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NON UNIFORM RATES OF CONVERGENCE TO NORMALITY FOR STRONG MIXING PROCESSES

By RATAN DASGUPTA
Indian Statistical Institute

SUMMARY. Non uniform rates of convergence to normality of standardised sample sum is studied for some non-stationary strong mixing process under suitable mixing conditions, including polynomial decay, when all the moments exist but m.g.f may not exist ; or the m.g.f may exist but the r.v's may not be bounded. Also the case when some finite moment of order higher than two exist has been considered. The technique is based on estimating the bounds of moments for sample sum which has its own importance. As application of these non uniform bounds probabilities of moderate and large deviations are computed. The bounds are further applied to have rates of general moment convergences of standardised sample sum to that of normal distribution.

1. Introduction

Let X_n, $n \geqslant 1$ be a non stationary strong mixing sequence of random variables defined on some probability space. It is well known that if a moment of order higher than two is uniformly bounded for the random variables X_n, then the distribution of standardised sample sum converges to the normal distribution. More explicitly, let $EX_n = 0 \ \forall\ n$ and $\sup_n E|X_n|^{2+\delta} < \infty$ for some $\delta > 0$, then defining $F_n(t) = P(\sigma_n^{-1} S_n \leqslant t)$ where $S_n = \sum_{i=1}^{n} X_i, \sigma_n^2 = V(S_n), -\infty < t < \infty$ we have $F_n \Rightarrow \Phi$, where Φ is the standard normal distribution function. Uniform rates of such convergences are studied by various authors, a sharp result is due to Tikhomirov.

The nonuniform rate of such convergences are of great importance with application to computation of probabilities of deviation. An elegant idea to study nonuniform rates is to break up the positive axis into two parts and then to obtain two different bounds for the difference $|F_n(t) - \Phi(t)|$ depending on the region where t^2 belongs. This idea was possibly implicit in Esseen (1945) but was explored very effectively by Michel (1976).

The deviation results when some finite moment of order higher than two exist were proved by Rubin and Sethuraman (1965) and then these results were generalised and extended by Michel (1976), Ghosh and Babu (1977), Ghosh and Dasgupta (1978), Ghosh, Babu and Singh (1978), Babu and Singh (1978) (to be referred GBS and BS later), etc. under different set up.

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In this paper, the case when some finite order moment higher than two exist and the strong mixing decaying coefficients are of polynomial type is considered. A moment inequality on sum of truncated random variables is proved in Section 2 along the lines of Doob and Ibragimov (1962). Subsequent deviation results are also proved.

The present paper also deals with the case when all the moments of the r.v's exist and the strong mixing decay coefficients are general in nature. In iid set up Cramér (1938) showed if the m.g.f of the r.v exist in a neighbourhood of origin then $1-F_n(t_n) \sim \Phi(-t_n) \sim F_n(-t_n)$, for $t_n = o(n^{1/6})$. Later these type of deviation results were obtained under relaxed assumptions by different authors, see Linnik (1961, 1962) Nagaov (1965), Dasgupta (1989) etc. In the last mentioned paper it was also shown that the necessary and sufficient conditions coincide for some special type large deviations even for triangular array of independent r.v's. Statulavicius (1966) has some results in general set up based on cummulants of the sum of r.v's, but it is not clear how these are dependent on the specific moment bounds of each random variable in mixing processes.

The situation when all the moments of the strong mixing radom variables exist are treated in Sections 3–5 of the present paper. We compute m-th absolute moment of S_n, the sample sum, in terms of the corresponding individual moments of X_i in Section 3. For computing nonuniform rates specifically we consider $E|X|^m \leqslant f(m)$ $m = 1, 2, 3 \ldots$ where

$$f(m) = L^m e^{\nu m \log m}, \; \nu \geqslant 1 \text{ for some } L > 1. \qquad \ldots \quad (1.1)$$

One may note that $\nu = 0$ implies that the random variables are bounded and $\nu = 1$ implies that m.g.f of X_n exists we also consider f of following type.

$$f(m) = L^m \exp(m^\nu) \text{ for some } L > 0 \text{ and } \nu > 1 \qquad \ldots \quad (1.2)$$

which is of quite higher order than the former. It is shown in appendix A1 that these are implied by some other condition which can be interpreted as m.g.f of some function of X. Although the non uniform rates are computed for f of the above two types, the technique adopted may be followed for f of more general form.

As already stated we break up the positive axis into two parts and find the non uniform rates of convergence of $F_n(t)$ to $\Phi(t)$ depending on the region where t^2 belongs. In the first zone we use the non uniform bounds to obtain a region of t for which $1-F_n(t_n) \sim \Phi(t_n) \sim F_n(-t_n)$, $t_n \to \infty$ as $n \to \infty$. This gives us a clear picture of the variation of the normal approximation zone as ν varies in both the cases of $f(m)$. We also show that these results can be sharpended under additional restriction $EX_i^3 = 0$. Next, combining the results of non uniform bounds in both parts of positive axis we find an overall non uniform bound and use this to have rates of convergence of $Eg(\sigma_n^{-1}S_n)$ to $Eg(T)$, for some functions g, where T is $N(0, 1)$ variable.

The last section indicates how the results can be extended for general decaying strong mixing coefficients including polynomial decay.

2. Deviation results when some finite order moment exist

Below we state a lemma on moments for the sum of the truncated random variables.

Lemma 2.1 : *Let X_n be a non stationary strong mixing process with polynomial decay ; $\alpha(t) \leqslant t^{-\lambda}$, $\lambda > 0$ and var $\left(\sum_{i=1}^{u} X_{i+h}\right) \ll u$ for any $h \geqslant 0$. Let $EX_n = 0$, $E|X_n|^{\delta+\epsilon^*} < N$ where $\delta > 2$, $0 < \epsilon^* < 2$. Fix a $v_0 \geqslant 2$. Let λ be so large that v_0 def $1+D_0/\log 2 = 1+10(2^{v_0}-2)$ $(v_0+\epsilon^*)/(\lambda\epsilon^* (\log 2)^2) \leqslant \delta/2$. Let ($\bullet$) $1 < d < u^{(v_0-\eta)/\delta^*}$, for some $\eta > 0$ and $Y_i = Y_{i,d} = X_i\, I(|X_i| < d)$. Then for any $v \leqslant v_0$, $\exists\, D(v) > 0$ not dependent on d such that for all $u \leqslant d^2$,*

$$E\left|\sum_{i=1}^{u} Y_{i+h}\right|^{v} \leqslant D(v)\,(u^{v/2}+u^{v}R(v)) \text{ where } R(v) = d^{v-\delta},\ 1 < v \leqslant v_0. \quad \ldots \quad (2.1)$$

(This means, for a fixed v_0 if the polynomial power λ is large enough such $v_0 \leqslant \delta/2\ (> 1)$, then (2.1) holds for $v \leqslant v_0$. Observe that $v_0 \to 1$ and therefore $v \to 1$ as $\lambda \to \infty$.)

Proof : The lemma can be proved using the general idea of the proof of lemma 2.2 of BS(1978). However more precise estimates are needed here as the lemma is stated under polynomial decay for the mixing function α. For each iteration of u of the form 2^r a crucial choice of constant is $\epsilon_1 = \epsilon^*/r$, so that the accumulated effect in moment power is ϵ^*, a finite quantity.

Remark 2.1 : Lemma 2.1 holds in the range $d \leqslant u^{1/2}(\log u)^k$ for every fixed $k > 0$, even if $R(v)$ *is redefined as* $R(v) = d^{v-\delta-\epsilon^*}$ and v is replaced by $v^* = v+\epsilon^*/2+\epsilon''$ where $\epsilon'' > 0$ is arbitrarily small and if v is stricly less than $\delta/2$.

We now proceed to prove deviation results when some finite moment of order greater than two exist and the decaying co-efficients satisfy

$$\alpha(n) \leqslant \exp(-n^{\epsilon}) \text{ for some } \epsilon > 0 \quad \ldots \quad (2.2)$$

which is higher than any polynomial decay but lower than exponential decay. In this case v_0 of Lemma 2.1 can be taken to be arbitrary near to one, for every fixed v_0 and ϵ^*. Following Remark 2.1 ν^* can be taken arbitrary near to 1 with arbitrary small choice of ϵ^*. Therefore, from Lemma 2.1 and Remark 2.1, we have

Proposition 2.1: *Under* (2.2) *and* $EX_n = 0$, $E|X_n|^{\delta+\epsilon^*} < N$; $(\delta+\epsilon^*) > 2$, *defining* $Y_t = Y_{t,d} = X_t I(|X_t| < d)$ *we have, for any* v, *a* $D(v) > 0$, *not dependent on* d, *such that for all* $u \leqslant d^2$

$$E|\sum_{i=1}^{u} Y_{t+h}|^v \leqslant D(v)(u^{v/2}+u^{\nu}R(v)) \text{ where } R(v) = d^{v-(\delta+\epsilon^*)} \quad \ldots \quad (2.3)$$

and ν can be taken arbitrarily near to 1, provided $d \leqslant u^{1/2}(\log u)^k$ for some $k > 0$.

Since $(\delta+\epsilon^*) > 2$ for simplicity in notation we shall write $\delta+\epsilon^* = 2+c$. Recall that $S_n = \sum_{i=1}^{n} X_i$, $\sigma_n^2 = V(S_n)$, $F_n(t) = P(\sigma_n^{-1} S_n \leqslant t)$ and assume that

$$\underline{\lim}\, n^{-1}\sigma_n^2 > 0. \quad \ldots \quad (2.4)$$

We now state

Theorem 2.1 : *Let* $\{X_n\}$ *be a non stationary strong mixing sequence satis fying* (2.2), (2.4) *and* $EX_n = 0$, $E|X_n|^{2+c} < N$ *for some real* N. *Then for all* $t^2 \leqslant M \log n$ *where* M *may be arbitrary large but fixed, we have*

$$|P(S_n \leqslant t\sigma_n)-\Phi(t)| \leqslant kn^{-\delta} exp(-t^2/2)+O(n^{-c/2}(1+|t|)^{-(2+c)}), \quad \ldots \quad (2.5)$$

where $k > 0$, $\delta >$ *are constants.*

Proof of the above is by usual blocking procedure on the truncated r.v's $X_i' = X_i I(|X_i| \leqslant tn^{1/2})$, application of Prop. 2.1 and exponential centering. See also BGS (1978), BS (1978). The remark 4.1 of BGS (1978) holds in this case. Consequently we have the following two theorems on probabilities of deviations.

Theorem 2.2 : *Under the assumptions of Theorem* 2.1, *for a sequence* $t_n \to \infty$ *in such a way that*

$$t_n^2 - c\, \log n-(c+1)\log\log n \to -\infty \text{ as } n\to\infty \quad \ldots \quad (2.6)$$

we have

$$P(S_n > t_n\sigma_n) \sim \Phi(-t_n) \sim P(S_n < -t_n\sigma_n) \text{ as } n\to\infty. \quad \ldots \quad (2.7)$$

Theorem 2.3 : *If* $\{X_n\}$ *is stationary or* $\{|X_n|^{2+c}\}$ *is uniformly integrable then under the assumption of Theorem* 2.1 ; (2.7) *holds even if r.h.s of* (2.6) *is bounded above.*

3. Non uniform rates of coverugence of strong mixing sequence when all the moments exists

We next consider the case when all the moments of X_n exists but the moment generating function of X_n may not exist or m.g.f exists but the r.v's are not necessarily bounded. Let X_n be a non stationary strong mixing sequence and

$$EX_n = 0 \text{ for all } n \geqslant 1 \qquad \dots \quad (3.1)$$

$$\alpha(n) \leqslant \exp(-\lambda n) \text{ for some } \lambda > 0 \qquad \dots \quad (3.2)$$

$$\sup_{i \geqslant 1} E|X_i|^m \leqslant f(m)\ m = 1,2, \dots \text{ where } f:(0,\infty) \to (0,\infty) \qquad \dots \quad (3.3)$$

is nondecreasing satisfying $\sup_{i=1..m} f(i+1)f(m+2-i) \leqslant f(2)f(m+1)$.

Now we prove a lemma, stating the order of m-th absolute moment of S_n in terms of $f(m)$.

Lemma 3.1 : *Let X_n be a non stationary strong mixing process satisfying (3.1)—(3.3). Then there exist a constant $L(>1)$ depending on $\{\alpha(t)\}$ and $\{f(1), \dots f(m_0)\}$ for some fixed m_0 such that for all positive integer u and $h \geqslant 0$*

$$E|\sum_{i=1}^{u} X_{i+h}|^m \leqslant u^{m/2} m!\, L^m f(m+1);\ 1 \leqslant m \leqslant u^{1/2}(\log u)^{-1-\delta} \text{ for some } \delta > 0 \qquad \dots \quad (3.4)$$

and

$$E|\sum_{i=1}^{u} X_{i+h}|^m \leqslant u^{m/2}\, m!\, (\log m)^m L^m f(m+1),\ \forall\ m \geqslant 1. \qquad \dots \quad (3.5)$$

Proof : Fix an integer $h \geqslant 0$, define $Z_u = \sum_{i=1}^{u} X_{i+h}$, $Z_{u,t} = Z_{2u+t} - Z_{u+t}$ $S_{u,t} = \sum_{i=1}^{t} X_{u+h+i}$, $c(u, m, h) = E|Z_u|^m$ and $c(u, m) = \sup_{h \geqslant 0} c(u, m, h)$ As all the moments exist, it follows from Lemma 2.1 that $c(u, m) \leqslant u^{m/2}k(m)$ since the decay is exponential. Hence, the lemma is true for any $m \leqslant m_0$ where m_0 may be taken sufficiently large by adjusting L with $K(m_0)$. Specifically we take $L = \max\,[1, \max_{m \leqslant m0} \{(m!\, f(m+1)^{-1/m} K^{1/m}(m)\}]$ By Davydov's (1970) inequality with ε_1 is to be chosen later, one gets,

$$|E|Z_u|^{m+1-j}|Z_{u,t}|^j - E|Z_u|^{m+1-j}E|Z_{u,t}|^j|$$

$$\leqslant 10(\alpha(t))^{\varepsilon_1/(m+1+\varepsilon_1)}[c(u, m+1+\varepsilon_1)]^{(m+1)/(m+1+\varepsilon_1)} \qquad \dots \quad (3.6)$$

Hence

$$E(|Z_u+Z_{u,t}|)^{m+1} \leqslant 2(1+10.2^m(\alpha(t))^{\epsilon_1/(m+1+\epsilon_1)})c(u, m+1+\epsilon_1)$$

$$+\sum_{j=1}^{m}\binom{m+1}{j}E|S_u|^{m+1-j}E|S_{u,t}|^j$$

$$\leqslant 2(1+10.2^m(\alpha(t))^{\epsilon_1(m+1+\epsilon_1)})c(u, m+1+\epsilon_1)$$

$$+u^{(m+1)/2}(m+1)\,!\,L^{m+1}mf(2)f(m+2) \quad \ldots \quad (3.7)$$

from 2nd part of (3.3).

Let $\epsilon_1 = \log 2/(\log u)$ then

$$10.2^m(a(t))^{\epsilon_1/(m+1+\epsilon_1)} \leqslant 1 \text{ if } \log 10+m\log 2+\frac{\log 2}{(m+2)\log u}(-\lambda t) \leqslant 0,$$

as $a(t) \leqslant e^{-\lambda t}$, for some $\lambda > 0$ which is true if $t = \lambda^* m^2 \log u$ for some large λ^*. With this choice of t we have

$$c(2u, m+1) \leqslant \sup_{\lambda\geqslant 0} E|Z_u+Z_{u,t}+S_{u,t}-S_{2u,t}|^{m+1}$$

$$\leqslant [\{4c(u, m+1+\epsilon_1)+u^{(m+1)/2}(m+1)\,!\,L^{m+1}mf(2)f(m+2)\}^{1/(m+1)}$$

$$+2t\sup_{t\geqslant 1} E^{1/(m+1)}|X_t|^{m+1}]^{m+1}$$

$$\leqslant [4c(u, m+1+\epsilon_1)+u^{(m+1)/2}(m+1)\,!\,L^{m+1}mf(2)f(m+2)]$$
$$(1+\epsilon^*)^{m+1} \quad \ldots \quad (3.8)$$

where

$$\epsilon^* = 2t\sup_{t\geqslant 1} E^{1/(m+1)}|X_t|^{m+1}/[u^{1/2}L\{f(m+2)\,(m+1)\,!\}^{1/(m+1)}]$$

$$\leqslant au^{-1/2}\, m\, \log u\; L^{-1} \text{ for some } a > 0. \quad \ldots \quad (3.9)$$

So ϵ^* can be made arbitrarily near to zero for $u^{-1/2}m \log u \leqslant 1$ with a large choic of L. Now write $b = (1+\epsilon^*)^{m+1}$. Let r_0 be such $(2^{r_0})^{-\frac{1}{2}}m \log 2^{r_0} \simeq 1$. Repeating (3.8), $(r-r_0)$ times we have, when $r > r_0$,

$$c(2^r, m+1) \leqslant 4bc(2^{r-1}, m+1+\epsilon_1)+(2^{r-1})^{(m+1)/2}bf_1(m+1) \quad \ldots \quad (3.10)$$

where

$$f_1(m+1) = (m+1)\,!\,L^{m+1}mf(2)f(m+2)$$

$$\leqslant (4b)^{r-r_0}c(r_0, m+1+(r-r_0)\epsilon_1)+f_1(m+1)b2^{(r-1)(m+1)/2}(1-4b\;2^{-(m+1)/2})^{-1}$$

if $4b2^{-(m+1)/2} < 1$. In order that the above is less than or equal to $u^{(m+1)/2}$ $L^{m+1}\,(m+1)\,!\,f(m+2)$ first we may need

$$(4b)^{r-r_0}\,c(r_0, m+1+(r-r_0)\epsilon_1) \leqslant \frac{1}{2}\,2^{r(m+1)/2}\,L^{m+1}\,(m+1)\,!\,f(m+2). \quad \ldots \quad (3.11)$$

Now using the trivial bound

$$c(r_0, m+1+(r-r_0)\epsilon_1) \leqslant (2^{r_0})^{m+1+(r-r_0)\epsilon_1} \sup_{i \geqslant 1} E|X_i|^{m+1+(r-r_0)\epsilon_1} < (2^{r_0})^{m+2} f(m+2) \quad \ldots \quad (3.12)$$

as $r\epsilon_1 = 1$, we have (3.11), if

$$(4b)^{r-r_0} (2^{r_0})^{m+2} \leqslant \tfrac{1}{2}\, 2^{r(m+1)/2} L^{m+1} (m+1)\,! \quad \ldots \quad (3.13)$$

Now from our choice of r_0, $2^{r_0(m+1)/2} \simeq m^{(m+1)} (\log 2^{r_0})^{m+1}$ and $r_0 = O_e(\log m)$. Therefore (3.13) is true if

$$(4b)^{r-r_0}\, 2^{r_0(m+3)/2} (r_0 \log 2)^{m+1} \leqslant \tfrac{1}{2}\, 2^{r(m+1)/2} L^{m+1} \text{ for some } L > 1 \quad \ldots \quad (3.14)$$

i.e. if

$$1 \leqslant \frac{1}{2}\left[\frac{1}{4}\left(\frac{\sqrt{2}}{(1+\epsilon^*) r_0^{1/(r-r_0)}}\right)^{m+1}\right]^{r-r_0} L^{m+1} \text{ as } r_0 = O_e(\log m) \quad \ldots \quad (3.15)$$

Since ϵ^* is arbitrarily small r.h.s of (3.15) goes to ∞ i.e, (3.15) holds if

$$r_0^{1/(r-r_0)} \to l < (\sqrt{2}-\delta) \text{ for some } \delta > 0 \ (r \to \infty)$$

i.e. if $(r-r_0) \geqslant M^{-1} \log r_0$ where $M = \log(\sqrt{2}-\delta) = \frac{1}{2}\log 2 - \delta'$, for some $\delta' > 0$. Since $r_0 = O_e(\log m)$ this is true if $2^r \geqslant 2^{r_0 + M^{-1} \log\log m}$ using the definition of r_0 this is true if $u \geqslant m^2 (\log m)^k$, $k = 2(1+\delta)$ for some $\delta > 0$ i.e. if

$$m \leqslant u^{\frac{1}{2}} (\log u)^{-(1+\delta)} \text{ for some } \delta > 0, \quad \ldots \quad (3.16)$$

which we assumed in (3.4). Consider the last term of r.h.s. of (3.10), note that $4b2^{-(m+1)/2} < 1$ if $4(1+\epsilon^*)^{(m+1)}\, 2-(m+1)/2 < 1$ which is true as ϵ^* is small enough for $r > r_0$. Hence $(1-4b2^{-(m+1)/2})^{-1} = \lambda_4 < \infty$. Therefore,

$$\lambda_1 f_1(m+1) b\, 2(r-1)(m+1)/2 < \tfrac{1}{2}\, 2^{r(m+1)/2} L^{m+1}(m+1)\,! f(m+2)$$

if $\lambda_4 f(2) < (2m)^{-1} (\sqrt{2}/(1+\epsilon^*))^{m+1}$ which is again true for small choice of ϵ^* as r.h.s.$\to \infty$ as $m \to \infty$. Hence (3.4) for $u = 2^r$; as for $r \leqslant r_0$, (3.16) is violated, we need to show (3.5). To prove (3.5), note that if $r \leqslant r_0$ we have to show (from 3.12)

$$(2^r)^{m+2} f(m+2) \leqslant 2^{r(m+1)/2} L^{m+1}(m+1)\,! (\log(m+1))^{m+1} f(m+2). \quad \ldots \quad (3.17)$$

This is true if,

$$2^r 2^{r(m+1)/2} \leqslant L^{m+1} (\log m)^{m+1} (m+1)\,! \quad \ldots \quad (3.18)$$

Since in that case $2^{r/2} \leqslant 2^{r_0/2} \simeq r_0 m \log 2$, (3.18) is true if

$$2^{r_0} r_0^{m+1} m^{m+1} (\log 2)^{m+1} \leqslant L^{m+1} (\log m)^{m+1} (m+1)\,! \quad \ldots \quad (3.19)$$

Since $r_0 = O_e(\log m)$ the above is true for all large m, choosing L large. For $r > r_0$ note that in place of (3.13) we have to show $(4b)^{r-r_0}(2^{r_0})^{m+2} \leqslant \frac{1}{2} 2^{r(m+1)/2} L^{m+1}(m+1)!(\log m)^{m+1}$ which is true if $(4b)^{r-r_0} 2^{r_0} r_0^{m+1} m^{m+1} (\log 2)^{m+1} \leqslant \frac{1}{2} 2^{(r-r_0)(m+1)} L^{m+1} (\log m)^{m+1}(m+1)!$ proceeding as in (3.10). This is true for all sufficiently large m in view of the fact that $r_0 = O_e(\log m)$. Hence the lemma holds for integer of the form 2^r. For general u, one uses binary decomposition of u.

4. Rates of convergence

We now proceed to study the rates of convergences for different choices of f of the type (1.1) and (1.2). It is easy to check that bounds of the above type satisfy the condition in (3.3) viz. $\sup_{i=1\cdots m} f(i+1)f(m+2-i) \leqslant f(2)f(m+1)$ i.e supremum of the product is obtained at the end points. This is quite intuitive to expect. Describe the following blocking procedure to be adopted later. Let

$$p = p(\alpha, n) = [n^\alpha], q = q(\beta, n) = [n^\beta],$$
$$k = k(\alpha, \beta, n) = [n/(p+q)] \text{ and } l = n-k(p+q)$$

where $0 < \beta \leqslant \alpha < 1$ will be chosen accordingly. Let

$$\xi_i = \xi_{ni} = \sum_{j=1}^{p} X_{(i-1)(p+q)+j}, \eta_i = \eta_{ni} = \sum_{j=1}^{q} X_{ip}+(i-1)q+j$$

$\xi_{k+1} = \xi_{n,k+1} = \Sigma X_{k(p+q)+l}$ or 0 according as $l \geqslant 1$ or not.

Also let $U_n = \sum_{i=1}^{k+1} \xi_i$ $U'_n = \sum_{i=1}^{k} \eta_i$ and $t_n = (t \pm n^{-\lambda})$ where $\lambda > 0$ to be chosen later. First consider the moment bound (1.1). The following theorem states non uniform rates of convergence in an interval containing the origin. Note that since $k = O_e(n^{1-\alpha})$ the result is for the *large deviation zone.*

Theorem 4.1 : *Let* $\{X_n, n \geqslant 1\}$ *be a nonstationary strong mixing process satisfying* (3.1)-(3.3), (2.4) *and* (1.1). *Then for*

$$t^2 \leqslant M \min(k^{1/(2\nu+1)}, n^{\alpha+\beta-1}), M > 0 \quad \dots \quad (4.1)$$

with $|t| < \epsilon k^{1/2}$, $\epsilon > 0$ *small, there exist constant* $b > 0$ *(depending on* M) *such that for any* $\lambda > 0$ *and some* $a > 0$ *(depending on* ν *and* $\alpha(t)$)

$$|F_n(t) - \Phi(t)| \leqslant b|t|^{-1}\exp(-t^2/2)|\exp(O(|t|^3 k^{-1/2}))-1| + bn^{-\lambda}\exp(-t^2/2)$$
$$+ b\exp[-a\{|t| n^{(\alpha-\beta-2\lambda)/2}\}^{1/(\nu+1)}] + b\exp[-a\{|t| n^{(1-\alpha)/2}\}^{1/(\nu+1)}]. \quad \dots \quad (4.2)$$

Sketch of the proof : w.o.l.g. let $t > 0$. The theorem follows if one shows $|\Phi(t_n)-\Phi(t)| \leqslant bn^{-\lambda}\exp(-t^2/2)$ for $t_n = (t \pm n^{-\lambda})$ which is trivial and

$$P(|U_n'| > tn^{-\lambda}\sigma_n) \leqslant b\exp[-a\{tn^{(\alpha-\beta-2\lambda)/2}\}^{1/(\nu+1)}], \qquad \dots \quad (4.3)$$

and

$$|P(U_n > t_n\sigma_n)-\Phi(-t_n)| \leqslant bt^{-1}\exp(-t^2)/2)\exp(O(t^3k^{-1/2}))-1)| + b\exp[-a\{tn^{(1-\alpha)/2}\}^{1/(\nu+1)}] \qquad \dots \quad (4.4)$$

Now

$$P(|U_n'| > tn^{-\lambda}\sigma_n) \leqslant (tn^{-\lambda}\sigma_n)^{-m}E|U_n'|^m \leqslant t^{-m}n^{-(\alpha-\beta-2\lambda)}L^m e^{(\nu+1)m\log m} \qquad \dots \quad (4.5)$$

from (3.4). Putting the appropriate value of m viz. $m = [n^{(\alpha-\beta-2\lambda)/2} t L^{-1}]^{1/(\nu+1)}e^{-1}$ so that r.h.s of (4.5) is a minimum we obtain (4.3)

To show (4.4) we define $\xi_i' = p^{-1/2}\xi_i$ and $\xi_{i0} = \xi_i' I(|\xi_i'| \leqslant st_n(k+1)^{1/2})$ where $s > 0$ to be chosen accordingly. Therefore

$$\left|P\left(\sum_{i=1}^{k+1}\xi_i' > t_n p^{-1/2}\sigma_n\right) - P\left(\sum_{i=1}^{k+1}\xi_{i0} > t_n p^{-1/2}\sigma_n\right)\right|$$

$$\leqslant \sum_{i=1}^{k+1} P(|\xi_i'| > s(k+1)^{1/2}t_n) \qquad \dots \quad (4.6)$$

$\leqslant (k+1)\exp\{-a'(tn^{(1-\alpha)/2})^{1/(\nu+\delta)}\}$ for some $a' > 0$, following (4.5)

$\leqslant \exp\{-a(tn^{(1-\alpha)/2})^{1/(\nu+\delta)}\}$ for some $a > 0$, which is 2nd part in r.h.s. of (4.4).

Now, on the main part, with r.v's ξ_{i0}, we use exponential centering. See Dasgupta (1989). Also see BGS (1978) and BS (1978).

As an application of Theorem 4.1, we have the following theorem on normal approximation zone, with appropriate choice of α, β and λ, ($\lambda = (2(2\nu+9))^{-1}$ for $\nu < 1$ and $\lambda = (2(6\nu+5))^{-1}$ for $\nu \geqslant 1$).

Theorem 4.2 : *Under the assumption of Theorem* 4.1, *we have* $1-F_n(t_n) \sim \Phi(-t_n)$, $F_n(-t_n) \sim \Phi(-t_n)$, $t_n \to \infty$, *for* $t_n = o(n^{c^*}(\log n)^{-(\nu+1)})$ *where* $c^* = \min(1/(2(6\nu+5)),\ 1/(2(2\nu+9)).)$

The next theorem states the rates of convergence in the complementary zone of Theorem 4.1.

Theorem 4.3 : *Under the assumption of Theorem* 4.1 *for* $t^2 \geqslant c_0 k^{1/(2\nu+2\delta-1)}$ $c_0 > 0$ *we have* $|F_n(t)-\Phi(t)| \leqslant b\exp\{-|t|^{1/(\nu+\delta+1)}n^{\epsilon}a\}$ *for some* ϵ, $a > 0$.

Proof : With the blocking procedure stated in the beginning of this section, define $U_n = \sum_{i=1}^{k}\xi_i$, $U_n' = \sum_{i=1}^{k}\eta_i$, $T_n = \xi_{k+1}$. w.l.o.g take $t > 0$. We shall be using the moment inequality (3.5) here, as we need the lemma for all values of m. Since $(\log m)^m = o(m\,!)$, we have $m\,!\,(\log m)^m = o([\delta m]\,!\,)$ where $\delta > 1$ may be taken arbitrary close to 1. Proceeding as in (4.5) with the above observation we have

$$P(|T_n| > tn^{1/2}) \leqslant b\exp[-a(tn^{(1-\alpha)/2})^{1/(\nu+\delta)}]. \qquad \dots \quad (4.7)$$

Define $y = t^{-\alpha'}n^{-1/2}k^{\alpha'/2(2\nu+2\delta-1)}$, $0 < \alpha' < 1$ to be chosen later, $\xi_j^* = \xi_j I(|\xi_j| < y^{-1})$ and $U_n^* = \sum_{j=1}^{k} \xi_j^*$. Then

$$|P(U_n > tn^{1/2}) - P(U_n^* > tn^{1/2})| \leqslant \sum_{i=1}^{k} P(|\xi_i| > y^{-1})$$

$$\leqslant b\exp\{-a(t^{\alpha'}n^{1/2-\alpha/2-\alpha'(1-\alpha)/2(2\nu+2\delta-1)}\}^{1/(\nu+\delta)}$$

along the lines of (4.7). Also

$$P(U_n^* > tn^{1/2}) \leqslant e^{-ytn^{1/2}}E(e^{yU_n^*}) \leqslant \exp(-t^{\,1-\alpha'}k^{\alpha'/2(2\nu+2\delta-1)})E\left(\prod_{j=1}^{k} e^{y\xi_j^*}\right)$$

where the last expectation is estimated by a strightforward modification of Lemma 2.2 of BS (1978) to non stationary process. $\log E\left(\prod_{j=1}^{k} e^{y\xi_j^*}\right) = O(1+kpy^2)$ where $kpy^2 = O_\epsilon(ny) = O_\epsilon(t^{-2}k^{1/(2\nu+2\delta-1)\}} = O(1)$. Hence the theorem.

Observe that for (4.1), $k^{1/(2\nu+1)} \leqslant n^{\alpha+\beta-1}$ if (•) $2(\nu+1)/(2\nu+1) \leqslant \beta/(1-\alpha)$. Since $\delta > 1$, the zone in Theorem 4.3 covers the complementary zone of Theorem 4.1 provided (•) holds.

Combining Theorem 4.1 and 4.3 we may have a overall nonuniform bound for $-\infty < t < \infty$. Take $\beta/(1-\alpha) = 2(\nu+1)/(2\nu+1) = \delta^*$ (say), $\alpha = (3\delta^*+1)/(3(1+\delta^*)+1)$ and $\lambda = (1-\alpha)/6 = (2(3(1+\delta^*)+1))^{-1}$, to have the following theorem.

Theorem 4.4 : *Under the assumption of Theorem 4.1 one has for any $\lambda^* > 0$ and $\delta > 1$*

$$|F_n(t) - \Phi(t)| \leqslant bn^{-\lambda+\epsilon}\exp(-\lambda^*|t|^{1/(\nu+\delta+1)}) \qquad \dots \quad (4.8)$$

where $\lambda = (2(3(1+\delta^*)+1))^{-1}$, $\nu^* = 2(\nu+1)/(2\nu+1)$ and $\epsilon > 0$ is arbitrary.

Remark 4.1 *:* Since this nonuniform bound is sharper than any uniform bound (the ideal being $O(n^{-1/2})$) for sufficiently large t, one may use the uniform bound by Tikhomirov near origin and Theorem 4.4 for t away from origin.

The above nonuniform bound may be utilised to have the following theorem on general moment type convergences, noting that $Eh(X) = \int_0^\infty h'(t)P(|X| > t)\,dt$, $h(0) = 0$.

Theorem 4.5 : *Let $g : (-\infty, \infty) \to (0, \infty)$ be even with $g'(X) = O(exp(\lambda^* x^{1/(v+\delta+1)}))$ for some $\lambda^* > 0$, $0 < x < \infty$, $g(0) = 0$. Then under the assumptions of Theorem 4.1 one has, for $T \sim N(0, 1)$*

$$|Eg(\sigma_n^{-1}S_n) - Eg(T)| = O(n^{-\lambda+\varepsilon}),\ \varepsilon > 0 \text{ arbitrary.} \qquad \ldots \quad (4.9)$$

Next consider the moment bound (1.2). Here we may use the moment inequality (3.5), as f in (1.2) is of quite high order compared to m !. The proof of the following theorem is similar to that of Theorem 5.1, the restriction $\alpha+\beta-1 > 0$ is required to show that the expectation of the product of exponentially centred r.v's ξ_{t_0} differs slightly from the product of expectation of centred r.v's.

Theorem 4.6 : *Let X_n be a non stationary strong mixing sequence satisfying (3.1)–(3.3), (2.4) and (1.2). Then for $t^2 \leqslant M(logn)^{v/(v-1)}$, there exists constant $b > 0$ depending on M such that for any $\lambda > 0$, $\varepsilon > 0$ and $\alpha+\beta-1 > 0$*

$$|F_n(t) - \Phi(t)| \leqslant b|t|^{-1}|exp(-t^2/2)\, exp(0(|t|^3/k^{1/2})) - 1| + bn^{-\lambda}exp(-t^2/2)$$

$$+ b\, exp\Big[-(v-1)\Big\{\frac{(1-\varepsilon)}{v}\Big(\frac{1}{2}(\alpha-\beta-2\lambda)\, logn + logt\Big)\Big\}^{v/(v-1)}\Big]$$

$$+ b\, exp\Big[-(v-1)\Big\{\frac{1-\varepsilon}{v}\Big(\frac{1}{2}(1-\alpha)logn + logt\Big)\Big\}^{v/(v-1)}\Big]. \quad \ldots \quad (4.10)$$

As a consequence of the above theorem we may obtain a normal approximation zone, equating $1-\alpha = \alpha-\beta$, letting $\lambda \to 0$ and $\alpha+\beta-1 \to 0$.

Theorem 4.7 : *Under the assumption of Theorem 4.6 we have* $1-F_n(t_n) \sim \Phi(-t_n) \sim F_n(-t_n)$, $t_n \to \infty$ if

$$t_n^2 \leqslant 2(v-1)(1-\varepsilon)((6v)^{-1}logn)^{v/(v-1)},\ v > 1 \qquad \ldots \quad (4.11)$$

where $\varepsilon > 0$ can be made arbitrarily small.

Remark 4.2 : The zone (4.11) is larger than moderate deviation zone as expected, since all the moments exist and have bound of the type (1.2).

We may also have non uniform rates in the complementary zone of Theorem 4.6. The proof of the following theorem is similar to that of Theorem 4.3.

Theorem 4.8 : *Under the assumptions of Theorem 4.6, for* $t^2 \geqslant c_0 (\log n)^{r/(v+1)}$ $c_0 > 0$ *and for any* $\varepsilon > 0$ *and* $\alpha+\beta-1 > 0$, *there exist* $b(>0)$ *depending on* c_0 *and* ε *such that*

$$|F_n(t)-\Phi(t)| \leqslant b\, exp\Big[-(v-1)\Big\{\frac{(1-\varepsilon)}{v}\Big(\frac{1}{2}\log n+\log|t|\Big)\Big\}^{v/(v-1)}\Big] \quad \ldots \quad (4.12)$$

It is possible to have a overall nonuniform bound combining (4.10) and (4.12). For that take $\alpha \simeq \beta \simeq \frac{1}{2}, \lambda \simeq \frac{1}{12}$, to have the following theorem.

Theorem 4.9 : *Under the assumptions of Theorem 4.6, for any* $\varepsilon > 0$ *and* $\lambda^* > 0$ *there exists a constant* $b > 0$ *depending on* ε *and* λ^* *such that*

$$|F_n(t)-\Phi(t)| \leqslant bn^{-1/12+\varepsilon}\, exp\Big[-(v-1)\Big\{\lambda+\frac{1-\varepsilon}{v}\log(1+|t|)\Big\}^{v/(v+1)}\Big] \quad \ldots \quad (4.13)$$

An analogous theorem of Theorem 4.5 is possible to obtain, proof is similar.

The normal approximation zone of Theorem 4.2 can be sharpened further if $EX_i^3 = 0 \forall i$. It follows from $A2$ that $|E(p^{-1/2}\zeta)^3| \leqslant bp^{-1/12+\varepsilon}$ if $EX_i^3 = 0 \forall i$, so $|E\xi_{i0}^3| = O(p^{-1/12+\varepsilon})+O(k^{-2})$. Hence expanding the expectation of the exponentially centred r.v ξ_0 upto third term and proceeding as in Theorem 4.1 we have the following in place of (4.2). (see also Dasgupta (1989)).

$$\begin{aligned} |F_n(t)-\Phi(t)| \leqslant\; & b|t^{-1}|\exp(-t^2/2)|\, \exp(O(|t|^3k^{-1/2}p^{-(1/12)+\varepsilon}+t^4k^{-1}))-1| \\ & +bk^{-1/2}\exp(-t^2/2+O|t|^3k^{-1/2}p^{-(1/12)+\varepsilon}+t^4k^{-1})) \\ & +nb^{-\lambda}e^{-t^2}/2+b\exp[-a\{|t|n^{(\alpha-\beta-2\lambda)/2}\}^{1/(\nu+1)}] \\ & +b\exp[-a\{|t|n^{(1-\alpha)/2}\}^{1/(\nu+1)}]. \end{aligned} \quad \ldots \quad (4.14)$$

The corresponding normal approximation zone becomes $t_n = O(n^{m^*-\varepsilon})$ where $m^* = \max\limits_{0<\beta\leqslant\alpha<1,\, \lambda>0} \{(6-5x)/36 \wedge \lambda \wedge (\alpha+\beta-1)/2 \wedge (1-\alpha)/2(2\nu+1)$

$$\wedge(\alpha-\beta-2\lambda)/2(2\nu+1)\wedge(1-\alpha)/4\},\ x \wedge y = \min(x, y)$$

and $\varepsilon > 0$ can be made arbitrarily small.

5. Extensions of the results for general decay (when all the moments of the r.v's exist)

Since the key role to find the nonuniform rates are played by the moment bounds of the sample sum, we shall generalize Lemma 4.1 for decay other than exponential decay.

Let $\alpha(t) \leqslant t^{-\lambda}$, $\lambda > 1$ so that $\sum\limits_{j=1}^{\infty} [\alpha(j)]^{1-2/(\delta+4^*)} < \infty$ implying $E|\sum\limits_{i=1}^{u} Y_{i+h}|^2 \leqslant u$. Since all the moments exists, δ in Lemma 2.1 may be taken arbitrary large. Hence by Lemma 2.1, $c(u, n) \leqslant u^{1/2}k(m)$ for every fixed m. Now let $\epsilon_1 = (\epsilon(u)\log u/\log 2)^{-1}$ instead of $\log 2/\log u$ in the proof of Lemma 3.1, then $10\, 2^m(\alpha(t))^{\epsilon_1/(m+1+\epsilon_1)} \leqslant k$ for some k, if $t = u^{m^2\epsilon(u)/\lambda}$ where k is independent of m and u. With this choice of t, ϵ^* in (3.9) becomes

$$\epsilon^* < a u^{m^2\epsilon(u)/\lambda - 1/2} m^{-1} L^{-1} \text{ for some } a > 0 \qquad \ldots \quad (5.1)$$

which can be made arbitrarily small uniformly in m and u if

$$\epsilon(u) \leqslant \lambda/(2m^2). \qquad \ldots \quad (5.2)$$

Now repeating (3.8) r times (*not* $(r-r_0)$ times as in Lemma 3.1), we have, in place of (3.10), for $u = 2^r$, the following

$$c(2^r, m+1) \leqslant (kb)^r\, c(1, m+1+r\epsilon_1) + f_1(m+1) b 2^{(r-1)(m+1)/2}(1-kb2^{-(m+1)/2})^{-1}. \qquad \ldots \quad (5.3)$$

Now

$$c(1, m+1+r\epsilon_1) = c(1, m+1+(\epsilon(u))^{-1}) \leqslant f(m+1+(\epsilon(u))^{-1}) \leqslant 2^{r(1-\epsilon)(m+1)/2} \qquad \ldots \quad (5.4)$$

if

$$f(m+(1/\epsilon(u)) \leqslant u^{(1-\epsilon)m/2} \qquad \ldots \quad (5.5)$$

for all sufficiently large m and u in the range (5.2), where $\epsilon > 0$ is arbitrary close to zero. Now proceeding as in Lemma 3.1, in view of (5.4), we have the following in place of (3.15)

$$1 \leqslant \frac{1}{2}\left[\frac{1}{k}\left\{\frac{\sqrt{2}}{(1+\epsilon^*)2^{(1-\epsilon)/2}}\right\}^{m+1}\right]^r L^{m+1} \qquad \ldots \quad (5.6)$$

which is true for all sufficiently large m and r since ϵ^* can be made arbitrary close to zero and since $\epsilon > 0$.

The calculation for the second part of r.h.s in (5.3) remains the same as that followed in Lemma 3.1.

Summarising the above we state the following theorem :

Theorem 5.1 : *Let* $\{X_n\}$ *be a non stationary strong mixing sequence with decaying coefficients* $\alpha(t) \leqslant t^{-\lambda}$, $\lambda > 1$. *Let* (3.1), (3.3), (2.4) *and* (5.5) *hold for some sequence* $\epsilon(u)$ ($\to 0$ *as* $u \to \infty$). *Then for all* m *satisfying* (5.2) *we have*

$$E\left|\sum_{t=1}^{u} X_{t+h}\right|^m \leqslant u^{m/2}\, m!\, L^m f(m+1), \text{ for some } L \geqslant 1. \qquad \ldots \quad (5.7)$$

We explain the above for some choice of f e.g. let $f(m) = e^{\nu m \log m} L_1^m$ for $\nu > 0$ and $L_1 \geqslant 1$ then letting $\epsilon^{-1}(u) = (\delta/\nu) m \log u/\log\log u$, where $\delta < \frac{1}{2}$, we see that (5.5) is satisfied for large m and u in the range (5.2), which states

$$m \leqslant (\lambda\delta/2\nu)\log u/\log\log u, \; \delta < \frac{1}{2}. \qquad \ldots \quad (5.8)$$

Hence with f of (1.1), we have (5.7), in the range (5.8). When the r.v's X_i's are bounded by L_1 then $\nu = 0$. Take $\varepsilon^{-1}(u) = \delta m \log u/\log L_1$, $\delta < \frac{1}{2}$, which satisfies (5.5) for large u. Also (5.2) states $m \leqslant \frac{\lambda\delta}{2} \log u/\log L_1$ where $|X_i| \leqslant L_1$. Hence the following corollary.

Corollary 5.1 : *If the strong mixing r.v's X_i's are bounded, $|X_i| \leqslant L_1$, (3.1) and (2.4) holds ; $\alpha(t) \leqslant t^{-\lambda}$, $\lambda > 1$, then for $1 \leqslant m \leqslant \frac{\lambda\delta}{2} \log u/\log L_1$, $\delta < \frac{1}{2}$ we have*

$$E\left|\sum_{i=1}^{u} X_{i+h}\right|^m \leqslant u^{m/2} m\,!\, L^m L_1^{m+1} \text{ for some } L > 1. \quad \ldots \quad (5.9)$$

For decaying coefficient $\alpha(t) \leqslant \exp(-\lambda t^{\varepsilon})$, $\lambda > 0$ and $0 < \varepsilon \leqslant 1$ we may take $t = \lambda^{*}(m^2 \log u)^{1/\varepsilon}$, so that we may have $10\; 2^m(\alpha(t))^{\varepsilon_1/(m+1+\varepsilon_1)} \leqslant 1$ with $\varepsilon_1 = \log 2/\log u$ as in the proof of Lemma 3.1. See (3.7) and (3.8). Then replacing $m\,!$ by $[\varepsilon^{-1}m]\,!$ in the r.h.s of (3.4) and (3.5) we may have ε^{*} in (3.9), as $\varepsilon^{*} \leqslant a u^{-1/2}(m \log u)^{1/\varepsilon}$. Now define r_0 as $(2^{r_0})^{1/2}(m \log 2^{r_0})^{1/\varepsilon} \simeq 1$. Then following the proof of Lemma 3.1 with this r_0, we have the following.

Theorem 5.2 : *If $\alpha(t) \leqslant exp(-\lambda t^{\varepsilon})$, $0 < \varepsilon < 1$ then under the assumptions* (3.1), (3.3) *and* (2.4) *we have* (3.4) *and* (3.5) *where $m\,!$ in the r.h.s of* (3.4) *and* (3.5) *is replaced by* $[\varepsilon^{-1}m]\,!$

As already stated, using the above moment inequalities one may obtain non uniform rates in more general situations.

Before we conclude, let us examine the sharpness of the moment bounds. In view of the fact that $u^{-1/2} \sum_{i=1}^{u} X_{i+h} \xrightarrow{a} N(0,.)$ It is natural to expect that moment bound of $u^{-1/2} \sum_{i=1}^{u} X_{i+h}$ should be free of u and m-th absolute moment of this should be close enough to the corresponding absolute normal moment i.e. $\leqslant (m/2)\,!\, L^m$ for some L. The moment bounds obtained are similar to that where $(m/2)\,!$ is replaced by $m\,!$ or $[\varepsilon^{-1}m]\,!$ depending on the decaying coefficients.

Appendix

A1. The following assertions are true :

$$E \exp(s|X|^{1/\nu}) < \infty \text{ for some } s > 0 \Rightarrow E|X|^m \leqslant L^m m^{m\nu} \quad \ldots \quad (1)$$

for some $L > 0$ where $\nu > 1$, $m = 1, 2, \ldots$, and

$$E \exp\{\log(1+|X|)\}^{\nu/(\nu-1)} < \infty \Rightarrow E|X|^m \leqslant L^m e^{m^{\nu}} \quad \ldots \quad (2)$$

for some $L > 0$, where $\nu > 1$, $m = 1, 2, \ldots$, *Proof*: For (1), note that

$$\exp(s|X|^{1/\nu}) \leqslant \frac{|X|^{p/\nu}}{p\,!\,s^{-p}}\quad p = 1, 2, \ldots.$$

Now for $|X| \geqslant 1, \dfrac{|X|^{p/\nu}}{p\,!\,s^{-p}} \geqslant \dfrac{|X|^m}{p\,!\,s^{-p}}$ if $p/\nu \geqslant m$, e.g., if $p = [\nu m]+1$.

Therefore, with this choice of p

$$\begin{aligned} E|X|^m &= E|X|^m I(|X| \leqslant 1)+E|X|^m I(|X| > 1) \\ &\leqslant 1+([\nu m]+1)\,!\,s^{[\nu m]+1} E\exp(s|X|^{1/\nu}). \qquad \ldots \quad (3) \end{aligned}$$

Now $E\ \exp(s|X|^{1/\nu}) < \infty$ and using Strirlings approximation for factorials, $([\nu m]+1)\,! = ([\nu m]+1)\,[\nu m]\,! \leqslant L^m m^{m\nu}$ for some $L > 0$. And hence

$$E|X|^m \leqslant L^m m^{m\nu} \text{ for some } L > 0.$$

For (2), also note

$$\exp(\log(1+|X|))^{\nu/(\nu-1)} = (1+|X|)^{\log(1+|X|)1/(\nu-1)}. \qquad \ldots \quad (4)$$

Now

$$\begin{aligned} E|X|^m < E(1+|X|)^m &= E(1+|X|)^m I(\log^{1/(\nu-1)}(1+|X|) \geqslant m) \\ &\quad +E(1+|X|)^m I(\log^{1/(\nu-1)}(1+|X|) < m). \quad \ldots \quad (5) \end{aligned}$$

The first term of the r.h.s of (5) is finite from l.h.s of (2) and from (4). Again note that

$$\log^{1/(\nu-1)}(1+|X|) < m \Rightarrow \log(1+|X|) < m^{\nu-1} \Rightarrow m\log(1+|X|) < m^\nu$$

$$\Rightarrow (1+|X|)^m < e^{m^\nu}.$$

Hence the second term of the r.h.s of (5) is bounded above by $\exp(m^\nu)$ and therefore (2) follows.

A2. If $E\,X_i^3 = 0$, then for the strong mixing process of Section 4, $E(n^{-1/2}S_n)^3 = O(n^{-1/12+\epsilon})$, where $\epsilon > 0$ can be made arbitrarily small.

Proof: Note that

$$E(n^{-1/2}\,S_n)^3 = \int_0^\infty x^3 dF_n(x) + \int_{-\infty}^0 x^3 dF_n(x) \qquad \ldots \quad (6)$$

Also

$$\begin{aligned} \int_0^\infty x^3 dF_n(x) &= x^3(1-F_n(x))\Big|_0^\infty + \int_0^\infty 3x^2(1-F_n(x))dx \\ &= \int_0^\infty 3x^2(1-F_n(x))dx \text{ from (4.13).} \qquad \ldots \quad (7) \end{aligned}$$

Similarly

$$\int_{-\infty}^{0} x^3 dF_n(x) = -\int_{-\infty}^{0} 3x^2 F_n(x)dx. \qquad \ldots \quad (8)$$

From (4.13), noting that $\int_{-\infty}^{\infty} x^3 d\,\Phi(x) = 0$ we have from (6)

$$|E(n^{-1/2}S_n)^3| \leqslant bn^{-1/12+\epsilon}. \qquad \ldots \quad (9)$$

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