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TAIL BEHAVIOUR OF DISTRIBUTIONS IN THE DOMAIN OF PARTIAL ATTRACTION AND SOME RELATED ITERATED LOGARITHM LAWS

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SUMMARY. Let F be a distribution function and let (S_n) be a partial sum sequence of i.i.d. random variables with the common distribution F. F is said to be in the domain of partial attraction iff there exists an integer sequence (n_j) such that (S_{n_j}), properly normalized, converges to a non degenerate random variable. Under certain assumptions on the sequence (n_j) we characterize the tail of F and obtain iterated logarithm laws for (S_n) and $(\max_{1 \leqslant k \leqslant n} |S_k|)$.

1. Introduction

Let (X_n) be a sequence of independent identically distributed (i.i.d.) random variables (r.v.) defined over a common probability space $(\Omega, \mathcal{F}, P)$ and let $S_n = \sum_{j=1}^{n} X_j$, $n \geqslant 1$. Let F denote the distribution function (d.f.) of X_1. Let (n_j) be an integer subsequence and let (a_{n_j}) and (B_{n_j}) be sequences of constants ($B_{n_j} \to \infty$ as $j \to \infty$). Set $Z_{n_j} = B_{n_j}^{-1} S_{n_j} - a_{n_j}$. When (n_j) coincides with the sequence of natural numbers (n), for proper selection of (a_n) and (B_n), if (Z_n) converges weakly, then it is wellknown that the limit law is stable (or possibly degenerate). For some subsequence (n_j) and for proper selection of (a_{n_j}) and (B_{n_j}), if (Z_{n_j}) converges weakly, then the limit law is known to be an infinitely divisible law (see, ex. Gnedenko and Kolmogorov (1954)). Kruglov (1972) considered sequences (n_j) satisfying (i) $n_j < n_{j+1}$, $j \geqslant 1$, and (ii) $\lim_{j \to \infty} n_{j+1}/n_j = r (\geqslant 1)$, and characterized the class $\mathcal{U}$ of all infinitely divisible distributions which are limit laws of (Z_{n_j}). He found that the members of $\mathcal{U}$ have many properties of stable laws. It may be noted that the class of all stable laws is included in $\mathcal{U}$. In particular, if $\lim n_{j+1}/n_j = 1$, Kruglov (1972) established that (i) the

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limit law of (Z_{n_j}) is a stable law and (ii) the sequence (Z_n), properly normalized, will itself converge to the same stable law. Consequently, the subsequences of our interest under Kruglov's setup are those subsequences (n_j) with $\lim_{j\to\infty} n_{j+1}/n_j = r$, $r > 1$. Here Kruglov has characterized the limit distribution G as either normal or as an infinitely divisible distribution with the characteristic function ϕ of the form

$$\log \phi(t) = i\gamma t + \int \left(e^{itx} - 1 - \frac{itx}{1+x^2}\right) dH(x),$$

where γ is some real constant and H is a spectral function with $H(-x) = x^{-\alpha}\, \theta_1 (\log x)$, $x > 0$, $H(x) = -x^{-\alpha}\, \theta_2(\log x)$, $x > 0$, $0 < \alpha < 2$ and θ_1 and θ_2 are periodic functions with a common period such that for all $x > 0$ and $h \geqslant 0$, $e^{\alpha h}\, \theta_i(x-h) - e^{-\alpha h}\, \theta_i(x+h) \geqslant 0$, $c_i \leqslant \theta_i(x) \leqslant d_i$, $x > 0$, $i = 1, 2$, $c_1 + c_2 > 0$.

When the d.f. $G \in \mathcal{U}$ is non-normal we denote it by G_α, $0 < \alpha < 2$. Throughout this paper, F is in the domain of partial attraction of G_α means that the sequence (Z_{n_j}) converges in distribution to G_α, where (n_j) satisfies the conditions $n_j < n_{j+1}$, $j = 1, 2, \ldots$ and $\lim_{j\to\infty} n_{j+1}/n_j = r(>1)$. This is denoted by $F \in DP(\alpha)$, $0 < \alpha < 2$.

In the next section we obtain an asymptotic expression for the tail of F when $F \in DP(\alpha)$. Assuming that $a_{n_j} = 0$, in Z_{n_j}, $j \geqslant 1$, we establish a law of the iterated logarithm (l.i.l.) for (S_n), which is similar to Chover (1966). Under a further assumption that X_1 is symmetric about zero, we prove a l.i.l. for $A_n = \max_{1\leqslant k\leqslant n} |S_k|$, $n \geqslant 1$, which is of the form of Theorem 1, Jain and Pruitt (1973). Even though the weak convergence is available only over the subsequence (n_j), the iterated logarithm results have been obtained for the sequences (S_n) and (A_n).

For any $u > 0$, by $[u]$ we mean the greatest integer $\leqslant u$. i.o. and a.s. stand for infinitely often and almost surely. Throughout the paper, c, ε, J (integer) and N (integer), with or without a suffix, stand for positive constants.

2. Tail behaviour of F

Theorem 1 : *Let $F \in DP(\alpha)$, $0 < \alpha < 2$. Then there exists a slowly varying function L and a function θ bounded in between two positive numbers b_1, b_2, $0 < b_1 \leqslant b_2 < \infty$, such that*

$$\lim_{x\to\infty} \frac{x^{\alpha}(1-F(x)+F(-x))}{L(x)\,\theta(x)} = 1.$$

Proof: From the fact that $F \epsilon DP(\alpha)$, by Gnedenko, and Kolmogorov, (1954) we have for any $y > 0$,

$$\lim_{j \to \infty} n_j F(-B_{n_j} y) = y^{-\alpha} \theta_1 (\log y)$$

and

$$\lim_{j \to \infty} n_j (F(B_{n_j} y) - 1) = -y^{-\alpha} \theta_2 (\log y).$$

For $x > 0$, which is large, choose an integer j and a fixed positive number y such that $B_{n_j} y \leqslant x \leqslant B_{n_{j+1}} y$. Define $T(x) = 1 - F(x) + F(-x)$ and $\phi_k(y) = \dfrac{\theta_1(\log y) + \theta_2(\log y)}{\theta_1(\log ky) + \theta_2(\log ky)}$ for any $k > 0$. We have for any $k > 0$,

$$\frac{T(B_{n_{j+1}} y)}{T(kB_{n_j} y)} \leqslant \frac{T(x)}{T(kx)} \leqslant \frac{T(B_{n_j} y)}{T(kB_{n_{j+1}} y)}$$

so that

$$\frac{n_j}{n_{j+1}} \cdot \frac{n_{j+1} T(B_{n_{j+1}} y)}{n_j T(kB_{n_j} y)} \leqslant \frac{T(x)}{T(kx)} \leqslant \frac{n_{j+1}}{n_j} \cdot \frac{n_j T(B_{n_j} y)}{n_{j+1} T(kB_{n_{j+1}} y)}.$$

Using the fact that $n_{j+1}/n_j \to r$ as $j \to \infty$, as $x \to \infty$ ($j \to \infty$), one gets

$$\frac{k^\alpha \phi_k(y)}{r} \leqslant \liminf_{x \to \infty} \frac{T(x)}{T(kx)} \leqslant \limsup_{x \to \infty} \frac{T(x)}{T(kx)} \leqslant r k^\alpha \phi_k(y).$$

Since $c_i \leqslant \theta_i(x) \leqslant d_i$, $x > 0$, $i = 1, 2$, we have

$$k^\alpha c^{-1} \leqslant \liminf_{x \to \infty} \frac{T(x)}{T(kx)} \leqslant \limsup_{x \to \infty} \frac{T(x)}{T(kx)} \leqslant k^\alpha c,$$

where $c = r(d_1 + d_2)/(c_1 + c_2)$.

Now set $T(x) = x^{-\alpha} H(x)$. Then we have the relation

$$c^{-1} \leqslant \liminf_{x \to \infty} \frac{H(x)}{H(kx)} \leqslant \limsup_{x \to \infty} \frac{H(x)}{H(kx)} \leqslant c \qquad \ldots (1)$$

By Drasin, and Seneta, (1986) one now finds that

$\lim_{x \to \infty} \dfrac{H(x)}{L(x)\theta(x)} = 1$, where L is slowly varying (s.v) at ∞ and θ is such that both $\theta(x)$ and $1/\theta(x)$ are bounded for large x. Hence we have $T(x) \simeq x^{-\alpha} L(x)\, \theta(x)$ and the proof of the theorem is complete.

3. Iterated logarithm laws

In this section we obtain two l.i.l. results. For Theorem 2 below we assume that $a_{n_j} = 0$ in Z_{n_j}. When $\alpha < 1$, a_{n_j} can always be chosen to be zero. When $\alpha > 1$, a_{n_j} becomes $n_j EX_1$. Hence one can make $a_{n_j} = 0$ by shifting EX_1 to zero. Consequently the condition $a_{n_j} = 0$ is no condition at all when $\alpha \neq 1$, $0 < \alpha < 2$. However when $\alpha = 1$, this assumption restricts only to symmetric d.f.s $F \in DP(1)$. For Theorem 3 below we further assume that the d.f. F is symmetric about zero. We first prove a lemma needed in presenting our main results.

Lemma: *Let B_n be the smallest root of the equation : $nT(x) = 1$. Then $B_n \simeq n^{1/\alpha}\, l(n)\eta(n)$, where l is a function s.v. at ∞ and η is a function such that both η and $1/\eta$ are bounded.*

Proof: For x large, we have by Theorem 1,

$$T(x) \simeq x^{-\alpha}L(x)\,\theta(x),\ b_1 \leqslant \theta(x) \leqslant b_2.$$

Hence there exists a X_0 such that for all $x > X_0$,

$$b_1 x^{-\alpha}L(x) \leqslant T(x) \leqslant b_2 x^{-\alpha}L(x) \qquad \dots \quad (2)$$

Let B_{1n} and B_{2n} be respectively the smallest roots of $nb_1x^{-\alpha}L(x) = 1$ and $nb_2x^{-\alpha}L(x) = 1$. Then by the properties of regularly varying functions, one gets $B_{in} = b_i^{1/\alpha} n^{1/\alpha} l(n)$ $i = 1, 2$, where l is s.v. at ∞. Relation (2) implies that $B_{1n} \leqslant B_n \leqslant B_{2n}$. Hence $B_n = n^{1/\alpha} l(n)\eta(n)$ where $\eta(n)$ is bounded between $b_1^{1/\alpha}$ and $b_2^{1/\alpha}$.

Theorem 2 : *Let $F \in DP(\alpha)$, $0 < \alpha < 2$. Then*

$$P\left(\limsup_{n \to \infty} |B_n^{-1}S_n|^{1/\log\log n} = e^{1/\alpha}\right) = 1 \qquad \dots \quad (3)$$

Proof: In order to establish the theorem, equivalently we show that for any ε with $0 < \varepsilon < 1$,

$$P(|S_n| > B_n(\log n)^{(1+\varepsilon)/\alpha}\, i.o.) = 0 \qquad \dots \quad (4)$$

and

$$P(|S_n| > B_n(\log n)^{\frac{(\varepsilon-1)}{\alpha}}\, i.o.) = 1 \qquad \dots \quad (5)$$

By Feller (1946) and by Kruglov (1972), (4) and (5) hold once we show that

$$P(|X_n| > B_n(\log n)^{(1+\varepsilon)/\alpha}\, i.o.) = 0 \qquad \dots \quad (6)$$

and

$$P(|X_n| > B_n(\log n)^{(1-\varepsilon)/\alpha}\, i.o.) = 1 \qquad \dots \quad (7)$$

From Theorem 1 above, one can find an integer N_1 such that for all $n \geqslant N_1$,

$$P(|X_n| > B_n(\log n)^{(1+\varepsilon)/\alpha}) \leqslant c_3 L(B_n (\log n)^{(1+\varepsilon)/\alpha})/B_n^\alpha (\log n)^{(1+\varepsilon)}$$

Using the fact that $L((\log n).^{(1+\varepsilon)/\alpha} B_n) = 0((\log n)^{\varepsilon/2} \ L(B_n))$ and $L(B_n)$ $l^{-\alpha}(n) = 0\,(1)$ which follows by the properties of s.v functions (see Feller, (1966) or Seneta (1976)) one can show that

$$\limsup_{n \to \infty} n(\log n)^{(1+\varepsilon/2)} P(|X_n| > B_n(\log n)^{(1+\varepsilon)/\alpha}) < \infty.$$

Consequently, $\sum_{n=1}^{\infty} P(|X_n| > B_n(\log n)^{(1+\varepsilon)/\alpha}) < \infty$, which in turn establishes (6) by Borel-Cantelli lemma.

Again by Theorem 1, there exists a N_2 such that for all $n \geqslant N_2$,

$$P(|X_n| > B_n(\log n)^{(1-\varepsilon)/\alpha}) \geqslant c_4 L(B_n(\log n)^{(1-\varepsilon)/\alpha})/B_n^\alpha (\log n)^{(1-\varepsilon)}.$$

By arguments similar to the above, one can show that

$$\lim_{n \to \infty} n(\log n)^{(1-\varepsilon/2)} P(|X_n| > B_n(\log n)^{(1-\varepsilon)/\alpha}) = \infty, \qquad \dots \quad (8)$$

Now (7) follows from (8) again by appealing to Borel-Cantelli lemma.

Theorem 3 : *Let F be a d.f. symmetric about zero and let $F \in DP(\alpha)$, $0 < \alpha < 2$. Let $\psi_n = B_{[n/\log \log n]}$, $n \geqslant 3$. Then there exists a finite positive constant c such that*

$$lim\ inf\ \psi_n^{-1} A_n = c \quad a.s.$$

Proof: We now establish that for some constants c_5 and c_6, $0 < c_5 \leqslant c_6 < \infty$,

$$c_5 \leqslant \liminf_{n \to \infty} \psi_n^{-1} A_n \leqslant c_6 \text{ a.s.} \qquad \dots \quad (9)$$

In view of Hewitt-Savage zero-one law (9) implies that $\liminf_{n \to \infty} \psi_n^{-1} A_n$ is a.s. a finite positive constant. The proof is on the lines of Jain and Pruitt (1973.) First we prove that

$$P(\psi_n^{-1} A_n \leqslant c_5 \text{ i.o.}) = 0 \qquad \dots \quad (10)$$

Since $F \in DP(\alpha)$, we know that for all $x \in (-\infty, \infty)$,

$$\lim_{j \to \infty} P(S_{n_j} \leqslant x B_{n_j}) = G_\alpha(x) \qquad \dots \quad (11)$$

where $n_j < n_{j+1}$, $j = 1, 2, \dots$ and $n_{j+1}/n_j \to r$ as $j \to \infty$.

Let m_j be an integer sequence such that $n_j = [m_j/\log\log m_j]$. Set $N_j = [m_j/n_j]$, $j = 1, 2, \ldots$ Then for any $c_5 > 0$,

$$\left(A_{m_j} \leqslant c_5\,\psi_{m_{j-1}}\right) \subset \bigcap_{i=1}^{N_j} \left(|S_{in_j} - S_{(i-1)n_j}| \leqslant 2\,c_5\,\psi_{m_{j-1}}\right).$$

Therefore

$$P\left(A_{m_j} \leqslant c_5\,\psi_{m_{j-1}}\right) \leqslant \left(P\left(|S_{n_j}| \leqslant 2\,c_5\psi_{m_{j-1}}\right)\right)^{N_j}$$

Now proceeding as in Jain and Pruitt (1973) one gets for all $j \geqslant J_1$,

$$P\left(A_{m_j} \leqslant c_5\,\psi_{m_{j-1}}\right) \leqslant e^{-\theta N_j}$$

where $\theta > 1$ is some constant. By Kruglov (1972) we have

$$n_j = r^{j\beta(j)} \qquad \ldots \quad (12)$$

where β is a s.v. function such that $\beta(j) \to 1$ as $j \to \infty$. Consequently one gets $N_j \sim \log\log n_j \sim \log j$. One can find a J_2 such that for all $j \geqslant J_2$,

$$P\left(A_{m_j} \leqslant c_5\,\psi_{m_{j-1}}\right) \leqslant j^{-\theta}.$$

Now $\theta > 1$, implies that $\sum_{j=1}^{\infty} P\left(A_{m_j} \leqslant c_5\,\psi_{m_{j-1}}\right) < \infty$. By Borel-Cantelli lemma one gets

$$P\left(A_{m_j} \leqslant c_5\,\psi_{m_{j-1}}\ \text{i.o.}\right) = 0. \qquad \ldots \quad (13)$$

Notice that for $m_{j-1} \leqslant n \leqslant m_j$, $j = 1, 2, \ldots$, $A_n/\psi_n \leqslant A_{m_j}/\psi_{m_{j-1}}$. Hence (13) implies that

$$P(A_n \leqslant c_5\psi_n\ \text{i.o.}) = 0 \qquad \ldots \quad (14)$$

To prove the other half of the theorem we proceed as follows. Let t_j be an integer sequence such that $n_j = [2t_j/\log\log t_j]$, $j \geqslant 1$ and let $M_j = [t_j/n_j]$ Define $A_{n_j}(k) = \max_{1 \leqslant i \leqslant n_j} |S_{kn_j+i} - S_{kn_j}|$, $k = 0, 1, 2, \ldots, M_j$.

For any $\varepsilon > 0$ and $\lambda > 0$, let

$$E_k = \left\{|S_{(k+1)n_j}| \leqslant \varepsilon\psi_{t_j},\ A_{n_j}(k) \leqslant \lambda\psi_{t_j}\right\},\ k = 0, 1, 2, \ldots, M_j.$$

Then we have

$$\bigcap_{k=0}^{M_j} E_k \subset \left\{A_{t_j} \leqslant (\varepsilon+\lambda)\psi_{t_j}\right\} \qquad \ldots \quad (15)$$

Using (15) we now obtain a lower bound for $P(A_{t_j} \leqslant (\varepsilon+\lambda)\psi_{t_j})$. Using the technique of iterated conditional expectations as in Jain and Pruitt (1973), one gets for all

$$\varepsilon > \varepsilon_1,\ \lambda > \lambda_1 \text{ and } j \geqslant J_2$$

$$P(A_{t_j} \leqslant (\varepsilon+\lambda)\psi_{t_j}) \geqslant (1/4)^{(M_j+1)} \qquad \ldots \quad (16)$$

Observe that $M_j \sim (\log\log\ n_j)/2$. Hence for a $\beta > 1$, but sufficiently close to one, there exists a J_3 such that for all $j \geqslant J_3$, $\varepsilon \geqslant \varepsilon_1$ and $\lambda \geqslant \lambda_1$,

$$P\Big(A_{t_j} \leqslant (\varepsilon+\lambda)\psi_{t_j}\Big) \geqslant (1/4)^{(\beta\log\ \log n_j)/2} = (\log\ n_j)^{-\delta} \qquad \ldots \quad (17)$$

where $\delta = (\beta\ \log\ 4)/2$. Note that $\delta < 1$. Choose $\gamma\ \varepsilon(1,\ \delta^{-1})$.

Define $q_j = t_{[j^\gamma]}$ and observe the relation

$$A_{q_j} \leqslant A_{q_{j-1}} + \max_{q_{j-1} \leqslant i \leqslant q_j} |S_i - S_{q_{j-1}}|. \qquad \ldots \quad (18)$$

Using (17) and proceeding as in Jain and Pruitt (1973) one can show that for some J_4 and $c_7 > \varepsilon_1+\lambda_1$,

$$P\Big(\max_{q_{j-1} < i \leqslant q_j} |S_i - S_{q_{j-1}}| \leqslant c_7\psi_{q_j}\Big) \geqslant \Big(\log\ n_{[j^\gamma]}\Big)^{-\delta}$$

whenever $j \geqslant J_4$.

From (12), there exists a J_5 such that for all $j \geqslant J_5$,

$$P\Big(\max_{q_{j-1} \leqslant i \leqslant q_j} |S_i - S_{q_{j-1}}| \leqslant c_7\psi_{q_j}\Big) \geqslant c_8/j^{\gamma\delta}(\log\ r)^{\delta} \qquad \ldots \quad (19)$$

Since $1 < \gamma < \delta^{-1}$ (i.e., $\gamma\delta < 1$), we find that $\sum_{j=1}^{\infty} j^{-\gamma\delta} = \infty$.

By appealing to Borel-Cantelli lemma, (19) implies that

$$P\Big(\max_{q_{j-1} < i \leqslant q_j} |S_i - S_{q_{j-1}}| \leqslant c_7\psi_{q_j} \text{i.o.}\Big) = 1 \qquad \ldots \quad (20)$$

We now show that for any constant $c_9 > 0$,

$$P\Big(A_{q_{j-1}} \geqslant c_9\psi_{q_j} \text{i.o.}\Big) = 0 \qquad \ldots \quad (21)$$

Since F is symmetric about zero, we have by weak symmetrization inequality

$$P\left(A_{q_{j-1}} \geqslant c_9\psi_{q_j}\right) \leqslant 2P\left(|S_{q_{j-1}}| \geqslant c_9\psi_{q_j}/2\right).$$

Let $z_j = c_9\psi_{q_j}/2B_{q_{j-1}}$ and observe that $z_j \to \infty$ as $j \to \infty$. Then we have

$$P\left(A_{q_{j-1}} \geqslant c_9\psi_{q_j}\right) \leqslant 2P\left(|S_{q_{j-1}}| \geqslant z_jB_{q_{j-1}}\right) \qquad \ldots \quad (22)$$

From Heyde, (1967) one gets that

$$\limsup_{j\to\infty} \frac{P\left(\left|S_{q_{j-1}}\right| \geqslant z_jB_{q_{j-1}}\right)}{q_{j-1}P\left(|X_1| \geqslant z_jB_{q_{j-1}}\right)} < \infty.$$

By Theorem 1 and by some elementary properties of a s.v. function, we get

$$P\left(\left|S_{q_{j-1}}\right| \geqslant z_jB_{q_{j-1}}\right) \leqslant c_{10}z_j^{-(\alpha-\epsilon)}.$$

Observing that $\sum_{j=1}^{\infty} z_j^{-(\alpha-\epsilon)} < \infty$, by Borel-Cantelli lemma and by (22) one gets

$$P\left(A_{q_{j-1}} \geqslant c_9\psi_{q_j} \text{ i.o.}\right) = 0 \qquad \ldots \quad (23)$$

and the proof of the theorem is complete.

Remark: As in Jain and Pruitt (1973) the exact value of lim inf A_n is not available here also.

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