theta_2'')$ and (**bii**) $g_2(\theta_1) - g_2(\theta_1'') = 0$. Now, (**ai**) & (**bi**) $\implies$ $g_1(\theta_2') - g_1(\theta_2'') > [z(2) - z(1)](\theta_2 - \theta_2'') \implies l_1(\theta'', \theta') < 0$ while (**aii**) & (**bii**) $\implies$ $g_2(\theta_1'') > g_2(\theta_1') \implies l_2(\theta'', \theta') > 0$. Therefore WGS *holds with strict inequality* for the pair θ', θ'' and hence, contradiction. In the same way, for any other combination of $>$

and $<$ amongst the inequalities (**aii**) and (**bi**); either WGS is violated or WGS *holds with strict inequality*. Therefore if (**ai**) holds with equality then (**bi**) holds with equality; and if (**aii**) holds with equality then (**bii**) holds with equality. Since $\theta, \theta', \theta''$ are chosen arbitrarily, this means that $\forall\, x, y \in \Re_+$ either (**c**) $g_1(x) - g_1(y) = [z(2) - z(1)][x - y]$ or (**d**) $g_2(x) = g_2(y)$[16].

If (**c**) holds, then it can be shown, as in Claim 4.6.2B that $\forall\, x', y' \in \Re_+$ $g_2(x') - g_2(y') = [z(2) - z(1)][x' - y']$. If (**d**) holds then it can be shown, as in Claim 4.6.2D that $\forall\, x', y' \in \Re_{++}$ $g_1(x') = g_1(y')$. There remains a possibility that $\lim_{x \to 0+} g_1(x) \stackrel{def}{=} \bar{T}_1 \neq g_1(0) = 0$. Pick any ζ, ζ' such that $0 < \zeta' < \zeta$. Now if $\bar{T}_1 > 0$ then $g_1(\zeta) > 0 \implies l_1((0,0),(\zeta',\zeta)) < 0$ while from (**d**) we know that $g_2(\zeta') = g_2(0) \implies l_2((0,0),(\zeta',\zeta)) > 0$. If $\bar{T}_1 < 0$ then $g_1(\zeta) < g_1(0) \implies l_1((\zeta',\zeta),(0,0)) < 0$ while $g_2(\zeta) = g_2(0) \implies l_2((\zeta',\zeta),(0,0)) > 0$. Thus in both cases WGS holds with strict inequality and hence, contradiction. Therefore, $\bar{T}_1 = 0$ which implies that $g_t(x) = 0, \forall\, x \geq 0, \forall\, t = 1, 2$. □

Proof of Only If in Theorem 4.3.1: If a pair of order preserving $\{1,2\}$-profiles exists such that WGS amongst them holds with strict inequality then from Proposition 4.6.1 we can obtain the $g(.)$ maps. This is captured by the expression in theorem, when η takes a finite positive value. If for all pairs of order preserving $\{1,2\}$-profiles, WGS holds with equality, then the corresponding

[16] It may be that WGS *holds with equality* in such a way that $g_1(\theta_2) - g_1(x) = [z(2) - z(1)][\theta_2 - x]$ & $g_2(\theta_1) - g_2(y) = 0$ where $x \in \{\theta_2', \theta_2''\}$ and $y \in \{\theta_1', \theta_1''\}$ respectively. Then arguing as above we could say that $\forall\, m, n > 0$, $g_1(m) - g_1(n) = [z(2) - z(1)][m - n]$ and $g_2(m) - g_2(n) = 0$. But then it is easy to check that $l_i((m,n),(m-\epsilon, n+\epsilon)) < 0, \forall i = 1, 2$ when $0 < \epsilon < m < n$.

$g(.)$ maps are given by Proposition 4.6.2. The expression in theorem captures the two possible implications of this possibility; when $\eta = 0$ or when (with a slight abuse of notation) $\eta = \infty$. □

Proof of If in Theorem 4.3.1: We need to show that the necessary result obtained above, ensures that WGS is not violated for any possible $\{1, 2\}$ deviations, *order preserving* or otherwise. All the logical arguments involved in establishing Propositions 4.6.1 & 4.6.2 are reversible. Therefore, given the expression of $g(.)$ maps in the theorem; if there existed any order preserving $\{1, 2\}$ deviation such that WGS is violated, then either of the two propositions mentioned above, would have been violated. Thus the $g(.)$ maps are sufficient to ensure WGS for order preserving $\{1, 2\}$ deviations.

Since there are only two agents, there is only one other possible type of $\{1, 2\}$ deviations; *order interchanging* deviations. Pick any such deviation, say, from $\beta \equiv (\zeta_1, \zeta_2)$ to $\alpha \equiv (\rho_1, \rho_2)$ where $\rho_1, \rho_2, \zeta_1, \zeta_2$ are any four arbitrary non-negative numbers such that (w.l.o.g.) $\rho_1 \geq \rho_2$ and $\zeta_1 < \zeta_2$. Therefore, 1 precedes 2 in the efficient order for α while 2 precedes 1 in the efficient order for β. Then **(i)** $l_1(\beta, \alpha) = [z(2) - z(1)][\rho_2 - \zeta_1] - g_1(\rho_2) + g_1(\zeta_2)$ and **(ii)** $l_2(\beta, \alpha) = [z(2) - z(1)][\zeta_2 - \zeta_1] + g_2(\zeta_1) - g_2(\rho_1)$. It will be shown that for any possible values of the arbitrarily chosen four numbers; there exists one agent $t^* \in \{1, 2\}$ such that t^* is not strictly better off in a deviation from β to α.

Now, there are two possible cases, namely;

Case A: $\rho_2 < \eta$

Now if $\zeta_2 < \eta$ then $g_1(\rho_2) - g_1(\zeta_2) = [z(2) - z(1)][\rho_2 - \zeta_2]$ which means that equation **(i)** $\Longrightarrow\ l_1(\beta, \alpha) = [z(2) - z(1)][\zeta_2 - \zeta_1] > 0$. Hence $t^* = j$.

If $\zeta_2 \geq \eta$ then $g_1(\rho_2) - g_1(\zeta_2) = [z(2) - z(1)][\rho_2 - \eta] \Longrightarrow\ l_1(\beta, \alpha) = [z(2) - z(1)](\eta - \zeta_1)$. If $\zeta_1 \leq \eta$ then $t^* = 1$. If $\zeta_1 > \eta$, then $l_1(\beta, \alpha) < 0$; but $g_2(\zeta_1) = [z(2) - z(1)]\eta$ and the fact that $g_2(\rho_1) \leq [z(2) - z(1)]\eta$ imply that $g_2(\zeta_1) - g_2(\rho_1) \geq 0$. From **(ii)**, it then follows that $l_2(\beta, \alpha) \geq [z(2) - z(1)][\zeta_2 - \zeta_1] > 0$ and so $t^* = 2$.

Case B: $\rho_2 \geq \eta$

If $\zeta_2 < \eta$ then $g_1(\rho_2) - g_1(\zeta_2) = [z(2) - z(1)][\eta - \zeta_2] \Longrightarrow\ l_1(\beta, \alpha) > 0$ and so $t^* = 1$.

If $\zeta_2 \geq \eta$ then $g_1(\rho_2) - g_1(\zeta_2) = 0 \ \Longrightarrow\ l_1(\beta, \alpha) = [z(2) - z(1)][\rho_2 - \zeta_1]$. Now if $\zeta_1 \leq \rho_2$ then $t^* = 1$. If $\zeta_1 > \rho_2$ then $l_1(\beta, \alpha) < 0$; but $\zeta_1 > \rho_2 \geq \eta$ and so, as in the previous case $g_2(\zeta_1) - g_2(\rho_1) \geq 0 \Longrightarrow\ l_2(\beta, \alpha) \geq [z(2) - z(1)][\zeta_2 - \zeta_1] > 0$ and so $t^* = 2$.

Thus, given the $g(.)$ maps, no order interchanging $\{1, 2\}$ deviation violates WGS. □

Remark 4.6.11. It is possible that in the (4.5); $\theta_1 = 0$[17]. In that case all the Claims other than 4.6.2A, 4.6.2B and 4.6.6 will go through. Since $g_1(0) = 0$; in case Claim 4.6.3i holds with $g_1(\theta_1) = g_1(\theta_2')$, the $g_1(.)$ map is a horizontal straight line along the x-axis. As in Claim 4.6.2D, it can then be proved that the $g_2(.)$ map, too, is a horizontal straight line. But there remains a possibility of jump discontinuity at the origin in $g_2(.)$ map. It can further be shown that such a jump, if present, can only occur in an

[17] Recall that 1 precedes 2 in both profiles θ and θ'. Therefore $\theta_1 = 0 \ \Longrightarrow\ \theta_2 = 0$ so that $\sigma_1(0, 0) < \sigma_2(0, 0)$.

upward direction[18]. To capture this possibility we would need to assume that $\tau_t(0,0,\ldots,0) = C_t > 0$ in the expression for the $g_t(.)$ map for all t. The relevant map containing the implications of $\theta_1 = 0$ would then be given by $\eta = 0$ and $\alpha_2 < 1$. Hence, our assumption of transfers independent of agent specific constants, that is $\tau_i(0,0,\ldots,0) = 0, \forall\, i \in N$, rules out the case where $\theta_1 = 0$[19].

Remark 4.6.12. Consider the possibility that $z(2) - z(1) = 0$. For any θ, θ' such that θ' is an order preserving $\{1,2\}$-profile of θ with $\sigma_1(\theta) = 1, \sigma_2(\theta) = 2$; if WGS holds with strict inequality then from (4.5) and (4.6) we get that $g_1(\theta_2) > g_1(\theta_2')$ and $g_2(\theta_1) < g_2(\theta_1')$. But then WGS gets violated in a $\{1,2\}$ deviation from (θ_1, θ_2') to (θ_1', θ_2). Then from Proposition 4.6.2 we get that $\eta \in \{0, \infty\}$.

Proof of Theorem 4.3.9: First we need to prove the following lemma, which says that given equations (4.9) and (4.10) there will always be an agent whose transfer turns out be independent of the announcements of all other players. For this purpose, we need to define the notation $\theta(r) := \{\theta_i | \sigma_i(\theta) = r\}$, $\forall\, \theta \in \Re^n_+$, $\forall\, r = 1, 2, \ldots, n$. Therefore $\theta(r)$ denotes the per unit time waiting cost of the agent ranked r in the efficient ranking for profile θ.

Lemma 4.6.13. If equations (4.9) and (4.10) hold then $\forall \theta \in \Re^n_+$, $\exists m(\theta) \in N$ such that $\tau_{m(\theta)}(\theta) = Constant$.

[18]If the jump is in downward direction then $l_t((x,y),(0,0)) < 0$ where $0 < y < x$.

[19]If $\theta_1 = 0$ then $\sigma_1(\theta) = 1 \implies \theta_2 = 0$. Putting $g_t(0) = 0$ for all t in equations (4.5) and (4.6) we get that (i) $g_1(\theta_2') < \theta_2'$ and (ii) $g_2(\theta_1') < 0$. But then WGS gets violated in a deviation from θ' to $\theta \equiv (0,0)$. Hence contradiction.

Proof: Suppose $N = \{1, 2, \ldots, n\}$. Define $s' = \min\{s = 1, 2, \ldots, n-1 : \theta(s) \leq \eta(s)\}$, $\forall\, \theta \in \Re^n_+$. Then choose the agent $m(\theta)$ so that

$$\sigma_{m(\theta)}(\theta) = \begin{cases} n & \text{when } \{s'\} = \emptyset \\ s' & \text{when } \{s'\} \neq \emptyset \end{cases}$$

We will show that $\tau_{m(\theta)}(\theta) = Constant$ in each of the following two cases;

Case 1: $\{s'\} = \emptyset$

Therefore the last ranked agent is chosen to be $m(\theta)$; hence (i) $\tau_{m(\theta)}(\theta) = g_{m(\theta)}(\theta_{-m(\theta)})$ and (ii) $\theta_{-m(\theta)}(s) = \theta(s) > \eta(s)$ for all $s = 1, 2, \ldots, n-1$. Therefore using (i) & (ii) we can write that

$$\begin{aligned} \tau_{m(\theta)}(\theta) &= \sum_{j \neq m(\theta)} \Delta z(\sigma_j(\theta_{-m(\theta)})) \min\left\{\theta_j, \eta(\sigma_j(\theta_{-m(\theta)}))\right\} \\ &= \sum_{k=1,2,\ldots,n-1} \Delta z(k) \ \min\left\{\theta_{-m(\theta)}(k), \eta(k)\right\} \\ &= \sum_{k=1,2,\ldots,n-1} \Delta z(k).\eta(k) = Constant \end{aligned}$$

Case 2: $\{s'\} \neq \emptyset$

In this case $\sigma_{m(\theta)}(\theta) = s' < n$; which implies that (**a**) $\theta_{-m(\theta)}(k) = \theta(k) > \eta(k)$, $\forall\ k = 1, 2, \ldots, s'-1$. Now from (4.10) we get that $\eta(s') \leq \eta(k)$, $\forall\ k = s'+1, s'+2, \ldots, n-1$ and so we can say that (**b**) $\theta_{-m(\theta)}(k) = \theta(k+1) \leq \theta(s') \leq \eta(s') \leq \eta(k)$, $\forall\ k = s', s'+1, \ldots, n-1$. Note that $\forall\, j \in P'_{m(\theta)}(\theta)$, $z(\sigma_j(\theta)) - z(\sigma_j(\theta_{-i})) = z(\sigma_j(\theta_{-i}) + 1) - z(\sigma_j(\theta_{-i})) = \Delta z(\sigma_j(\theta_{-i}))$. Therefore using (**a**) and (**b**), we can write that

$$
\begin{aligned}
\tau_{m(\theta)}(\theta) &= -\sum_{j \in P'_{m(\theta)}(\theta)} \Delta z(\sigma_j(\theta_{-m(\theta)}))\theta_j + \sum_{j \neq m(\theta)} \Delta z(\sigma_j(\theta_{-m(\theta)})) \min\left\{\theta_j, \eta(\sigma_j(\theta_{-m(\theta)}))\right\} \\
&= -\sum_{k=s',s'+1,\ldots,\mathbf{n}-1} \Delta z(k)\theta_{-m(\theta)}(k) + \sum_{k=1,2,\ldots,\mathbf{n}-1} \Delta z(k) \min\left\{\theta_{-m(\theta)}(k), \eta(k)\right\} \\
&= \sum_{k=1,2,\ldots,s'-1} \Delta z(k)\eta(k) = Constant
\end{aligned}
$$

□

Pick any non-empty $S \subseteq N$ and $\theta, \theta' \in \Re^n_+$ such that θ is an S-profile of θ'. Suppose coalition S deviates from θ' to θ and this deviation violates WGS. For notational simplicity we suppress the argument θ in the term $m(\theta)$ and write just m.

Claim 4.6.14. $m \notin S$

Proof of Claim: Say $m \in S$. Lemma 4.6.13 implies that $\tau_m(\theta)$ is independent of the announcements of its coalition partners. Therefore, for agent m, this coalitional deviation is only as good as a unilateral deviation. But then strategyproofness contradicts WGS being violated. □

Identify the agent $a \stackrel{def}{=} argmax\{\sigma_j(\theta) | j \in S \cap P_m(\theta)\}$ and the rank $r_a \stackrel{def}{=} \sigma_a(\theta)$. Similarly, agent $b \stackrel{def}{=} argmax\{\sigma_j(\theta) | j \in S \cap P'_m(\theta)\}$ and the rank $r_b \stackrel{def}{=} \sigma_b(\theta)$. Therefore, a is the *last ranked member of S preceding agent m* and b is the *first ranked member of S succeeding agent m*; in the efficient ranking for cost profile θ. Also note that if not both, either of the agents a and b, must exist.

Now, from the definition of m, there are two possible cases;

Case 1: $\sigma_m(\theta) = n$

In this case, only a is well defined. As before we can write

$$\tau_a(\theta) = -\sum_{k=r_a,r_a+1,\ldots,n-1} \Delta z(k)\theta_{-a}(k) + \sum_{k=1,2,\ldots,n-1} \Delta z(k)\min\{\theta_{-a}(k),\eta(k)\}$$

Note that $\sigma_m(\theta) = n \implies$ either $\{s'\} = \emptyset$ or $s' = n \implies \theta(k) > \eta(k), \forall\, k = 1,2,\ldots,n-1$. Therefore since $a' < n$; (**i**) $\theta_{-a}(k) = \theta(k) > \eta(k)$, $\forall\, k = 1,2,\ldots,r_a - 1$ and from (4.10); (**ii**) $\theta_{-a}(k) = \theta(k+1) > \eta(k+1) \geq \eta(k)$, $\forall\, k = r_a, r_a+1,\ldots,n-2$. Also (**iii**) $\theta_{-a}(n-1) = \theta_m$. Using (**i**), (**ii**) and (**iii**); the equation above can be reduced to the following;

$$\begin{aligned}\tau_a(\theta) &= \sum_{k=1,2,\ldots,r_a-1} \Delta z(k)\eta(k) + \sum_{k=r_a,r_a+1,\ldots,n-2} \Delta z(k)\left[\eta(k) - \theta(k+1)\right] \\ &+ \Delta z(n-1)\left[\min\{\theta_m,\eta(n-1)\} - \theta_m\right]\end{aligned}$$

By the definition of a; the numbers $\{\theta(k+1)\}_{k=r_a}^{n-2}$ are waiting costs of agents who are not members of S. Given $m \notin S$, this means that $\tau_a(\theta)$ does not depend on the misreports of members of $S - \{a\}$. Therefore arguing as in Claim 4.6.14, we can arrive at a contradiction.

Case 2: $\sigma_m(\theta) = s' < n$

Once again, if a exists;

$$\begin{aligned}\tau_a(\theta) &= -\sum_{k=r_a,r_a+1,\ldots,n-1} \Delta z(k)\theta_{-a}(k) + \sum_{k=1,2,\ldots,n-1} \Delta z(k)\min\{\theta_{-a}(k),\eta(k)\} \\ &= \sum_{k=a,r_a+1,\ldots,s'-2} \Delta z(k)\left[\min\{\theta_{-a}(k),\eta(k)\} - \theta_{-a}(k)\right] \\ &+ \sum_{k=s',s'+1,\ldots,n-1} \Delta z(k)\left[\min\{\theta_{-a}(k),\eta(k)\} - \theta_{-a}(k)\right] \\ &+ \Delta z(s'-1)\left[\min\{\theta_{-a}(s'-1),\eta(s'-1)\} - \theta_{-a}(s'-1)\right] \\ &+ \sum_{k=1,2,\ldots,r_a-1} \Delta z(k)\min\{\theta_{-a}(k),\eta(k)\}\end{aligned}$$

By definition; (**a**) $s' < n \implies \theta(k) > \eta(k)$, $\forall\, k = 1, 2, \ldots, s' - 1$. By construction, $r_a < s'$ and so (**i**) $\theta_{-a}(s' - 1) = \theta(s') = \theta_m$. Then using (**a**) we can say that (**ii**) $\theta_{-a}(k) = \theta(k) > \eta(k)$, $\forall\, k = 1, 2, \ldots, r_a - 1$. From the construction of the rank s' and (4.10), it follows that (**iii**) $\theta_{-a}(k) = \theta(k+1) > \eta(k+1) \geq \eta(k)$, $\forall\, k = r_a, r_a + 1, \ldots, s' - 2$ and (**iv**) $\theta_{-a}(k) = \theta(k+1) \leq \theta(s') \leq \eta(s') \leq \eta(k)$, $\forall\, k = s', s' + 1, \ldots, n - 1$. Using conditions **(i)-(iv)**, we can write that

$$\begin{aligned}\tau_a(\theta) &= \sum_{1,2,\ldots,r_a-1} \Delta z(k)\eta(k) + \sum_{r_a,r_a+1,\ldots,s'-2} \Delta z(k)\left[\eta(k) - \theta(k+1)\right] \\ &+ \Delta z(s'-1)\left[\min\{\theta_m, \eta(s'-1)\} - \theta_m\right]\end{aligned}$$

Arguing as in Case 1; we can see that $\tau_a(\theta)$ in independent of reports of members of $S - \{a\}$. Therefore as in Claim 4.6.14, we reach a contradiction.

Similarly, if b exists;

$$\begin{aligned}\tau_b(\theta) &= -\sum_{k=r_b,r_b+1,\ldots,n-1} \Delta z(k)\theta_{-b}(k) + \sum_{k=1,2,\ldots,n-1} \Delta z(k)\min\{\theta_{-b}(k), \eta(k)\} \\ &= \sum_{k=1,2,\ldots,r_b-1} \Delta z(k)\min\{\theta_{-b}(k), \eta(k)\} \\ &+ \sum_{k=r_b,r_b+1,\ldots,n-1} \Delta z(k)\left[\min\{\theta_{-b}(k), \eta(k)\} - \theta_{-b}(k)\right]\end{aligned}$$

By definition $r_b > s'$. Then using (4.10) and the condition (**a**) we get that **(v)** $\theta_{-b}(s') = \theta(s') = \theta_m \leq \eta(s')$; **(vi)** $\theta_{-b}(k) = \theta(k) > \eta(k)$, $\forall\, k = 1, 2, \ldots, s' - 1$; **(vii)** $\theta_{-b}(k) = \theta(k) \leq \theta(s') \leq \eta(s') \leq \eta(k)$, $\forall\, k = s' + 1, s' + 2, \ldots, r_b - 1$ and **(viii)** $\theta_{-b}(k) = \theta(k+1) \leq \theta(s') \leq \eta(s') \leq \eta(k)$, $\forall\, k = r_b, r_b + 1, \ldots, n - 1$. Conditions **(v)-(viii)** then imply that

$$\tau_b(\theta) = -\sum_{k=1,2,\ldots,s'-1} \Delta z(k)\eta(k) + \Delta z(s)\theta_m + \sum_{k=s'+1,s'+2,\ldots,r_b-1} \Delta z(k)\theta(k)$$

By definition of b, the numbers $\{\theta(k)\}_{k=s'+1}^{r_b-1}$ are waiting costs of members of $N - S$. Since $m \notin S$, it can therefore be said that $\tau_b(\theta)$ is independent of misreports of members of $S - \{b\}$. Thus as in Claim 4.6.14, we reach a contradiction. $\square$

Bibliography

[1] S. Barbera, D. Berga, and B. Moreno. Individual versus group strategyproofness: when do they coincide? *Journal of Economic Theory*, 145:1648–1674, 2010.

[2] Michael R. Baye, D. Kovenock, and Casper G. de Vries. Rigging the lobbying process: An application of the all-pay auction. *American Economic Review*, 83:289–294, 1993.

[3] Michael R. Baye, D. Kovenock, and Casper G. de Vries. The all-pay auction with complete information. *Economic Theory*, 8:291–305, 1996.

[4] A. Bogomolnaia and H. Moulin. Random matching under dichotomous preferences. *Econometrica*, 72:257–279, 2004.

[5] B. Caillaud and P. Jehiel. Collusion in auctions with externalities. *RAND Journal of Economics*, 29:680–702, 1998.

[6] K. Chatterjee, B. Dutta, Ray D., and K. Sengupta. A noncooperative theory of coalitional bargaining. *Review of Economic Studies*, 60:463–477, 1993.

[7] Y. Chun. A pessimistic to the queueing problem. *Mathematical Social Sciences*, 51:171–181, 2006.

[8] Y. Chun and E. J. Heo. Queueing problems with two parallel problems. *International Journal of Economic Theory*, 4:299–315, 2008.

[9] E. H. Clarke. Multipartpricing of public goods. *Public Choice*, 11:17–33, 1971.

[10] R. J. Dolan. Incentive mechanisms for priority queueing problems. *Bell Journal of Economics*, 9:421–436, 1978.

[11] D. Foley. Resource allocation and the public sector. *Yale Econ Essays*, 7:45–98, 1967.

[12] Y. Fujinaka and T. Sakai. The manipulability of fair solutions in assignment of an indivisible object with monetary transfers. *Journal of Public Economic Theory*, 9:993–1011, 2007.

[13] Y. Fujinaka and T. Sakai. The positive consequence of strategic manipulation in indivisible good allocation. *International Journal of Game Theory*, 38:325–348, 2009.

[14] R. Fullerton and P. McAfee. Auctioning entry into tournaments. *Journal of Political Economy*, 107:573–605, 1999.

[15] D. A. Graham and R. C. Marshall. Collusive bidder behavior at single-object second price and english auctions. *Journal of Political Economy*, 95:1217–1239, 1987.

[16] T. Groves. Incentives in team. *Econometrica*, 41:617–631, 1973.

[17] R. Hain and M. Mitra. Simple sequencing problems with interdependent costs. *Games and Economic Behavior*, 48:423–447, 2004.

[18] K. Hashimoto and H. Saitoh. Strategyproof and anonymous rule in queueing problems: A relationship between equity and efficiency. *Social Choice and Welfare*. forthcoming.

[19] K. Hatsumi and S. Serizawa. Coalitionally strategyproof rules in allotment economies with homogenous indivisible goods. *Social Choice and Welfare*, 33:423–447, 2009.

[20] K. Hendricks, R. Porter, and G. Tan. Equilibria in networks. *Econometrica*, 67:1407–1434, 1999.

[21] B. Holmstr'om. Groves' schemes on restricted domains. *Econometrica*, 47:1137–1144, 1979.

[22] P. Jehiel and B. Moldovanu. Strategic nonparticipation. *RAND Journal of Economics*, 27:84–98, 1996.

[23] P. Jehiel, B. Moldovanu, and E. Stacchetti. How (not) to sell nuclear weapons. *American Economic Review*, 86:814–829, 1996.

[24] A. Kar, M. Mitra, and M. Mutuswami. On the coincidence of prenucleolus and the shapley value. *Mathematical Social Sciences*, 57:16–25, 2009.

[25] A. Katta and J. Sethuraman. Cooperation in queues. Working paper, Columbia University, 2005.

[26] C. Kayi and E. Ramaekers. Efficiency, fairness, and strategyproofness in queueing problems: Characterizations and impossibility. *Games and Economic Behavior*, 68:220–232, 2010.

[27] G. Mailath and P. Zemsky. Collusion in second price auctions with heterogeneous bidders. *Games and Economic Behavior*, 3:467486, 1991.

[28] F. Maniquet. A characterization of shapley value in queueing problems. *Journal of Economic Theory*, 109:90–103, 2003.

[29] R. C. Marshall and L. M. Marx. Bidder collusion. *Journal of Economic Theory*, 133:374–402, 2007.

[30] R. P. McAfee and J. McMillan. Bidding rings. *American Economic Review*, 82:579–599, 1992.

[31] P. Milgrom. Auction theory. In Truman Bewley, editor, *Advances in Economic Theory: Fifth World Congress*, pages 1–32. Cambridge University Press, London, 1987.

[32] M. Mitra. Mechanism design in queueing problems. *Economic Theory*, 17:277–305, 2001.

[33] M. Mitra and S. Mutuswami. Group strategyproofness in queueing models. *Games and Economic Behavior*, 72:242–254, 2011.

[34] M. Mitra and A. Sen. Efficient allocation of heterogeneous commodities with balanced transfers. *Social Choice and Welfare*, 35:29–48, 2010.

[35] H. Moulin. Proportional scheduling, split-proofness, and merge-proofness. *Games and Economic Behaviour*, 63:567–587, 2008.

[36] S. Ohseto. Strategyproof and efficient allocation of an indivisible good on finitely restricted preference domains. *International Journal of Game Theory*, 29:365–374, 2000.

[37] S. Pápai. Groves sealed bid auctions of heterogeneous objects with fair prices. *Social Choice and Welfare*, 20:371–385, 2003.

[38] D. Ray. *A Game-Theoretic Perspective on Coalition Formation.* Oxford University Press Inc., New York., 2004.

[39] D. Ray and Vohra R. A theory of endogenous coalition structures. *Games and Economic Behavior*, 26:286–336, 1999.

[40] M. S. Robinson. Collusion and the choice of auction. *RAND Journal of Economics*, 16:141–145, 1985.

[41] A. Rubinstein. Perfect equilibrium in a bargaining model. *Econometrica*, 50:97–109, 1982.

[42] J. Suijs. On incentive compatibility and budget balancedness in public decision making. *Economic Design*, 2:193–209, 1996.

[43] Lars-Gunnar Svensson and Bo Larsson. Strategyproof and nonbossy allocation of indivisible goods and money. *Economic Theory*, 20:483–502, 2002.

[44] K. Tadenuma and W. Thomson. No-envy and consistency in economies with indivisible goods. *Econometrica*, 59:1755–1767, 1991.

[45] K. Tadenuma and W. Thomson. The fair allocation of an indivisble good when monetary compensations are possible. *Mathematical Social Sciences*, 25:117–132, 1993.

[46] H. Varian. Equity, envy, and efficiency. *Journal of Economic Theory*, 9:63–91, 1974.

[47] W. Vickerey. Counterspeculation, auctions and competitive sealed bidders. *Journal of Finance*, 16:8–37, 1961.