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RATES OF CONVERGENCE TO NORMALITY FOR SOME VARIABLES WITH ENTIRE CHARACTERISTIC FUNCTION

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SUMMARY. Nonuniform rates of convergence to normality are studied for standardised sum of independent random variables in a triangular array when m.g.f. of the random variables necessarily exist but the r.v's may not be bounded. The assumed condition (2.1) implies that each variable has an entire characteristic function of order $\leqslant 2$. As application of these results, rates of moment type convergences and non-uniform L_p version of Berry-Esseen theorem are obtained. The results are generalised to the general non-linear statistics. As for example linear process is considered.

1. Introduction

Consider a double sequence $\{X_{ni} : 1 \leqslant i \leqslant n, n \geqslant 1\}$ of r.v's where variables in each array are independently distributed and satisfy $EX_{ni} = 0$. Then defining

$$S_n = \sum_{i=1}^{n} X_{ni}, s_n^2 = \sum_{i=1}^{n} EX_{ni}^2 \text{ and } F_n(t) = P(s_n^{-1} S_n \leqslant t)$$

we have, under very moderate assumption that $F_n \Longrightarrow \Phi$. In i.i.d case the uniform rate of convergence of $|F_n(t) - \Phi(t)|$ to zero is provided by classical Berry-Esseen theorem and was later extended by Katz (1963).

Through very helpful, these uniform rates are inappropriate for many purposes, e.g. since $F_n \Longrightarrow \Phi$ it is natural to ask when does a L_p version of Berry-Esseen theorem holds, or given that $Eg(T) < \infty$ where T is a normal deviate and g is a real valued non negative, even and non decreasing function over $[0, \infty)$, when does $|Eg(s_n^{-1} S_n) - Eg(T)| \to 0$ and at what rate? Note that $Eg(T) < \infty$ if $g(x) = O((1+|x|)^{-\delta} \exp(x^2/2))$ for some $\delta > 1$. We explain further in the followings.

Consider the double sequence X_{ni} which along with $EX_{ni} = 0$ also satisfies

$$\sup_{n \geqslant 1} n^{-1} \sum_{i=1}^{n} EX_{ni}^2 \, g(X_{ni}) < \infty \qquad \ldots \quad (1.1)$$

where g is non negative, even, non decreasing on $[0, \infty)$.

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The whole spectrum of g can be broadly classified into three categories :

(i) $g(x) \leqslant |x|^k$ for some $k > 0$.

(ii) $|x|^k \leqslant g(x) \leqslant \exp(s|x|), \forall\, k > 0$ and some $s > 0$

(iii) $g(x) \geqslant \exp\,(s|x|), \forall\, s > 0$.

The first case where a finite moment higher than second exists has been dealt by various authors. Von Bahr (1965) considered convergence of moments with $g(x) = |x|^c, c > 0$. Michel (1976) derived non-uniform rates with same g in i.i.d case and used these to find a normal approximation zone, i.e. a zone of t_n where $1-F_n(t_n) \sim \Phi\,(-t_n) \sim F_n(-t_n), t_n \to \infty$ and to find out rate of moment convergences. His results were extended to triangular array of independent random variables with slightly more general g by Ghosh and Dasgupta (1978), the results were also extended to non-linear statistis in general. A non-uniform L_p version of Berry Esseen theorem was also derived.

The situation (ii) has also been studied extensively, e.g., see Linnik (1961, 62), Nagaev (1979) in the intermediate case and under the assumption of existance of m.g.f by Chernoff (1952), Plachky (1971), Plachky and Steinebach (1975), Bahadur and Rao (1960), Statulevicius (1966), Petrov (1975) etc. That the necessary and sufficient assumptions for the normal approximation zones are the same is shown in Dasgupta (1989) with allied results.

In this paper we study the situation (iii) when m.g.f. of the r.v's exist but the r.v's may not be bounded. We only partially cover the spectrum (iii) as it turns out that better result may not be possible in general even when the r.v's are bounded, see remark 1. Also since it is known that normal approximation zone, i.e., the zone of t_n such that $1-F_n(t_n) \sim \Phi(-t_n) \sim F_n(-t_n)$ $t_n \to \infty$, cannot be extended in general, even for bounded r.v's compared to weaker assumption of the existance of m.g.f. (see e.g. Feller p-520, (6.21)) we shall not proceed to study normal approximation zone in this case which has already been considered in Dasgupta (1989).

We shall assume without loss of generality

$$E\,X_{ni} = 0 \;\forall\; n \geqslant 1, \quad 1 \leqslant i \leqslant n \qquad \dots \quad (1.2)$$

and

$$\underline{\lim}\, n^{-1}\, s_n^2 \geq 0 \text{ where } s_n^2 = \sum_{i=1}^{n} E\,X_{ni}^2 \qquad \dots \quad (1.3)$$

With the assumption that all the odd order moments are vanishing i.e.,

$$E\,X_{ni}^{2m+1} = 0 \;\forall\; n \geqslant 1,\; 1 \leqslant i \leqslant n,\; m = 1, 2, 3, \ldots \qquad \ldots \quad (1.4)$$

we shall show that a sharper result is possible. As one may note this is satisfied for symmetric r.v's.

In section 2 we prove the results for independent r.v's in a triangular array and these are generalised to non linear statistics in Section 3. As for example linear process is considered in Section 4. The implications of the assumptions made and some examples are discussed in Section 5.

2. The results on the row sums of random variables in a triangular array

We start with the following theorem :

Theorem 2.1. *Let* $\{X_{ni} : 1 \leqslant i \leqslant n, n \geqslant 1\}$ *be a triangular array of r.v's where variables within each array are independent and satisfy* (1.2)—(1.4) *and*

$$\sup_{n\geqslant 1}\; (n/s_n^2)^m \frac{1}{n} \sum_{i=1}^{n} E\,X_{ni}^{2m} \leqslant l^{-m}\,(2m)\,!\,/\,m\,!$$

$$1 < l \leqslant 2,\; m = 2, 3, \ldots \qquad \ldots \quad (2.1)$$

then there exist a constant $b\ (> 0)$ *such that*

$$|F_n(t) - \Phi(t)| \leqslant b \exp\left(-t^2(1 - l^{-1})\right),\; -\infty < t < \infty. \qquad \ldots \quad (2.2)$$

Remark 1. The bound in (2.2) cannot, in general, be substantially improved even for bounded r.v's is evident from the fact that $F_n(t) - \Phi(t) = 1 - \Phi(t)$ for $t > a\, n^{1/2}$, 'a' being sufficiently large and X_{ni}'s bounded. $\sim (2\pi)^{-1/2}\, t^{-1} \exp(-t^2/2),\; t \to \infty$.

For a particular r.v X, (2.1) implies $E \exp(cX^2) < \infty$ for some $c > 0$, which in turn implies that the c.f of X is an entire function of order $\leqslant 2$, possibly having zeroes (see Feller, 1969, 498-499). In the followings b represents a generic positive constant.

Proof of the theorem. Since $1 - \Phi(t) \leqslant b\,|t|^{-1} \exp(-t^2/2)$ sufficient to show that

$$P(s_n^{-1} S_n > t) \leqslant \exp\left(-t^2(1 - l^{-1})\right),\; t > 0. \qquad \ldots \quad (2.3)$$

Now

$$P(s_n^{-1} S_n > t) \leqslant \prod_{i=1}^{n} \beta_i \exp(-h\, s_n\, t) \qquad \ldots \quad (2.4)$$

where

$$\beta_i = E[\exp(h X_{ni})],\ i = 1, 2, \ldots, n. \tag{2.5}$$

Let $h = t/s_n$ then

$$P(s_n^{-1} S_n > t) \leqslant \left(\prod_{i=1}^{n} \beta_i \right) \exp(-t^2). \quad \ldots \quad (2.6)$$

Now

$$\prod_{i=1}^{n} \beta_i \leqslant \exp(h^2 s_n^2/l) \quad \ldots \quad (2.7)$$

since

$$(\pi \beta_i)^{1/n} \leqslant \frac{1}{n} \Sigma\beta_i \leqslant \sum_{m=0}^{\infty} \frac{h^{2m}}{(2m)!} \left(\frac{1}{n} \sum_{i=1}^{n} EX_{ni}^{2m}\right) \leqslant \sum_{m=0}^{\infty} \left(\frac{h^2 s_n^2}{n}\right)^m \frac{1}{m!}$$

from (1.4) and (2.1). Hence (2.3).

Remark 2. If odd order moments are non-zero, still we may have

$$\beta_i \leqslant E(e^{-hx_{ni}} + e^{-hx_{ni}}) \leqslant 2 \sum_{m=0}^{\infty} \frac{h^2}{(2m)!} E X_{ni}^{2m}$$

$$P(s_n^{-1} S_n > t) \leqslant 2^n \exp(-t^2(1-l^{-1})). \quad \ldots \quad (2.8)$$

Hence we have the following.

Theorem 2.2. *If the assumption* (1.4) *is omitted in Theorem* 2.1 *then*

$$|F_n(t) - \Phi(t)| \leqslant b\, 2^n\, exp(-t^2(1-l^{-1})) \quad \ldots \quad (2.9)$$

In the following we continue to assume that odd order moments are zero, our next theorem states moment type convergences of $Y_n = |s_n^{-1} S_n|$ to that of $T = |N(0, 1)|$.

Theorem 2.3. *Let the assumptions of Theorem* 2.1 *alongwith* $\inf n^{-1} s_n^2 > 0$ *be satisfied, let* $g : (-\infty, \infty) \rightarrow [0, \infty)$, $g(x)$ *even,* $g(0) = 0$ *be such that* $E\, g(T) < \infty$ *and*

$$g'(x) = O\{(1+x)^{-1-\delta} exp(x^2(1-l^{-1}))\},\ x > 0,\ \delta > 0. \quad \ldots \quad (2.10)$$

Then

$$|E\, g(Y_n) - E\, g(T| = O(n^{-\delta^*}) \quad \ldots \quad (2.11)$$

where

$$\delta^* = \left\{\frac{\delta}{3+(\delta \vee 1)} \wedge \frac{1}{2}\right\}$$

Proof. Under (2.1) since the m.g.f. of X_{ni} exist (as $\beta_i < \infty$ for fixed h) we have, in view of $\inf_{n \geqslant 1} n^{-1} s_n^2 > 0$ and (1.4) with $m = 1$, by an application of Theorem 2.2 of Dasgupta (1989),

$$|F_n(t)-\Phi(t)| \leqslant b \exp(-t^2/2)\, r_n,\ 0 < |t| < M_n \qquad \ldots \quad (2.12)$$

where $\quad r_n = (n^{-1} M_n^3)\, V(n^{-1/2}),\ M_n = O(n^{1/4}).$

To see this note that from (2.6) of Dasgupta (1989) for $0 < t \leqslant \alpha n^{1/4}$ where $\alpha > 0$ one has $|t|^{-1}\ |\exp(O(n^{-1}t^4))-1| = O(n^{-1}t^3) = O(n^{-1}M_n^3)$ with $t \leqslant \alpha n^{1/4} = O(M_n)$. In the same region of t, $\exp(-t^2/2+O(n^{-1}t^4))n^{1/2} \leqslant b \exp(-t^2/2)\, n^{-1/2}$ and $\sum_{i=1}^{n} P(|X_{ni}| > rs_n|t|) \leqslant b|t|^{-2}\, e^{-r^* t n^{1/2}}$ for some $r^* > 0$ since m.g.f. exists $\leqslant bn^{-1/2}\, e^{-t^2/2}$ for $t \leqslant \alpha n^{1/4}$.

Again from Theorem 2.1,

$$|F_n(t)-\Phi(t)| \leqslant b \exp(-t^2(1-l^{-1})),\ M_n \leqslant |t| < \infty. \qquad \ldots \quad (2.13)$$

Hence with the representation

$$|E\, g(Y_n)-E\, g(T)| \leqslant \int_0^\infty g'(t)\, |P(|s_n^{-1} S_n| \leqslant t)-P(|N(0,1)| \leqslant t)|\ dt \qquad \ldots \quad (2.14)$$

and that

$$\int_0^\infty (1+x)^{-1-\delta}\, dx < \infty,\ \int_{M_n}^\infty (1+x)^{-1-\delta}\, dx = O(M_n^{-\delta})$$

we have

$$|E\, g(Y_n)-E\, g(T)| = O(r_n)+O(M_n^{-\delta}) \qquad \ldots \quad (2.15)$$

Equating the order of $M_n^{-\delta}$ and r_n, the result follows.

The following theorem provides a non uniform L_q version of the Berry-Esseen theorem.

Theorem 2.4. *Under the assumptions of theorem* 2.1 *for any* $q \geqslant 1$

$$\|exp(t^2(1-l^{-1}))\,(1+|t|)^{-(\delta+1/q)}\,(F_n(t)-\Phi(t))\|q = O(n^{-\delta^*}) \ \ldots \quad (2.16)$$

where $\delta > 0$ *and* δ^* *is defined in Theorem* 2.3.

Remark 3. The bound in (2.16) is quite sharp. For symmertic point binomial variable this asserts

$$\left\| \exp\left(\frac{t^2}{2}\right)(1+|t|)^{-(\delta+1/q)}(F_n(t)-\Phi(t)) \right\|_q = O(n^{-1/2})$$

whereas the weaker known result given in Bhattacharya and Rao (1976) states

$$|F_n(t)-\Phi(t)| \leqslant \exp\left(-\frac{t^2}{4}\right).$$

The proof of (2.16) follows along the lines of theorem 2.3 expressing l.h.s. of (2.16) in integral form.

Next we consider moment convergences and L_q version of the Berry-Esseen theorem when the assumption (1.4) is not satisfied i.e., odd order moments are non-zero.

Theorem 2.5. *Let the assumptions* (1.2), (1.3) *and* (2.1) *be satisfied. Then for any* $g:(-\infty,\infty)\to[0,\infty)$, $g(x)$ *even* $g(0)=0$ *such that* $E\,g(T)<\infty$ *and*

$$g'(x) = O(exp(x^2\nu)),\ x>0 \qquad \ldots \quad (2.17)$$

and $0<\nu<min(c^*,(1-l^{-1}))=\nu^*$ *the following holds*

$$|Eg(Y_n)-Eg(T)| = O(n^{-1/2}) \qquad \ldots \quad (2.18)$$

$c^*(>0)$ *being a constant depending on distributions of* $\{X_{ni}\}$, *via*

$$c^{**} = \min_{0<r<\infty}\ \sup_{n\geqslant 1}\frac{1}{n}\sum_{i=1}^{n}[(2r/3)E|X_{ni}|^3exp(2r|X_{ni}|)-1]r.$$

[Note that since $k(r)=\sup\limits_{n\geqslant 1}\frac{1}{n}\sum\limits_{i=1}^{n}E|X_{ni}|^3\exp(2r|X_{ni}|)\uparrow\infty$ as $r\uparrow\infty$ there exists a r^* such that $r^*=\frac{3}{4k(r^*)}$, hence $c^{**}<-r^*/2<0$.

Proof. First of all we shall show

$$|F_n(t)-\Phi(t)| \leqslant bn^{-1/2}\exp(-\nu^*t^2),\ -\infty<t<\infty \qquad (2.19)$$

then the theorem will follow from (2.17) with the representation (2.14)

Without loss of generality let $t<0$. Since the m.g.f. of X_n exist under (2.1) delating the last term of r.h.s. of (2.2) Theorem 2.1, of Dasgupta (1989) and following the proof of Theorem 2.6 of Dasgupta (1989) we have

$$|F_n(t)-\Phi(t)| \leqslant b\,n^{-1/2}\exp(-at^2) \qquad (2.20)$$

for $t^2 \geqslant (p-2a)^{-1}\log n$, $0<a<\frac{p}{2}$, $0<p<1$ and $t\leqslant f(p)n^{1/2}$ where $f(p)>0$. Similarly for $t^2\leqslant(p-2a)^{-1}\log n$ from theorem 1 of Ghosh and Dasgupta (1978) choosing c therein sufficiently large,

$$|F_n(t)-\Phi(t)| \leqslant b\,n^{-1/2}\exp(-a_1t^2),\ 0<a_1<\frac{1}{2}. \qquad \ldots \quad (2.21)$$

There c is taken to be sufficiently large to make $(p-2a)^{-1} \leqslant cK/2$ and the order of the second term of (2.1) of Ghosh and Dasgupta (1978) is

$$\sum_{i=1}^{n} P(|X_{ni}| < rs_n|t|) \leqslant b|t|^{-2}e^{-(rsnt)^2 c/c_n^2}, \text{ see (5.1), (5.2) ; } 0 < c < l/4.$$
$$\leqslant bt^{-2}\, e^{-rcnt^2}$$
$$\leqslant bn^{-1/2}\, e^{-t^2}/2, \text{ for } t^2 \leqslant (p-2a)^{-1} \log\, n.$$

Also from Theorem 2.2

$$|F_n(t) - \Phi(t)| \leqslant b\, 2^n \exp(-t^2(1-l^{-1})$$
$$\geqslant b\, n^{-1/2} \exp(-a_2 t^2),\, a_2 > 0 \qquad \ldots \quad (2.22)$$

if $t^2 \geqslant \lambda^2 n$ for some λ depending on l and a_2, $0 < a_2 < (1-l^{-1})$.

Finally for the zone $f(p)n^{1/2} < t < \lambda\ n^{1/2}$ we imitate the proof of Theorem 2.5 of Dasgupta (1989) with $g(x) = \exp(|x|)$ and $h = 2t^{-1} s_n^{-1} \log (tg(rs_nt)) \doteq 2r,\ 0 < r < \infty$ to obtain

$$|F_n(t) - \Phi(t)| \leqslant b\, \{t\, g(rs_nt)\}^{-1+hk(r)/3}$$
$$\leqslant b\, e^{rs_nt(2rk(r)/3-1)}$$

where $k(r)$ is defined in the Theorem 2.5

$$\leqslant b\, e^{s_nct^{**}},\ c^{**} < -r^*/2 < 0\ ; \qquad \ldots \quad (2.23)$$

So for $f(p)\, n^{1/2} < t < \lambda\, n^{1/2}$, $t = O_e(n^{1/2}) = O_e(s_n)$, one gets

$$|F_n(t) - \Phi(t)| \leqslant b\, n^{-1/2}\, e^{-c^*t^2}, \qquad \ldots \quad (2.24)$$

$c^* > 0$ depends on c^{**} and λ. (2.19) follows from (2.20), (2.21) and (2.24). Hence the theorem.

The following corrollary on a nonuniform L_q version of Berry-Esseen theorem is also immediate from (2.19).

Corrollary 2.1. *Let the assumptions* (1.2), (1.3) *and* (2.1) *be satisfied. Then for any* $\delta > 1$ *and* $q \geqslant 1$.

$$\|(1+|t|)^{-\delta/q}\ exp\,(v^*t^2)\,(F_n(t) - \Phi(t))\|_q = O(n^{-1/2}) \qquad \ldots \quad (2.25)$$

where v^* *is defined in Theorem* 2.5.

3. Rates of convergence for general non-linear statistics

This section generalises the results of section 2 for non-linear statistics of the form

$$T_n = s_n^{-1}\, S_n + R_n \qquad (3.1)$$

where $\quad S_n = \sum_{i=1}^{n} X_{ni},\ s_n^2 = \sum_{i=1}^{n} E\, X_{ni}^2,\ \inf_{n \geqslant 1} n^{-1} s_n^2 > 0$

$X_{n1}, X_{n2}, \ldots, X_{nn}$ being independent r.v's with vanishing expectation. Also let

$$E(R_n^{2m}) \leqslant c(2m)\ n^{-m}\ (\log\ n)^{hm} \qquad \ldots \quad (3.2)$$

for some $h \geqslant 0, m = 1, 2, 3 \ldots$ where $c(2m) \leqslant L^m\, m\,!$ for some $L > 1$. It may not be out of place to mention that similar type of analysis are carried out for T_n with $c(2m) = O(1)$ for some $m \geqslant 1$ in Ghosh and Dasgupta (1978) and with $c(2m) \leqslant L^m\ (2m)\,!$ in Dasgupta (1989).

Because of (3.2) with $c(2m) \leqslant L^m\, m\,!$ we have the following

$$\sup_n E\Big[\exp\ \Big(\lambda n^{1/2}\ (\log\ n)^{-h/2} |R_n|\Big)^2\Big]$$

$$= \sup_n \Big[1 + \sum_{m=1}^{\infty} \Big(\lambda\ n^{1/2}\ (\log\ n)^{-h/2}\Big)^{2m} E\, R_n^{2m}/m\,!\Big]$$

$$\leqslant 1 + \sum_{m=1}^{\infty} (\lambda\, L)^{2m} < \infty \text{ if } 0 < \lambda < L^{-1}. \qquad \ldots \quad (3.3)$$

Consequently

$$P(|R_n| > a_n(t)) = O\Big(\exp\ \Big(-\Big(\lambda n^{1/2}\ (\log n)^{-h/2}\, a_n(t)\Big)^2\Big)\Big),\ 0 < \lambda < L^{-1}. \qquad \ldots \quad (3.4)$$

Also note that due to representation (3.1)

$$|P(T_n \leqslant t) - \Phi\,(t)| \leqslant |P(s_n^{-1}\, S_n \leqslant t + a_n(t)) - \Phi(t \pm a_n(t))|$$
$$+ |\Phi\,(t \pm a_n(t)) - \Phi\,(t)| + P(|R_n| > a_n\,(t)). \qquad \ldots \quad (3.5)$$

w.o.l.g. let $t > 0$ and take $a_n\,(t) = n^{-1/2}\,(\log\ n)^{(h+1)/2}\, l\, \lambda^{-1}$ then

$$|P(T_n \leqslant t) - \Phi\,(t)| \leqslant b\, n^{-1/2}\, t^2 \exp(-t^2/2) + b\, n^{-1/2}$$
$$(\log n)^{(h+1)/2}\, t \exp(-t^2/2) + b\, n^{-1/2}\, \exp\,(-t^2/2) \qquad \ldots \quad (3.6)$$

for $t^2 \leqslant k \log n$, using Theorem 2.1 of Dasgupta (1989) and (3.4), where k may be taken to be arbitrarily large.

For $t^2 \geqslant k \log n$, under the assumptions of Theorem 2.1 one has, using (2.2) and (3.5) with the same choice of $a_n(t)$ as above

$$|P(T_n \leqslant t) - \Phi(t)| \leqslant b\, n^{-1/2}\, \exp(-t^2(1 - l^{-1})p)$$
$$+ b\, n^{-1/2}\,(\log n)^{(h+1)/2} \exp(-t^2/2)$$
$$+ b\, n^{-1/2}\, \exp(-t^2/2) \qquad (3.7)$$

where $0 < p < 1$ since $\exp(-t^2(1 - l^{-1})) \leqslant n^{-1/2}\ \exp(-t^2(1 - l^{-1})p)$ if $t^2 > (2a(1-p))^{-1} \log n$, which can be ensured choosing k sufficiently large.

As a consequence of (3.6) and (3.7) we have the following non-uniform bound over the entre range of t.

Theorem 3.1. *Under the assumptions of Theorem* 2.1 *and* (3.1), (3.2) *there exists a connstant* $b(>0)$ *depending on* $0<p<1$ *such that*

$$|P(T_n \leqslant t)-\Phi(t)| \leqslant b\, n^{-1/2}(log\, n)^{(h+1)/2}\, exp(-t^2(1-l^{-1})p),\ -\infty<t<\infty. \quad \dots \quad (3.8)$$

Subsequently the following two theorems are immediate from (3.8) taking $p>p^*$.

Theorem 3.2. *Under the assumptions of Theorem* 3.1 *for any* $g:(-\infty,\infty)\rightarrow[0,\infty)$, $g(x)$ *even, such that* $Eg(T)<\infty, g(0)=0, T=N(0,1)$ *and*

$$g'(x)=O\,(exp(x^2(1-l^{-1})p^*)),\ 0<x<\infty \quad \dots \quad (3.9)$$

and for some $p^*, 0<p^*<1$, *the following holds*

$$|Eg(T_n)-Eg(T)| = O\,(n^{-1/2}(log\, n)^{(h+1)/2}). \quad \dots \quad (3.10)$$

Proof of the above follows from (3.8) along the lines of (2.14) since the representation (2.14) remains valid even if $Y_n = s_n^{-1} S_n$ is replaced by a general nonlinear statistics T_n converging weakly to a $N(0,1)$ variable T.

Theorem 3.3. *Under the assumption of Theorem* 3.1

$$\|exp(t^2(1-l^{-1})p)\,(P(T_n \leqslant t)-\Phi(t)\|_q$$

$$= O(n^{-1/2}(log\, n)^{(h+1)/2})\ for\ any\ q \geqslant 1\ and\ 0<p<1. \quad \dots \quad (3.11)$$

Next we consider the case when odd order moments of X_{nt} are non-vanishing. As before for $t^2 \leqslant k \log n$ it is possible to obtain (3.6). However for $t^2 \geqslant k \log n$ one may use (2.19) in (3.5) with the same choice of $a_n(t)$ viz.,

$a_n(t) = n^{-1/2}(\log n)^{(h+1)/2}\,|t|\,\lambda^{-1}$ to obtain

$$|P(T_n \leqslant t)-\Phi(t)| \leqslant b\, n^{-1/2} \exp(-\nu^* t^2)$$

$$+bn^{-1/2}(\log n)^{(h+1)/2}\,|t|\,\exp(-t^2/2)+bn^{-1/2}\exp(-t^2/2) \quad \dots \quad (3.12)$$

For $t^2 \geqslant k \log n$, where ν^* is defined in Theorem 2.5. Hence combining (3.6) with (3.12) it is possible to obtain the following non-uniform bound.

Theorem 3.4. *Under the assumptions* (1.2), (1.3), (2.1), (3.1) *and* (3.2) *there exists a constant* $b\,(>0)$ *such that*

$$|P(T_n \leqslant t)-\Phi(t)| \leqslant b\ n^{-1/2}\ (log\ n)^{(h+1)/2}\,|t|\ exp\ (-\nu^* t^2),\ -\infty<t<\infty$$

where ν^* *is defined in Theorem* 2.5. ... (3.13)

Hence we have the following theorem from the representation (2.14) and (3.13).

Theorem 3.5. *Under the assumptions of Theorem* 3.4 *and g satisfying the conditions of Theorem* 2.5, *the following holds*

$$|Eg(T_n)-Eg(T)| = O(n^{-1/2} (\log n)^{(h+1)/2}). \quad \ldots \quad (3.14)$$

The next theorem is also immediate from (3.13).

Theorem 3.5. *Under the assumptions of Theorem* 3.4, *for any* $\delta > 2$

$$\|(1+|t|)^{-\delta/q} \exp(\nu^* t^2)(P(T_n \leqslant t)-\Phi(t))\|_q$$
$$= O(n^{-1/2} (\log n)^{(h+1)/2}) \quad \ldots \quad (3.15)$$

for any $q \geqslant 1$, *where* ν^* *is defined in Theorem* 2.5.

4. Rates of convergence for linear processes

For a sequence of constants a_i with $\sum_{i=1}^{\infty} a_i^2 < \infty$ consider

$$X_n = \sum_{i=1}^{\infty} a_i \xi_{n-i+1} \text{ or } X_n = \sum_{i=1}^{\infty} a_i \xi_{n+i-1} \quad \ldots \quad (4.1)$$

where ξ_i's are pure white noise. w.o.l.g. assume $E\xi = 0$ and $E\xi^2 = 1$. Under the assumption of finiteness of (C^2+2)th moment of ξ Babu and Singh (1978) proved the moderate deviation result decomposing the sum $S_n = \sum_{i=1}^{\infty} X_i$ as follows

$$S_n = \sum_{i=1}^{n} X_{ii} + \sum_{i=1}^{n} (X_i - X_{ii}) \quad \ldots \quad (4.2)$$

where

$$X_{m,n} = \sum_{i=1}^{m} a_i \xi_{n-i+1}$$

The representation (4.2) is clearly of the type (3.1). Now assume

$$E\xi_1^{2m} \leqslant l^{-m}(2m)!/m!. \quad \ldots \quad (4.3)$$

Then by Minkowski's inequality

$$E\left|\sum_{i=1}^{n} (X_i - X_{ii})\right|^{2m} \leqslant \left(\sum_{i=1}^{\infty} i|a_i|\right)^{2m} E\xi_1^{2m} \leqslant L^m m! \quad \ldots \quad (4.4)$$

for some $L > 0$ assuming

$$\sum_{i=1}^{\infty} i|a_i| < \infty. \quad \ldots \quad (4.5)$$

Again for $S_n' = \sum_{i=1}^{n} X_{ii} = \sum_{i=1}^{n} t_{n-i+1}\xi_i \sim \sum_{i=1}^{n} t_i \xi_i$

where

$$t_i = \sum_{j=1}^{i} a_j$$

one may use the results of section 2 for the independent $r.v$'s $t_i\xi_i$. Observing that

$$\frac{1}{n} \sum_{i=1}^{n} t_i^{2m} \to Z^{2m} \qquad \dots \quad (4.6)$$

where

$$Z = \sum_{i=1}^{\infty} a_i \neq 0,$$

which for $m = 1$ implies $\lim n^{-1} V(S_n') = Z^2$, one may also check that (2.1) with sup replaced by lim is satisfied for $r.v$'s $t_i \xi_i$ under the assumption (4.3). Normalisation of S_n in (4.2) may be done by $[V(S_n)]^{-1/2}$ since for $nZ_n^2 = V(S_n')$, $|Z_n^2 - Z^2| = O(n^{-1})$ as shown in Babu and Singh (1978). Therefore from (4.2)

$$[V(S_n)]^{-1/2} S_n = [V(S_n)]^{-1/2} S_n' + R_n \qquad \dots \quad (4.7)$$

where

$$R_n = [V(S_n)]^{-1/2} \sum_{i=1}^{n} (X_i - X_{ii})$$

satisfies (3.2) with $h = 0$. Consequently all the results of Section 3 hold for S_n.

5. Discussion and some examples

The assumption (2.1) imples that each of the random variables in the triangular array has an entire characteristic functon. To see this write $c_n = (s_n^2/n)^{1/2}$, $c_n > 0 \ \forall\ n$. Then from (2.1) one gets for $c > 0$

$$\sup_{n \geqslant 1} n^{-1} \sum_{i=1}^{n} E \exp(c(X_{ni}/c_n)^2) = n^{-1} \sum_{i=1}^{n} \left(1 + \sum_{m=1}^{\infty} E\, c^m (X_{ni}/c_n)^{2m}/m\,!\right)$$

$$= 1 + \sum_{m=1}^{\infty} \left(n^{-1} \sum_{i=1}^{n} E c^m (X_{ni}/c_n)^{2m}/m\,!\right) \leqslant 1 + \sum_{m=1}^{\infty} c^m l^{-m} (2m)\,!/(m\,!)^2 \quad \dots \quad (5.1)$$

Since for large m, $m\,! \sim (2\pi)^{1/2} m^{m+1/2} e^{-m}$ the above sum is finite if c is sufficiently small e.g. if $0 < c < l/4$, $1 < l \leqslant 2$. This is turn states that there exist $c^* = c_n^* = c/c_n^2 > 0$ for which

$$E \exp(c^* X_{ni}^2) < \infty, \quad i = 1, \dots, n \qquad \dots \quad (5.2)$$

which implies that the characteristic function of X_{ni} is an entire function of order $\leqslant 2$ (see page 498, Feller Vol. II); (5.2) also specifies the tail behaviour of the distribution of X_{ni}.

$$P(|X_{ni}| > x) = o(\exp(-c^* x^2)), x \to \infty. \quad \ldots \quad (5.3)$$

Some examples are provided below where (2.1) are satisfied.

Example 1. X_{ni} are uniform on the range $[-k_{ni}, k_{ni}]$ where $k_{ni} \in [a, b]$ $a, b > 0$ are to be specified later. Then

$$EX_{ni}^{2m} = k_{ni}^{2m}/(2m+1).$$

The l.h.s. of (2.1) in this case turns out to be

$$\sup_{n \geqslant 1} \left(n^{-1} \sum_{i=1}^{n} k_{ni}^{2m}/(2m+1)\right) \div \left(n^{-1} \sum_{i=1}^{n} \frac{k_{ni}^2}{3}\right)^m \leqslant \frac{3^m}{2m+1} (b/a)^{2m} \quad \ldots \quad (5.4)$$

Require it to be $\leqslant l^{-m} (2m)!/m!$ so that (2.1) is satisfied. Therefore one may require

$$\frac{3^m}{2m+1} (l^{-1/2}(b/a))^{2m} \leqslant (2m)!/m! \quad m = 2, 3, 4, \ldots.$$

From Stirling's approximation it is easy to see that r.h.s. of the above has higher order of growth than that of l.h.s. So the restriction on a b comes from first few m. For $m = 1$, (2.1) is trivially satisfied. For $m = 2$ this states

$$l^{-1/2}(b/a) \leqslant (20/3)^{1/4} = 1.6068$$

For $m = 3$ $\quad l^{-1/2}(b/a) \leqslant (280/9)^{1/6} = 1.773$

For $m = 4$ $\quad l^{-1/2}(b/a) \leqslant (1680/9)^{1/8} = 1.923$

For $m = 5$ $\quad l^{-1/2}(b/a) \leqslant (12320/9)^{1/10} = 2.059$

As expected upper bound increases with m and therefore the restriction on a and b comes from the first bound for $m = 2$. This states

$$b/a \leqslant 1.6068\ l^{-1/2}.$$

For $l = 2$ one gets $b/a \leqslant 1.13622$; $a \leqslant b$. Here the choice of k_{ni}'s are completely arbitrary ; $k_{ni} \in [a, b]$ with the restriction that $b/a \leqslant 1.6068$ $l^{-1/2}$; $a \leqslant b$. Theorem 2.1 holds for X_{ni}'s with $1 < l \leqslant 2$ in this region of a and b.

Example 2. X_{ni}^{m} has probability density function

$$f(x) = k_{ni}^{-2}(k_{ni} - |x|) ; \quad |x| \leqslant k_{ni}$$
$$= 0 \qquad\qquad \text{otherise}$$

where $k_{ni} \epsilon [a, b]$. This means that the density function has a triangular shape with vertices $(k_{ni}, 0), (-k_{ni}, 0)$ and $(0, k_{ni}^{-1})$.

Here $EX_{ni} = 0$, $EX_{ni}^{2m} = k_{ni}^{2m}/((m+1)(2m+1))$.

The r.h.s. of (2.1) then becomes

$$\sup_{n \geq 1} \left\{ n^{-1} \sum_{i=1}^{n} k_{ni}^{2m}/((m+1)(2m+1)) \right\} \div \left((n^{-1} \sum_{i=1}^{n} k_{ni}^{2}/6 \right)^{m}$$

$$\leqslant \frac{6^m}{(m+1)(2m+1)} \left(\frac{b}{a}\right)^{2m}$$

In order that this is $\leqslant l^{-m}(2m)!/m!$ one may need

$$\frac{6^m}{(m+1)(2m+1)}(l^{-1/2}\, b/a)^{2m} \leqslant (2m)!/m! \qquad \dots \quad (5.5)$$

As in Example 1 the restriction on a and b comes from first a few m. For $m = 2$, (5.5) states

$$l^{-1/2}\, b/a \leqslant 5^{1/4} = 1.4953$$

For $m = 3$ $\quad l^{-1/2}\, b/a \leqslant (140/9)^{1/6} = 1.5799$

For $m = 4$ $\quad l^{-1/2}\, b/a \leqslant (175/3)^{1/8} = 1.6624.$

The restriction for $m = 2$ is most stringent : $b/a \leqslant 1.4953\, l^{-1/2}$, $a \leqslant b$. Theorem 2.1 holds for X_{ni} in this region of a, b. For $l = 2$ this states $b/a \leqslant 1.05737$.

This bound for b/a is more restrictive than that in Example 1. This is due to the change of the type of denstiy. In both the examples, for i.i.d set up with $a = b$ i.e., $k_{ni} \equiv k$, the upper bound of $|X|$'s may be taken arbitrarily large.

Example 3. (i) $X_{nt} = X_t$ where X_t is symmetric point bionomial variable i.e., $X_t = \pm 1$ with probability 1/2. $EX^{2m}/(EX^2)^m/(EX^2)^m = 1$, (2.1) is satisfied with $l = 2$.

(ii) $X_{nt} = X_t$, where X_t is asymmetric point binomial variable

$X_t = -\alpha$ with probability $\beta/(\alpha+\beta)$, $\alpha, \beta > 0$

$= \beta$ with probability $\alpha/(\alpha+\beta)$.

The mean is zero and the variance is $\alpha\beta$. Without loss of generality take $\beta = \alpha^{-1}$ so that the variance is 1. Then

$$EX^{2m} = (\alpha^{2m} + \alpha^{2-2m})/(1+\alpha^2).$$

We need $EX^{2m} \leqslant l^{-m}(2m)!/m!$. This imposes some restriction on α. As before r.h.s. has higher order of growth than l.h.s. For $m = 2$ one needs $\alpha^4+\alpha^{-2} \leqslant 12\, l^{-2}(1+\alpha^2)$. For $l = 2$ one gets $\alpha^4+\alpha^{-2} \leqslant 3(1+\alpha^2)$ i.e., $0.518 \leqslant \alpha \leqslant 1.9305$.

For $m = 3$ the restriction with $l = 2$ is $\alpha^6+\alpha^{-4} \leqslant 15(1+\alpha^2)$. This is satisfied by α on $.518 \leqslant \alpha \leqslant 1.9305$. The restriction on α is more stringent for $m = 2$. In this case the random variable is not symmetric unless $\alpha = \beta = 1$. Theorem 2.5, Corollary 2.1 hold for X_{ni} with $l = 2$ whenever $518 \leqslant \alpha \leqslant 1.9305$.

Example 4. Truncated cauchy distribution: $X_{ni} = X_i$ where $X_i \sim$ $f(x) = (2 \tan^{-1}k)^{-1} \dfrac{1}{1+x^2}$, $|x| \leqslant k$

$$EX^2 = (k/\tan^{-1}k)-1,\ EX^{2m} \leqslant k^{2m-2}[(k/\tan^{-1}k)-1]$$

$$EX^{2m}/(EX^2)^m \leqslant k^{2m-2}[(k/\tan^{-1} k)-1]^{-(m-1)} \leqslant (k \tan^{-1}k)^{m-1}.$$

As m increases $l^{-m}(2m)!/m!$ has higher order of growth than that of $(k\tan^{-1}k)^m$. So the restriction on k comes from first few m. As for example, for $m = 2$ with $l = 2$ one may need $k\tan^{-1}k \leqslant 2^{-2}\, 4!/2! = 3$.

For $m = 3$ one requires $(k \tan^{-1}k)^2 \leqslant 15$ or, $k \tan^{-1}k \leqslant 3.87$.

The first restriction is more stringent: $k \tan^{-1}k \leqslant 3$ or, $k \leqslant 2.5158$. Therefore Theorem 2.1 holds for X_i with $k \leqslant 2.5158$ and $l = 2$. Incidentally for a standard cauchy variable Y,

$$P(Y \epsilon(-k, k)) = \frac{2}{\pi} \tan^{-1}k > 0.759 \text{ for } k = 2.5158.$$

Example 5. Let $X = i$ with probability $C^{-1}. e^{-i^2\beta}$, $i = 0, \pm1, \pm2, \pm3, \pm4, \ldots$, where $C = \sum_{i=-\infty}^{\infty} e^{-i^2\beta} < \infty$; $\beta > 0$ and let X_{ni} be i.i.d copies of X.

The following result of Poisson (1827) may be found in Whittaker and Watson, page 124, chapter VI :

$$\sum_{n=-\infty}^{\infty} e^{-n^2\beta-2n\alpha\beta} = (\pi/\beta)^{1/2} e^{\alpha^2\beta} \left\{1+2 \sum_{n=1}^{\infty} e^{-n^2\pi^2/\beta} \cos 2n\pi\alpha\right\};$$

$$\sum_{n=-\infty}^{\infty} e^{-n^2\beta} = (\pi/\beta)^{1/2} \left\{1+2 \sum_{1}^{\infty} e^{-n^2\pi^2/\beta}\right\} \qquad \ldots (5.6)$$

where the second summations in brackets go to zero very fast as β decreases (see Davis, page 117).

This gives

$$\sum_{n=-\infty}^{\infty} e^{-n^2\beta+hn} = (\pi/\beta)^{1/2} e^{n^2/(4\beta)} \left\{1+2 \sum_{n=1}^{\infty} e^{-n^2\pi^2/\beta} \cos(n\pi h/\beta)\right\} \quad \dots \quad (5.7)$$

For fixed β denoting the l.h.s. as $f(h)$ one gets $f'(0) = 0$ as expected since $EX = 0$. Also

$$\text{var}(X) = EX^2 = C^{-1}f''(0),$$

where

$$C = \sum_{i=-\infty}^{\infty} e^{-i^2/\beta} = (\pi/\beta)^{1/2}(1+o(1)) \quad \dots \quad (5.8)$$

where $o(1)$ represents negligible term for small β ($\beta < 1/2$ suffices). Now

$$f''(0) = (\pi/\beta)^{1/2}. \frac{1}{2\beta} + (\pi/\beta)^{1/2} 2\pi^2/\beta^2 \sum_{n=1}^{\infty} n^2 e^{-n^2\pi^2/\beta} + (\pi/\beta)^{1/2} \sum_{n=1}^{\infty} e^{-n^2\pi^2/\beta} \frac{1}{2\beta}$$

$$= (\pi/\beta)^{1/2} \frac{1}{2\beta}(1+o(1))$$

Therefore

$$\text{var}(X) = \frac{1}{2\beta}(1+o(1)). \quad \dots \quad (5.9)$$

From (5.7) and (5.8) one gets

$$Ee^{hX} = e^{h^2/4\beta}(1+o(1))$$

In view of the above, (5.9) and (2.7), Theorem 2.1 holds for X_{ni} with any $0 < l < 2$. It may be mentioned here that we essentially used (2.7) to obtain Theorem 2.1.

Example 6. Linear combination of random variables satisfying Theorem 2.1 : Let $\{X_{ni}, Y_{ni}, n \geqslant 1, 1 \leqslant i \leqslant n\}$ be two independent triangular array of random variables satisfying the assumption of Theorem 2.1, then for any fixed real co-efficients α_1 and α_2, the theorem holds for $Z_{ni} = \alpha_1 X_{ni} + \alpha_2 Y_{ni}$. Now X_{ni} and Y_{ni} have mean zero. Theorefore Z_{ni} also have zero expectation. Note that to prove Theorem 2.1 we used only the equation (2.7) i.e.,

$$\prod_{i=1}^{n} \beta_i \leqslant \exp(h^2 s_n^2/l), \forall\, h \in (-\infty, \infty) \quad \dots \quad (5.10)$$

where $\beta_i = E\exp(hX_{ni})$, $s_n^2 = \sum_{i=1}^{n} EX_{ni}^2$.

Similarly for the second set of variables Y_{ni}

$$\prod_{i=1}^{n} \beta_i^* \leqslant \exp(h^2 s_n^{*2}/l) \quad \ldots \quad (5.11)$$

where
$$\beta_i^* = E\exp(hY_{ni}),\ s_n^{*2} = \sum_{i=1}^{n} EY_{ni}^2$$

Then denoting $\beta_i^{**} = E\exp(hZ_i)$, one gets

$$\prod_{i=1}^{n} \beta_i^{**} = \prod_{i=1}^{n} Ee^{h\alpha_1 X_{ni}} \prod_{i=1}^{n} E\, e^{h\alpha_2 Y_{ni}} \leqslant \exp(h^2(\alpha_1^2 s_n^2 + \alpha_2 s_n^{*2})/l)$$

from (5.10) and (5.11).

Now denoting $s_n^{**2} = \sum_{i=1}^{n} E\, Z_{ni}^2$,

$$s_n^{**2} = \alpha_1^2 \sum_{i=1}^{n} EX_{ni}^2 + \alpha_2^2 \sum_{i=1}^{n} EY_{ni}^2 = \alpha_1^2 s_n^2 + \alpha_2^2 s_n^{*2}$$

Hence

$$\prod_{i=1}^{n} \beta_i^{**} \leqslant \exp(h^2 s_n^{**2}/l).$$

Therefore Theorem 2.1 holds for the random variables Z_{ni}. Although shown for the linear combination of two arrays of random variables it obviously holds for arbitrary number of combinations of variables in triangular arrays. The proof is similar.

Since (2.1) with $l = 2$ is satisfied for $N(0, \sigma^2)$ variables, one may take $N(0, \sigma^2)$ variables in the linear combinations with other variables satisfying (2.1). This makes the range of the combined variables unbounded in both directions.

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