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THREE WAY ELIMINATION OF HETEROGENEITY WITH NON-ORTHOGONAL INCIDENCE STRUCTURE FOR EVERY TWO DIRECTIONS

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SUMMARY. The present article deals with a study of optimal designs in the setup admitting three way elimination of heterogeneity, where incidence structure of every two directions is non-orthogonal in the sense that level combinations along the principal diagonal are infeasible, all other level combinations being feasible. For $b \equiv 1 \pmod v$ universally optimal designs are available and for $b \equiv 0 \pmod v$ A-, D-and E-optimal designs have been characterized. This characterization is very much similar to that in the two way setting of block designs involving $b \times b$ arrays where all the cells along a transversal are infeasible, though the construction of such optimal designs are more involved. In this paper, we mainly consider construction of optimal designs so characterized for the case $v = 3$, $v = 2^t$, $t > 2$ integer, and $v = tk+1$, odd prime or prime power, k odd, $t > 2$ integer. Also cited is a design constructed by extending Aggarwal's method (1966b) to higher dimensions which has a simple structure compared to the actual optimal designs. Though these may not be optimal, they possess high efficiency relative to the optimal designs. Thus the objective of the paper is three fold: (i) to obtain achievable conditions for optimal designs in the given non-orthogonal set up (ii) to prove the existence of such optimal designs in some cases by actual construction and (iii) to calculate the efficiency of some simple and easily constructible designs which may have been in use in the context for a long time, justifying their usability.

1. Introduction

Available optimality results (studied by Kiefer, 1958, 1975; Cheng, 1978; Mukhopadhyay and Mukhopadhyay, 1984) deal exclusively with the orthogonal framework that is to say the incidence pattern of every pair of directions is taken to be represented (except for a multiplier $\geqslant 1$) by the matrix $J = ((1))$ of all 1's. SahaRay (1986) initiated a study on optimal designs underlying two way elimination of heterogeneity set-ups with non-orthogonal incidence structure for the pair of directions. Specifically non-orthogonality was assumed in the sense of empty cells along a transversal of a square array of two way layout. In the present article we take up a study of optimal designs in the set up admitting three way elimination of heterogeneity where feasible experimental units are those for which levels along

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any two directions are not the same. That is to say, for every two directions, level combinations along the principal diagonal are infeasible. Let each of the three directions assume b levels and (i_1, i_2, i_3) denote the level combination for any experimental unit, $i_1, i_2, i_3 = 1, 2, \ldots, b$. Following Cheng (1978), therefore if we assume the incidence structure for every pair of directions as $(b-2)$ $(J-I)$, the number of feasible experimental units for allocation of v treatments becomes $b(b-1)$ $(b-2)$ which may be too large to be available to the experimenter in practice. On the other hand following techniques similar to those adopted by Mukhopadhyay and Mukhopadhyay (1984), the number of experimental units can be reduced to $b(b-1)$, thereby producing the incidence structure as $J-I$ for every pair of directions. Let E be an $OA(b^2, 3, b, 2)$* such that the three constraints have the same level combinations in each of the first b columns, i.e. the first b columns of the OA are of the type $\begin{pmatrix} i \\ i \\ i \end{pmatrix}$, $i = 1, 2, \ldots, b$. Then the remaining $b(b-1)$ columns of this orthogonal array serve as experimental units for us in the three way elimination of heterogeneity set up, where the entry in the i-th row of the u-th column denotes the level of i-th direction in the u-th experimental unit. We will deal with this latter incidence structure for the three way elimination of heterogeneity designs.

The usual fixed effects model e.g.

$$Y_{jj'k}(h) = \mu+\alpha_j+\beta_{j'}+\nu_k+\tau_h+e_{jj'k}, \quad 1 \leqslant h \leqslant v, \; 1 \leqslant j \neq j' \neq k \leqslant b$$

where μ, α_j, $\beta_{j'}$, ν_k, τ_h, stand respectively for general effect, effect of j-th level in 1st direction, effect of j'-th level in 2nd direction, effect of kth level in 3rd direction and hth treatment effect (if the h-th treatment is applied to the experimental unit), $e_{jj'k}$'s are i.i.d. $N(0, \sigma^2)$.

For a specified design d, $N_{di} = ((n_{dhl}^{(i)}))$, $d = 1, 2, 3$, $h = 1, 2, \ldots, v$, $l = 1, 2, \ldots, b$ stands for the incidence of treatment $\times$ i-th direction. Incidence structure is $J-I$ for any two directions.

As in the case of two way non-orthogonal set-up, we are interested in linear inferential problems involving treatment contrasts only and as such we refer to the underlying C-matrix of the design. Let $r_d = (r_{d1}, r_{d2}, \ldots, r_{dv})'$

*We use the notation $OA(N, k, S, t)$ for an orthogonal array of size N, k constraints, s levels, and strength t. Sometimes when the parameters are understood from the context we will denote the orthogonal array by simply OA.

be the vector of treatment replications, and $D_{r_d} = \text{Diag}\ (r_{d1}, \ldots, r_{dv})$. One can see that for any design d,

$$C_d = D_{r_d} - (N_{d_1}, N_{d_2}, N_{d_3}) \begin{pmatrix} (b-1)I & J-I & J-I \\ J-I & (b-1)I & J-I \\ J-I & J-I & (b-1)I \end{pmatrix}^{-} \begin{pmatrix} N'_{d1} \\ N'_{d2} \\ N'_{d3} \end{pmatrix}$$

where '—' over a matrix denotes its generalised inverse. With a particular choice of g-inverse, we get

$$C_d = D_{r_d} - \frac{1}{b(b-3)} (N_{d_1}, N_{d_2}, N_{d_3})$$

$$\begin{bmatrix} (b-2)I - \frac{2}{b-1}\frac{J}{b} & I-\frac{J}{b} & I-\frac{J}{b} \\ I-\frac{J}{b} & (b-2)\left(I-\frac{J}{b}\right) & I-\frac{J}{b} \\ I-\frac{J}{b} & I-\frac{J}{b} & (b-2)\left(I-\frac{J}{b}\right) \end{bmatrix} \begin{bmatrix} N'_{d1} \\ N'_{d2} \\ N'_{d3} \end{bmatrix}$$

After simplification, C_d becomes,

$$C_d = D_{r_d} - \frac{b-4}{b(b-3)} \sum_{i=1}^{3} N_{di} N'_{di}$$

$$- \frac{1}{b(b-3)} \sum_{i<j}\sum (N_{di}+N_{dj})(N_{di}+N_{dj})' + \frac{2}{(b-1)(b-3)} r_d r'_d \quad \ldots \quad (1.1)$$

The hth diagonal element of C_d, denoted by C_{dhh} turns out to be

$$C_{dhh} = r_{dh} - \frac{b-1}{b(b-3)} \sum_{i=1}^{3} \sum_{l=1}^{b} n_{dhl}^{(i)2}$$

$$- \frac{1}{b(b-3)} \sum_{i<j}\sum \sum_{l=1}^{b} (n_{dhl}^{(i)} + n_{dhl}^{(j)})^2 + \frac{2}{(b-1)(b-3)} r_{dh}^2 \quad \ldots \quad (1.2)$$

Developing similar tools as in the case of two way non-orthogonal set-up (SahaRay, 1986), we can here also characterize optimal designs under this three way non-orthogonal set-up. In Sections 2 and 3 we take up the problem of characterization and construction of optimal designs for the case $b = mv+1$ and $b = mv$ respectively. In Section 4, we calculate efficiency of a class of simple designs originally proposed by Aggarwal (1966b) in the same context.

2. Universal optimality results for $b = mv+1$

Let Ω be the relevant class of connected designs in the given set-up. Let d^* be a design in $\Omega\,(b = mv+1)$ which assigns each treatment m times to each level of every direction. Then as in the case of two way set-up (SahaRay, 1980), we can show that such a d^* maximizes the trace of C_d (tr C_d) over the class Ω. Referring to (1.1),

$$\text{tr } C_d = \sum_{h=1}^{v} C_{dhh} = \sum_{h=1}^{v} r_{dh} - \frac{mv-3}{(mv+1)(mv-2)} \sum_{i=1}^{3} \sum_{h=1}^{v} \sum_{l=1}^{mv+1} n_{dhl}^{(i)2}$$

$$-\frac{1}{(mv+1)(mv-2)} \sum_{i<j} \sum_{h=1}^{v} \sum_{l=1}^{mv+1} (n_{dhl}^{(i)}+n_{dhl}^{(j)})^2 + \frac{2}{mv(mv-2)} \Sigma r_{dh}^2$$

$$= \sum_{h=1}^{v} r_{dh} - \frac{mv-3}{(mv+1)(mv-3)} \sum_{i=1}^{3} \sum_{h=1}^{v} \left\{ \sum_{l=1}^{mv+1} \left(n_{dhl}^{(i)} - \frac{r_{dh}}{mv+1} \right)^2 \right\}$$

$$-\frac{1}{(mv+1)(mv-2)} \sum_{i<j} \sum_{h=1}^{v} \left\{ \sum_{l=1}^{mv} \left(n_{dhl}^{(i)} + n_{dhl}^{(j)} - \frac{2r_{dh}}{mv+1} \right)^2 \right\}$$

$$-\frac{1}{mv(mv+1)} \Sigma r_{dh}^2 \qquad \ldots \quad (2.1)$$

since $\sum_{l=1}^{mv+1} n_{dhl}^{(i)} = r_{dh} \quad \forall\, i = 1, 2, 3$ and $h = 1, 2, \ldots, v$.

Clearly $\sum_{h=1}^{v} r_{dh}^2$ assumes the least value for d^* as d^* is equireplicate with $r_{d^*1} = r_{d^*2} = \ldots = r_{d^*v} = (mv+1)m = r_{d^*}$ (say). Moreover, each sum of squares in (2.1) is '$\geqslant 0$' for any competing design and '$= 0$' for d^* since in d^*, $n_{d^*hl}^{(i)}$'s are each equal to $m \left(= \frac{r_{d^*}}{mv+1} \right)$. This settles the part on trace maximization. Next, it is also evident that C_{d^*} is completely symmetric ($C.S$). Hence an application of Proposition 1 of Kiefer (1975) asserts that d^* is universally optimal.

Such types of designs can be easily constructed. Except in case $mv+1 = 6$, we can in all other cases construct an $OA((mv+1)^2, 4, mv+1, 2)$ (Please note that existence of this OA is equivalent to a Graeco-Latin square of order $mv+1$) such that the first constraint is arranged in the order 0, 0, ...,

$0, 1, 1, \ldots, 1, 2, \ldots, 2, mv, \ldots, mv$, each symbol being repeated $(mv+1)$ times. Further, each of the remaining three constraints has the same symbol in a column for the first $(mv+1)$ columns. We identify the first constraint as denoting the treatments and the remaining three constraints as the three directions. Then by construction, the level combination (i, i, i), $i = 0, 1, 2, \ldots, mv$ receives treatment 0. We delete these first $(mv+1)$ columns from this OA and reduce the rest of the symbols in row number 1 mod v. This resulting array will give d^*.

3. Specific optimality in the case $b = mv$

Let Δ be the relevant class of connected designs for $b = mv$. As in the case of two way set-up, the C-matrix of an A–, D–, and E–. Optimal desing here is completely symmetric, but it does not necessarily produce maximum trace of C_d in Δ. Take for example $b = 9$, $m = 3$, $v = 3$. The design d_1 has larger trace than d^*, which will be shown to be E-optimal, where

$$d_1 = \left[\begin{array}{ccccccccc}
A_1 & A_3 & A_2 & B_3 & B_1 & B_2 & C_2 & C_1 & C_3 \\
- & 0 & 1 & 2 & 0 & 1 & 2 & 0 & 1 \\
A_3 & A_2 & A_1 & B_1 & B_2 & B_3 & C_1 & C_3 & C_2 \\
1 & - & 0 & 1 & 2 & 0 & 1 & 2 & 0 \\
A_2 & A_1 & A_3 & B_2 & B_3 & B_1 & C_3 & C_2 & C_1 \\
0 & 2 & - & 0 & 1 & 2 & 0 & 1 & 2 \\
B_3 & B_2 & B_1 & C_1 & C_3 & C_2 & A_3 & A_2 & A_1 \\
1 & 2 & 0 & - & 0 & 2 & 1 & 2 & 0 \\
B_1 & B_3 & B_2 & C_3 & C_2 & C_1 & A_2 & A_1 & A_3 \\
2 & 0 & 1 & 1 & - & 0 & 0 & 1 & 2 \\
B_2 & B_1 & B_3 & C_2 & C_1 & C_3 & A_1 & A_3 & A_2 \\
0 & 1 & 2 & 0 & 1 & - & 2 & 0 & 1 \\
C_3 & C_2 & C_1 & A_2 & A_1 & A_3 & B_1 & B_2 & B_3 \\
0 & 1 & 2 & 2 & 0 & 1 & - & 0 & 1 \\
C_2 & C_1 & C_3 & A_3 & A_2 & A_1 & B_2 & B_3 & B_1 \\
2 & 0 & 1 & 0 & 1 & 2 & 2 & - & 0 \\
C_1 & C_3 & C_2 & A_1 & A_3 & A_2 & B_3 & B_1 & B_2 \\
1 & 2 & 0 & 1 & 2 & 0 & 0 & 1 & -
\end{array}\right]$$

$$
d^* = \begin{bmatrix}
A_1 & A_3 & A_2 & B_3 & B_1 & B_2 & C_2 & C_1 & C_3 \\
- & 1 & 2 & 0 & 2 & 0 & 0 & 1 & 2 \\
A_3 & A_2 & A_1 & B_1 & B_2 & B_3 & C_1 & C_3 & C_2 \\
1 & - & 0 & 2 & 2 & 1 & 1 & 2 & 0 \\
A_2 & A_1 & A_3 & B_2 & B_3 & B_1 & C_3 & C_2 & C_1 \\
2 & 0 & - & 0 & 1 & 1 & 2 & 0 & 1 \\
B_3 & B_2 & B_1 & C_1 & C_3 & C_2 & A_3 & A_2 & A_1 \\
2 & 0 & 1 & - & 1 & 2 & 0 & 0 & 1 \\
B_1 & B_3 & B_2 & C_3 & C_2 & C_1 & A_2 & A_1 & A_3 \\
0 & 1 & 2 & 1 & - & 0 & 0 & 2 & 2 \\
B_2 & B_1 & B_3 & C_2 & C_1 & C_3 & A_1 & A_3 & A_2 \\
1 & 2 & 0 & 2 & 0 & - & 1 & 2 & 1 \\
C_3 & C_2 & C_1 & A_2 & A_1 & A_3 & B_1 & B_2 & B_3 \\
0 & 1 & 2 & 2 & 0 & 1 & - & 1 & 2 \\
C_2 & C_1 & C_3 & A_3 & A_2 & A_1 & B_2 & B_3 & B_1 \\
1 & 2 & 0 & 0 & 1 & 2 & 1 & - & 0 \\
C_1 & C_3 & C_2 & A_1 & A_3 & A_2 & B_3 & B_1 & B_2 \\
2 & 0 & 1 & 1 & 2 & 0 & 2 & 0 & -
\end{bmatrix}
$$

(In d_1 as well as in d^*, the three directions are given by rows, columns and the Roman letters $A_1, A_2, A_3, C_1, C_2, C_3, B_1, B_3, B_2$). Thus Proposition 1 of Kiefer (1975) is not directly applicable as regards universal optimality. So we look for specific optimality results.

Now we quote a well known lemma (without proof) which we will use in the derivation of optimal designs for the case $b = mv$.

Lemma 1 : *For any positive integers S and t, the minimum of* $\sum_{i=1}^{S} n_i^2$ *subject to* $\sum_{i=1}^{S} n_i = t$ *where n_i's are non-negative integers is obtained when* $t - S\left[\frac{t}{S}\right]$ *of the of n_i's are each equal to* $\left[\frac{t}{S}\right]+1$ *and the others are each equal to* $\left[\frac{t}{S}\right]$, *where* $\left[\frac{t}{S}\right]$ *is the largest integer contained in* $\left[\frac{t}{S}\right]$.

Let d^* be an equireplicate design for which (a) C_{d^*} is $C.S.$ and (b) the diagonal components of C_{d^*} are such that

$$C_{d^*hh} = \left\{ \max_{d\epsilon\Delta,\, r_{dh}=\bar{r}} \right\} C_{dhh} \text{ for every } h.$$

Here $\bar{r}$ stands for the constant replication number of the treatments under d^* i.e. $\bar{r} = m(mv-1)$.

Following essentially the technique in Kiefer (1975) and the steps developed for the case of two way heterogeneity set up in SahaRay (1986) it can be established that

(i) d^* is D-optimal for $v \geqslant 8$, $m \geqslant 2$ and hence A-optimal as d^* is $C.S.$ For the case $v \leqslant 7$, A-optimality is still unsettled.

(ii) d^* is E-optimal for $v \geqslant 3$, $m \geqslant 2$.

Construction of such d^* is rather involved and more complicated than in the case of two way set-up. For d^* to exist, a sufficient condition will be that the sufficient conditions required for two way set-up as elaborated in SahaRay (1986) should hold for every pair of directions viz. (i) For each level of a direction $(v-1)$ treatments occur m times each and exactly one treatment occurs $(m-1)$ times. (ii) Considering any pair of directions the treatment which occurs $(m-1)$ times for the ith level of the 1st direction cannot occur $(m-1)$ times in the ith level of 2nd direction in the pair $i = 1, 2, \ldots, mv$. (iii) Take any pair of directions in a definite order. The level combination (i, i) for the pair of direction gives an ordered pair of distinct treatments the first of which occurs $(m-1)$ times in the i-th level of the 1st direction and the second occurs $(m-1)$ times in the i-th level of the 2nd direction. Then we get a series of mv ordered treatment pairs. These mv pairs should exhaust all possible $\binom{v}{2}$ pairs of treatments each occurring a constant number of times. This result should be true for every pair of directions chosen. Properties (i) and (ii) ensure property (b) of C_{d^*} viz $C_{d^*hh} = \{ \max_{d\epsilon\Delta:\, r_{dh}=\bar{r}} \} C_{dhh}$ as it is clear from application of Lemma 1 on the expression of C_{dhh} (vide (1.2)). Now to achieve property (a) of C_{d^*} viz. complete symmetry of C_{d^*}, we first note that properties (i) and (ii) readily lead to complete symmetry of $N_{d^*i}N'_{d^*i}$. It remains to achieve complete symmetry of $(N_{d^*i}+N_{d^*j})\,(N_{d^*i}+N_{d^*j})'$ $\forall\; i < j$, $i, j = 1, 2, 3$, (vide (1.1)). A little algebraic simplification will show that condition (iii) implies this.

A necessary condition for realisation of condition (iii) is $mv = \lambda \binom{v}{2}$, λ is a positive integer. We have already observed that for two way elimination of heterogeneity in such cases, the required d^* does not exist for $m = 1$.

So we assume $m \geqslant 2$. We will present here construction of d^* for (i) $v = 3$, $m = 3$, (ii) $v = 2^t$, $t > 1$ integer, $m = v-1$, (iii) $v = tk+1$, odd prime or prime power, $t \geqslant 3$ integer, $k \geqslant 3$ odd integer, and $m = v-1$. We also exhibit how these d^*'s can be employed to get optimal designs for the multiples of m already chosen.

(i) $v = 3$, $m = 3$.

$$d^* = \begin{bmatrix}
\overset{A_1}{-} & \overset{A_3}{1} & \overset{A_2}{2} & \overset{B_3}{0} & \overset{B_1}{2} & \overset{B_2}{0} & \overset{C_2}{0} & \overset{C_1}{1} & \overset{C_3}{2} \\
\overset{A_3}{1} & \overset{A_2}{-} & \overset{A_1}{0} & \overset{B_1}{2} & \overset{B_2}{2} & \overset{B_3}{1} & \overset{C_1}{1} & \overset{C_3}{2} & \overset{C_2}{0} \\
\overset{A_2}{2} & \overset{A_1}{0} & \overset{A_3}{-} & \overset{B_2}{0} & \overset{B_3}{1} & \overset{B_1}{1} & \overset{C_3}{2} & \overset{C_2}{0} & \overset{C_1}{1} \\
\overset{B_3}{2} & \overset{B_2}{0} & \overset{B_1}{1} & \overset{C_1}{-} & \overset{C_3}{1} & \overset{C_2}{2} & \overset{A_3}{0} & \overset{A_2}{0} & \overset{A_1}{1} \\
\overset{B_1}{0} & \overset{B_3}{1} & \overset{B_2}{2} & \overset{C_3}{1} & \overset{C_2}{-} & \overset{C_1}{0} & \overset{A_2}{0} & \overset{A_1}{2} & \overset{A_3}{2} \\
\overset{B_2}{1} & \overset{B_1}{2} & \overset{B_3}{0} & \overset{C_2}{2} & \overset{C_1}{0} & \overset{C_3}{-} & \overset{A_1}{1} & \overset{A_3}{2} & \overset{A_2}{1} \\
\overset{C_3}{0} & \overset{C_2}{1} & \overset{C_1}{2} & \overset{A_2}{2} & \overset{A_1}{0} & \overset{A_3}{1} & \overset{B_1}{-} & \overset{B_2}{1} & \overset{B_3}{2} \\
\overset{C_2}{1} & \overset{C_1}{2} & \overset{C_3}{0} & \overset{A_3}{0} & \overset{A_2}{1} & \overset{A_1}{2} & \overset{B_2}{1} & \overset{B_3}{-} & \overset{B_1}{0} \\
\overset{C_1}{2} & \overset{C_3}{0} & \overset{C_2}{1} & \overset{A_1}{1} & \overset{A_3}{2} & \overset{A_2}{0} & \overset{B_3}{2} & \overset{B_1}{0} & \overset{B_2}{-}
\end{bmatrix}$$

where $A_1, A_2, A_3, C_1, C_2, C_3, B_1, B_3, B_2$ denote the 9 levels of the 3rd direction, the first two directions being the rows and the columns as usual, and 0, 1, 2 denote the three treatments.

In the following two cases $v = 2^t$ and $v = tk+1$, we use some common notation. Let L be a Latin Square written with v symbols of $GF(v)$ such that the symbols appearing along the diagonal are all distinct. Let $(L+x_i)$, $x_i \in GF(v)$ denotes the Latin Square obtained from L by replacing each symbol y of L by the symbol $(y+x_i)$. L^* is obtained from L after deleting the diagonal (i.e. the diagonal positions of L^* are all empty) and $(L+x_i)^{**}$ is obtained from $(L+x_i)$ by only replacing its diagonal by the diagonal of L.

(ii) $v = 2^t$, $t > 2$ integer, $m = v-1$.

Define $x_i = 1+\alpha+\alpha^2+\ldots+\alpha^{i-1}$, $i = 1, 2, \ldots, m$. Where α is a primitive element of $GF(v)$. Obviously, x_i's are all distinct elements of $GF(v)$, and

$x_m = 0$. We define a set of mv symbols divided into m sets, the ith set written as $A_i(x)$, $x \in GF(v)$, $i = 1, 2, \ldots, m$. Let A_i be a Latin Square written with the v symbols $A_i(x)$, such that along the diagonal of A_i, all the symbols are distinct. Each A_i, so constructed when super-imposed on L is assumed to form a Graeco Latin Square. Let $\widetilde{L}$ be any other Latin Square with elements of $GF(v)$, which need not have distinct elements along the diagonal and will form a Graeco Latin Square when superimposed on A_i. Now d^* can be constructed as follows.

First make an $m \times m$ Latin Square treating $\mathcal{A}_i$'s, $i = 1, 2, \ldots, m$ as m distinct symbols, such that along the principal diagonal and along the second direct diagonal (i.e. just above the principal diagonal) the symbols are all distinct. Now in place of symbol $\mathcal{A}_i$, insert the Latin Square A_i. The symbols $A_i(x)$, $i = 1, 2, \ldots, m$, $x \in GF(v)$ in the order now they occur along principal diagonal of this resulting $mv \times mv$ square will form the definite order of the levels of the third direction. Now arrange v treatments in this $mv \times mv$ square as follows :

Originally we started with an $m \times m$ Latin Square with m symbols $\mathcal{A}_i$, then $\mathcal{A}_i$'s are replaced by $v \times v$ squares A_i's as already indicated. Considering the original $m \times m$ square, its diagonal has m distinct elements $\mathcal{A}_1$, $\mathcal{A}_2$, ..., $\mathcal{A}_m$, in some order. Superimpose the treatment square L^* on each of the A_i's (written in place of $\mathcal{A}_i$'s) on the principal diagonal. Then superimpose $(L+x_i)^{**}$, in the given order $i = 1, 2, \ldots, m$ on A_i's along the second direct diagonal of the $m \times m$ original square. Superimpose $\widetilde{L}$ on the remaining A_i's of this $m \times m$ square. Then d^* looks like this.

$$d^* = \begin{bmatrix} A_1 & A_2 & A_3 & A_4 & \cdots & \cdots & \cdots & A_{m-1} & A_m \\ L^* & (L+x_1)^{**} & L & \widetilde{L} & \cdots & \cdots & \cdots & \widetilde{L} & \widetilde{L} \\ A_2 & A_3 & A_4 & A_5 & \cdots & \cdots & \cdots & A_m & A_1 \\ \widetilde{L} & L^* & (L+x_2)^{**} & \widetilde{L} & \cdots & \cdots & \cdots & \widetilde{L} & \widetilde{L} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ A_{m-1} & A_m & A_1 & A_2 & \cdots & \cdots & \cdots & A_{m-3} & A_{m-2} \\ \widetilde{L} & \widetilde{L} & \widetilde{L} & \widetilde{L} & \cdots & \cdots & \cdots & L^* & (L+x_{m-1})^{**} \\ A_m & A_1 & A_2 & A_3 & \cdots & \cdots & \cdots & A_{m-2} & A_{m-1} \\ L & \widetilde{L} & \widetilde{L} & \widetilde{L} & \cdots & \cdots & \cdots & \widetilde{L} & L^* \end{bmatrix}$$

Clearly, from the construction procedure, d^* satisfies condition (i). As x_i's $i = 1, 2, \ldots, m-1$ are non-null distinct elements, d^* satisfies condition (ii) also. To verify condition (iii) we intrroduce vector notatior $d(L)$, and $d(L+x_i)$, $i = 1, 2, \ldots, m-1$, to denote the v distint treatment symbols along the diagonals of L and $(L+x_i)$ respectively. The mv treatments (not all distinct) occuring with $(m-1)$ replications along the mv levels of row and column directions are given by the two vectors

$$(d(L+x_1), \quad d(L+x_2), \quad d(L+x_3), \quad \ldots\ldots, \quad d(L))$$

and

$$(d(L), \quad d(L+x_1), \quad d(L+x_2), \quad \ldots\ldots, \quad d(L+x_{m-1}))$$

respectively. As each of $d(L+x_i)$, $i = 1, 2, \ldots m$ contsists of v distint treatment symbols and $(x_1, x_2, x_1, x_3-x_2, \ldots, x_{m-1}-x_{m-2}, -x_{m-1})$ consists of all non-null elements of $GF(v)$ each appearing exactly once, the mv treatment pairs occurring $(m-1)$ times along the (i, i), $i = 1, 2, \ldots mv$ level combinations of row and column directions respectively exhaust all possible $\frac{v(v-1)}{2}$ pairs, each occurring equal number of times. Verification of this property for row vs. third direction and column vs. third direction is immediate.

(iii) $v = tk+1$ odd prime or prime power, k odd (>1), $t>2$ integer, $m = v-1$.

Let x be a primitive element of $GF(v)$. Let A_i^j, $i = 1, \ldots, t, j = 1, \ldots k$ be Latin Squares of size v with symbols $A_i^j\,(\alpha)$, $\alpha \epsilon\, GF(v)$ such that along the principal diagonal, the symbols appearing are distinct and L and A_i^j's when superimposed form Graeco Latin Squares. Let A_i's, $i = 1, \ldots t$ be Latin Squares of size k, formed with symbols A_i^j, $i = 1, \ldots k$, such that the symbols A_i^j's appearing along the principal diagonal and also the second direct diagonal are distinct. Since k is odd, this type of configuration is always possible. Let $\tilde{L}$ be any Latin Square of size v, with symbols from $GF(v)$, such that A_i^j superimposed on $\tilde{L}$ is a Graeco Latin Square. Now d^* can be constructed as follows.

First make a $t \times t$ Latin Square with symbols $\mathcal{A}_i$, $i = 1, \ldots t$ such that along the principal diagonal the symbols are distinct. Now in place of symbols $\mathcal{A}_i$'s, put the Latin Squares A_i's with symbols $\mathcal{A}_i^j$'s, $j = 1, \ldots, k$. Then in place of symbols $\mathcal{A}_i^j$'s put the corresponding Latin Squares A_i^j's. Now the order in which the symbols $\mathcal{A}_i^j(\alpha)$, $i = 1, \ldots, t, j = 1, \ldots, k$, $\alpha \epsilon GF(v)$, occur along the principal diagonal of this $mv \times mv$ square represent the definite order of the levels of the third direction.

We can now arrange v treatments on this square as follows. In the original $t \times t$ Latin Square, the symbols $\mathcal{A}_i$'s are repalced by the Latin Squares

A_i's of order $k\times k$ with symbols $\mathcal{A}_i^j$'s. Superimpose L^* on each of the Latin Squares A_i^j's (written in place of $\mathcal{A}_i^j$'s) occurring in the principal diagonal of the resulting $tk\times tk$ square. Consider the second direct diagonal of each of the $k\times k$ suqares occurring in the principal diagonal positions of the original $t\times t$ square. Take a representative $k\times k$ square, say A_{i_1}. Then superimpose the treatment squares

$$(L+x^{i_1-1})^{**},\ (L+x^{(i_1-1)+t})^{**},\ \dots\ (L+x^{(i_1-1)+t(k-1)})^{**}$$

in that order on the Latin Squares written corresponding to the elements occurring on the second direct diagonal of the square specified. The same is done for each such $k\times k$ square in the principal diagonal. Superimpose $\widetilde{L}$ on each of the remaining $v\times v$ Latin Squares written corresponding to the elements of the $tk\times tk$ square. The method is illustrated below :

Let

$$C_t = \left[\begin{array}{llll}
A_i^1 & A_i^2 & \dots\dots\dots & A_i^k \\
L^* & (L+x^{t-1})^{**} & & \widetilde{L} \\
A_i^2 & A_i^3 & \dots\dots\dots & A_i^1 \\
\widetilde{L} & L^* & & \widetilde{L} \\
\dots & \dots & \dots\dots\dots & \\
\dots & \dots & \dots\dots\dots & \\
A_i^{k-2} & A_i^{k-1} & \dots\dots\dots & A_i^{k-2} \\
\widetilde{L} & \widetilde{L} & & (L+x^{t-1+t(k-2)})^{**} \\
A_i^{k-1} & A_i^1 & \dots\dots\dots & A_i^{k-1} \\
(L+x^{t-1+t(k-1)})^{**} & \widetilde{L} & & L^*
\end{array}\right]$$

$$B_t = \left[\begin{array}{llll}
A_i^1 & A_i^2 & \dots\dots & A_i^k \\
\widetilde{L} & \widetilde{L} & & \widetilde{L} \\
A_i^2 & A_i^3 & \dots\dots & A_i^1 \\
\widetilde{L} & \widetilde{L} & & \widetilde{L} \\
\dots\dots & \dots\dots & \dots\dots & \dots\dots \\
A_i^k & A_i^1 & & A_i^{k-1} \\
\widetilde{L} & \widetilde{L} & & \widetilde{L}
\end{array}\right],\quad i = 1, 2, \dots, t.$$

The design d^* is given by the matrix D, where

$$D = \begin{bmatrix} C_1 & B_t & B_3 & \ldots\ldots & B_2 \\ B_2 & C_2 & B_t & \ldots\ldots & B_4 \\ \ldots & \ldots & \ldots & \ldots\ldots & \ldots \\ B_{t-1} & B_2 & B_4 & \ldots\ldots & C_t \end{bmatrix}$$

The sets $\{x, x^{i+t}, \ldots x^{i+t(k-1)}\}$ $i = 0, 1, \ldots, t-1$ have the following property: The sets $(x^{i+t}-x^{i}, \;\; x^{i+2t}-x^{i+t} \ldots, \;\; x^{i+t(k-1)}-x^{i+t(k-2)}), \;\; x^{i}-x^{i+t(k-1)}), \; i = 0, 1, \ldots, t-1$ exhaust all the distinct non-null elements of $GF(v)$, each occurring exactly once. This ensures the property (iii) of d^* along row vs. column directions. Automatically the same property holds for row vs. third and column vs. third direction. The other property is obviously satisfied.

In each of the above three cases, suppose the design is constructed for some m and we want to construct it for some multiple of m say $m^* = pm$ $p > 2$. The $(mv \times mv)$ optimal d^* can then be used to construct $m^*v \times m^*v$ optimal designs. Let $d_1^*, d_2^*, \ldots, d_p^*$ be p optimal designs of size $mv \times mv$, constructed in the above manner with p different sets of mv symbols. d_i^{**}'s are designs obtained from d_i^*'s by replacing L^* and $(L+x)^{**}$, $x \in GF(v)$ by $\tilde{L}$ or L simply. Then d^* $(m^*v \times m^*v)$ can be constructed by making a $p \times p$ Latin Square with symbols d_i^* such that along the diagonal the symbols appearing are distinct. Then replace all the off diagonal d_i^*'s by d_i^{**}'s, and in place of d_i^* or d_i^{**} symbols, put the corresponding $(mv \times mv)$ squares.

4. Efficiency of a simple class of designs in the line of Aggarwal 1966(b)

The above discussion reveals that construction of these optimal designs d^* are highly involved but can be settled in some cases atleast. But the structure of the optimal design d^* considered by us requires m to be a multiple of $\frac{v-1}{2}$ and in fact in all our constructions in the preceding section we have chosen $m = v-1$. This imposes a restriction on the size of the square and consequently the number of observations which may not be always easy to meet. We give in the following lines a simple design which happens to be a trivial generalisation to three way set-up of Aggarwal's (1966b) design considered in the context of two way set-up. These designs are simple, easy to use, can be constructed for all $m \geqslant 2$, (except when $mv = 6$) and have very high efficiency with regard to A-optimality criterion, even though the designs are not definitely optimal in cases where the optimal designs constructed by us exist.

Let us define a set of mv symbols divided into m sets, the ith set written as α_j^i, $j = 1, 2, \ldots, v$, $i = 1, 2, \ldots, m$. Let GL_i be a $v \times v$ Graeco-Latin Square formed with two sets of symbols $\{1, 2, \ldots, v\}$ and $\{\alpha_1^i, \alpha_2^i, \ldots, \alpha_v^i\}$ such that along the diagonal $\{1\alpha_1^i, 2\alpha_2^i, \ldots, v\alpha_v^i\}$ occurs in the same order. Let $\widetilde{G}L_i$ be any Graeco Latin Square formed with the above two sets of symbols. GL_i^* is obtained from $\widetilde{G}L_i$ after deleting its diagonal. Then a new $mv \times mv$ design d_0 can be formed as follows. First make a $m \times m$ Latin Square treating $\widetilde{G}L_i$, $i = 1, 2, \ldots, m$ as m symbols such that along the diagonal the symbols appearing are distinct. Then replace the diagonal $\widetilde{G}L_i$ symbols by the Graeco Latin Square GL_i^*, and off diagonal $\widetilde{G}L_i$'s by the corresponding Graeco Latin Square $\widetilde{G}L_i$. For m odd, one representation of d_0 can be

$$d_0 = \begin{bmatrix} GL_1^* & \widetilde{G}L_2 & \widetilde{G}L_3 & \ldots\ldots & \widetilde{G}L_m \\ \widetilde{G}L_2 & GL_3^* & \widetilde{G}L_4 & \ldots\ldots & \widetilde{G}L_1 \\ \widetilde{G}L_m & \widetilde{G}L_1 & \widetilde{G}L_2 & \ldots\ldots & GL_{m-1}^* \end{bmatrix}$$

In d_0, the symbols α_j^i, $j = 1, \ldots, v$, $i = 1, \ldots, m$ denote the levels of third direction, the other two directions are represented by rows and columns as usual. However, it possesses a high degree of efficiency as is demonstrated below with respect to A-optimality criterion. Recall from expression (1.2),

$$C_{dhh} = r_{dh} - \frac{mv-4}{mv(mv-3)} \sum_{i=1}^{3} \sum_{l=1}^{mv} n_{dhl}^{(i)2}$$

$$-\frac{1}{mv(mv-3)} \sum_{i<j}\sum \sum_{l=1}^{mv} (n_{dhl}^{(i)}+n_{dhl}^{(j)})^2 + \frac{2}{(mv-1)(mv-3)} r_{dh}^2$$

As in the case of two way elimination of heterogeneity, one can derive the expression for $g(r_h)$ i.e. max of C_{dhh} subject to the condition $\sum_{l=1}^{mv} n_{dhl}^{(i)} = r_{hi}$ $\forall$ $i = 1, 2, 3$. Denoting $\bar{r}$ by $\bar{r} = (m-1)\ mv + m(v-1) = umv + t$ (say) the expression of $g(r)$ for $r = \bar{r}$ is as follows :

$$g(\bar{r}) = A(\bar{r}) + B_2(\bar{r})$$

where $$A(\bar{r}) = \bar{r} + \frac{2\bar{r}^{-2}}{(mv-1)(mv-3)} - \frac{3(mv-4)}{mv(mv-3)}(-u^2mv + \bar{r} + 2u\bar{r} - mvu)$$

$$-\frac{3}{mv(mv-3)}(8u\bar{r} - 4\ mvu^2)$$

and $$B_2(\bar{r}) = -\frac{3}{mv(mv-3)}(6\bar{r} - 6\ mvu - 2mv). \qquad \ldots (4.1)$$

Note that for the optimal design d^* as well as for the above mentioned extended Aggarwal's design d_0, the C-matrices are completely symmetric. Hence we get for the efficiency of d_0 the expression,

$$E_{d_0} = \frac{\sum\limits_{i \neq j} \sum V(\tau_i - \tau_j) \text{ using } d^*}{\sum\limits_{i \neq j} \sum V(\tau_i - \tau_j) \text{ using } d_0} = \Sigma \frac{1}{\lambda_{d_i^*}} \Big/ \Sigma \frac{1}{\lambda_{d_0 i}}$$

$$= \frac{a_0}{a^*} \text{ using the representations } C_{d^*} = a^* (I - J/v) \text{ and } C_{d_0} = a_0 \ (I - J/v).$$

From expression (4.1), we get

$$(1-1/v)\, a^* = A(\bar{r}) - \frac{6m(2v-3)}{mv(mv-3)}$$

and

$$(1-1/v)\, a^0 = A(\bar{r}) - \frac{12m(v-1)}{mv(mv-3)}$$

$$= E(\bar{r}) \text{ say.}$$

$$E_{d_0} \text{ now simplifies to } \frac{E(\bar{r})}{E(\bar{r}) + \dfrac{6}{v(mv-3)}}.$$

Calculations indicated that even for moderate values of v and m, E_{d_0} is close to unity.

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