THE THEORY OF EQUITABLE QUALITY LEVEL AND ERROR-AREAS UNDER THE OPERATING CHARACTERISTIC CURVES OF LOT ACCEPTANCE SAMPLING PLANS

BY

M. T. SUBRAHMANYA
University of Ghana, Legon, Ghana
and
Indian Statistical Institute, Calcutta.

1969

PREFACE

This monograph deals with the theory of Equitable Quality Level and error-areas under the operating characteristic curves of lot acceptance sampling plans. A brief summary of the contents is given in Sections 1.2b (pages 15-16) and 2.9 (pages 107-111). A fairly good idea of the topics considered in this monograph can be obtained from the detailed list of contents given in pages iv- xi.

Most of the results given in the monograph are new and have not been published anywhere. They are based on author's own research work and are contained in his doctoral dissertation.

Most of the work connected with the monograph was done during my stay at the Indian Statistical Institute, Calcutta, while I was on leave from the University of Ghana, Legon, Ghana. I am grateful to the Vice-Chancellor, University of Ghana for granting me study leave during the preparation of this monograph and also to Professor C. R. Rao, F. R. S., Director of the Research and Training School, Indian Statistical Institute, Calcutta, for providing me all the necessary fecilities for research.

I am indebted to Professor S. K. Mitra of the Indian Statistical Institute for many useful discussions I have had with him during my stay at the Indian Statistical Institute, Calcutta.

Thanks are also due to Dr. V. J. Chacko of the United Nations, at present attached to the University of Ghana, for helpful suggestions.

Finally I wish to thank Shree G. M. Das of the Indian Statistical Institute, Calcutta, for his efficient typing of the manuscript.

M. T. SUBRAHMANYA

C O N T E N T S

CHAPTER 3: (contd.)

Page

CHAPTER 7: (contd.)

LIST OF TABLES

Page

Poisson single sampling plans (n, c)

LIST OF TABLES (contd.)

AN INDEX TO THEOREMS AND LEMMAS

CHAPTER 1

THE EQUITABLE QUALITY LEVEL

1.1 The Operating Characteristic (OC)

1.1a: -Let w measure the lot quality in such a way that increasing values of w correspond to lots of poorer and poorer quality. The proportion of defectives in the lot, the average number of defects in the lot, the proportion of items in the lot whose measurement exceed a stipulated value are all examples for w. Let θ stand for the sampling procedure of an acceptance sampling plan, or alternatively for the set of parameters which uniquely determine the plan. θ is a vector whose components are the 'elements' of the plan: e.g., (n,c), (n_1,n_2,c_1,c_2) (n,k) etc., in Chapters 3 to 7. Let $L(w,\theta)$ - or simply $L(w)$ for brevity - denote the probability of accepting a lot of quality w under the operation of the pla θ. $L(w)$ considered as a function of lot quality w is said to be the operating characteristic (OC) of the plan. Every plan has got its own OC which is uniquely determined by it.

1.1b: Classification of OCs:-

The OCs can be classified in various ways:

(i) Poisson, binomial and normal · OCs

An acceptance sampling plan sets up a decision variable whose value as obtained on the basis of sample observations determines the acceptability or otherwise of the lot. We can speak of Poisson OCs, binomial and inverse binomial OCs, normal OCs etc., according as the decision variable follows a Poisson, a binomial, an inverse binomial or a normal distribution. For instance, the case of Poisson OC-curves arises when the decision variable, namely, the number of defects observed in the sample follows a Poisson distribution. The decision variable corresponding to an inverse binomial OC is the number of items to be sampled from the lot in order to produce a prescribed number of defectives. Similarly, the name normal OC' implies that the decision variable - sample average - follows a normal distribution. It may be noted that the Poisson, the binomial and the normal are the three most important and widely used OC curves.

(ii) Single, double and multiple OCs

In a single sampling plan a decision of accepting or not accepting a lot is always arrived at on the basis of only one sample of items selected from the lot. In a double sampling procedure, a first sample is taken from every lot and the evidence is used to accept the lot, to reject the lot or to reserve decision until further information from a secon sample is obtained. Thus from each lot while one sample is taken and never more than two, only some of the lots are give a second chance. Multiple sampling is an extension of double sampling. An OC curve may therefore be called a single sampling, double sampling or a multiple sampling OC according as the maximum number of times samples have to be drawn from the lot before coming to a final decision is one, two or more than two.

(iii) Univariate and multivariate OCs

In cases where a lot has to satisfy quality requirement with respect to k quality characteristics, the overall lot quality may be expressed as a k-dimensional vector $w = (w^{(1)}, w^{(2)}, \ldots, w^{(k)})$ where $w^{(i)}$ refers to the measurement of lot quality with respect to the i-th quality characteristic. Since, in general, quality varies from lot-to-lot, the point w may be considered as varying in a k-dimensional space. A lot may be defined to be a good lot if space. In the simplest case considered in Sections 1.9 and Chapter 7, $w^{(i)}$s are chosen such that an increase in the value of atleast one of them means a deterioration in lot quality; further the region of good quality is rectangular i.e., bounded by the planes $w^{(i)} \leq W^{(i)}$; $i = 1,2,\ldots, k$, where $W^{(i)}$ are the specified quantities. For example, a lot may contain k types of defectives with respect to each of which it has to meet certain requirements. $w^{(i)}$ can be taken as the proportion of defectives in the lot belonging to the i-th category. $\underset{\sim}{w}$ varies in the k-dimensional cube, $0 \leq w^{(i)} \leq 1$. In certain situations, the region of good quality can be taken as the rectangle $0 \leq w^{(i)} \leq W^{(i)}$

In general, the OC L ($\underset{\sim}{w}$) is a function of k variables $w^{(1)}, w^{(2)},\ldots, w^{(k)}$. We can thus have univariate, bivariate or multivariate OCs according as w is a one , two -, or a multidimensional vector.

(iv) <u>Composite OCs</u>

All OCs belonging to a particular category can be generated by varying θ. For instance, all univariate single sampling Poisson OCs can be generated by varying the two elements, namely, the sample size n and the acceptance number c. However OCs can be generated in practice by so: other methods also. Suppose a lot is finally accepted if and only if two inspectors pass the lot after carrying out sampli inspections independently - may be with respect to the same quality characteristic or may be with respect to two differen characteristics. Then the ultimate OC takes the form

$$L = L_1 L_2 \qquad (1.1)$$

where L_1 and L_2 are the OCs corresponding to the first and second inspectors respectively. If the lot is finally accepted when either one or both of them have passed the lot, then the ultimate OC can be written as

$$L = 1 - (1 - L_1)(1 - L_2) \qquad (1.2)$$

Such OC curves may be called composite OC curves

Yet another example of a composite OC is given by

$$L = aL_1 + (1 - a)L_2, \; 0 \leq a \leq 1 \qquad (1.3)$$

This case arises when exact specifications lead to a non-integral value for the sample size or the acceptance number that are nearest to the desired value.

Unless otherwise stated; in future the word 'OC' alone will refer to a univariate OC.

1.1c: <u>The hypothetical random variable ξ :-</u>

In any sampling plan, lots of poorer (better) quality should have lower (higher) probability of being accepted. Since an increase in the value of w means a deterioration in lot quality, it is reasonable to assume that

(a) L(w) is a non-increasing function of w

(b) L(w) takes the value 1, when the lot quality is at its best, that is, when w is at its lowest value A, and that

(c) L(w) takes the value 0, when the lot quality is at its worst, that is, when w is at its highest value B.

We make an additional assumption that

(d) L(w) is continuous to the right.

It may be noted that if L_1 and L_2 satisfy these conditions, the composite OCs defined in (1.1), (1.2) and

(1.3) also satisfy them.

The conditions (a) to (d) ensure that L(w) can be looked upon as an upper cumulative distribution function. In other words, there exists a random variable ξ_θ such that

$$L(w, \theta) = P\{\xi_\theta > w\} \tag{1.4}$$

The subscript θ is generally dropped for the sake of convenience. It is clear that the range of ξ is contained within that of w: $A < \xi \leq B$.

Subject to conditions similar to (a) - (d), a k-variate OC $L(\underset{\sim}{w})$ is represented as the upper cumulative distribution function of a k-variate random vector $\xi = (\xi^{(1)}, \xi^{(2)}, \ldots, \xi^{(k)})$ i.e.,

$$L(\underset{\sim}{w}, \theta) = P\{\xi^{(i)} > w^{(i)},\ i = 1,2,\ldots, k\} \tag{1.5}$$

It may be remarked that the conditions (a)- (d) are no at all severe restrictions. Almost all the OCs used in practice do obey them. In fact, they are absolutely continu so that is a continuous random variable having a density function.

The (lower) cumulative distribution function (c.d.f.) of ξ is denoted by $G(\cdot)$ and the probability density function (p.d.f.) - when it exists - by $g(\cdot)$

$$G(w) = 1 - L(w),$$

and

$$g(w) = - \frac{dL(w)}{dw} \tag{1.6}$$

The p.d.f. of ξ is thus the absolute value of the slope of the OC curve at w. It is of interest to note that its value at the point of control (see (1.9) below) has been employed as a parameter in constructing certain systems of sampling plans (Hamacker, 1959)

1.1d: The moments of ξ :-

Moments of ξ. are of course finite when the range of w is finite. This covers all the cases where the lot quality is measured in terms of proportions. The moments are finite also in the case of Poisson OCs where w can range from 0 to infinity (Section 3.1). The mean, the variance and the mean deviation about the mean of ξ are denoted by $m(\xi)$, $V(\xi)$ and $D(\xi)$ or simply by m, V and D respectively. The central moment of order j is denoted by μ_j and the j-th cumulant by λ_j.

When ξ is continuous, we have (1.7)

$$m = \int_A^B xg(x)\,dx = -[wL(w)]_A^B + \int_A^B L(w)\,dw$$

$$D = \int_A^B |x-m|\,g(x)\,dx = -2[(w-m)L(w)]_A^B + 2\int_A^{B} L(w)\,dw$$

and

$$V = \int_A^B (x-m)^2 g(x)\,dx = -[w^2L(w)]_A^B + 2\int_A^B wL(w)\,dw - m^2$$

provided that the relevant integrals exist. The expressions in [ ] vanish when w stands for a proportion and also i the case of Poisson OC (Section 3.1) where w being the average number of defects can range from 0 to infinity. In such cases, m, D and V can be obtained by formally integrating the OC curve itself i.e., as suitable areas under the OC curve: (1.8)

$$m = \int_A^B L(w)\,dw$$

$$D = 2\int_A^B L(w)\,dw = 2\int_A^m \{1-L(w)\}\,dw$$

and

$$V = 2\int_A^B wL(w)\,dw - m^2$$

It should not however be supposed that the above equations a valid whenever ξ has finite moments. For example, in the case of normal OC (Section 5.1), if the lot quality is measured by the normal mean, μ, then the area under the OC, $\int L(\mu)d\mu$ becomes infinite, eventhough ξ has finite moments of all orders.

1.1e: The Indifference Quality Level (IQL):-

The IQL, W_0 - also called the point of control - is defined in literature by the relation

$$L(W_0) = \frac{1}{2} \tag{1.9}$$

Because of (1.5), W_0 turns out to be the median of ξ.

If ξ has higher cumulants such that the j-th cumulant λ_j is of the order of n^{1-j}, $j > 1$, $n > 1$, — n being the total sample size, this is a case of fairly common occurrence and includes the case of Poisson, binomial and normal OCs — then Fisher - Cornish expansions show that (Kendall and Stuart, 1963)

$$W_0 - m = -\frac{1}{6}\lambda_3 V^{-1} + \left(\frac{1}{40}\lambda_5 V^{-2} - \frac{1}{12}\lambda_3\lambda_4 V^{-3} + \frac{17}{324}\lambda_3^3 V^{-4}\right) + O(n^{-3}) \tag{1.10}$$

Also

$$L(m) = \frac{1}{2} - \frac{1}{6\sqrt{2\pi}} \lambda_3 v^{-3/2} - \frac{1}{\sqrt{2\pi}}\left(\frac{1}{40}\lambda_5 v^{-5/2} - \frac{5}{48}\lambda_3\lambda_4 v^{-7/2} + \frac{35}{432}\lambda_3^3 v^{-9/2}\right) + o(n^{-5/2}) \quad \text{(I.11)}$$

1.1f:- The concepts of EQL, the error areas and the EQL system of sampling plans to be presented later are based on $m(\xi)$ and $D(\xi)$. The problems of the process characteristic and that of choosing a suitable OC are briefly deal with in the following section from the point of view of providing a preliminary background and motivation for developing the concepts of EQL and the error areas.

1.2 The Process Characteristic (PC) and the problem of choosing a plan

1.2a:- A sampling plan is characterized by its OC i the sense that it tells us what happens when lots with a prescribed quality level were repeatedly subjected to the operation of that particular plan. In practice, the lot quality varies from lot-to-lot and can itself be considered a random variable following a certain distribution. The cumulative distribution function (c.d.f.) of lot quality will be denoted by $F(\cdot)$. This distribution is often referred to as the process characteristic (PC). The probability density function (p.d.f.) - if it exists - will be denoted by $F_1(\cdot)$ or $f(\cdot)$. It may therefore be expected that a satisfactory solution to the problems of choosing a suitable plan and evaluating the performance of a chosen plan in terms of risks and costs met by the producer and the consumer depends not only on the OC but also on the PC.

We should have some means of taking both the OC and the PC into our discussions. Here we meet with some difficulties, because OC and PC are curves and hence do not give single value measures of the performance of the plan.

With regard to OC, various summary figures like the Acceptable Quality Level (AQL), the Average Outgoing Quality Limit (AOQL), the Indifference Quality Level (IQL), the slope of OC etc., have been considered in literature. No one of these concepts can be considered the best for all situations. What is good in one situation may not be applicable in another situation and therefore there is ample justification for exploring the possibility of employing new concepts - the EQL and the error areas, for instance to aid us in selecting suitable plans.

Since it is known how to generate all possible OC curves (ex: .Poisson, binomial etc.) in a given situation, the problem of finding an OC with required properties is essentially the same as answering the questions: what are the properties of a particular OC with respect to a given PC? how are they related to those of other OC curves?

With regard to PC, it is not known how far and under what circumstances the use of process average alone (Dodge and Romig, 1959) is really adequate. Further, the problem of determining PC is rather intractable. Production processes change very rapidly even within a short interval of time. It may not be possible to obtain precise information as to the shape of the process characteristic or to the nature of the factors of cost for a sufficiently long time so as to demand the use of a plan which is mathematically worked out to be optimum.

In practice, sampling plans can be constructed so as to cover adequately the space of OC curves and a system of sampling plans may be viewed as a convenient way of indexing a collection of OC curves (Hamacker,1950). If an experienced inspector knows what OC he wants by just looking at it, the corresponding plan is determined. Otherwise the plan has to be selected on the basis of its indexing parameters (ex: a plan with 5 % AOQL, 10 % IQL or 2 % EQL etc.)

Ideally a chosen plan should reflect the views of the consumer and the producer on the cost of making wrong decisions. In other words, the concepts involved in constructing plans should be capable of being interpreted in terms of cost parameters. Here the question of determining the lot size becomes important, because the acceptance of a bad lot (or rejection of a good lot) is much more serious when the lot size is large as compared with when it is small.

If, during the operation of a particular plan, a considerable deterioration (or an improvement) of incoming quality sets in, the producer may be 'punished' (or favoured) by shifting to tightened (or reduced) inspection .

In cases where the given specifications have to be met with exactly - but the exact solution leads to non-integral values for the sample size or the acceptance number it may be necessary to employ the technique of randomization between two plans of the system.

It may also be noted that, in general, a given plan operates under the assumption that the technique of inspection-the method of assessing the quality of an item — is exac When the technique is not exact, the OC will also be effected and as a consequence the original specifications may not be satisfied.

1.2b: A summary:

To sum up, it would be ideal to use only those measures or parameters which are (a) simple to compute, (b) admit good interpretations and (c) which are reasonably insensitive to a slight change in the process characteristic. It is shown in the following sections of this chapter that the concept of EQL as defined in Section 1.3 - has these properties (Sections 1.3 and 1.4.). Besides (d) it is close to the IQL in the univariate case (Section 1.5); (e) it can be interpreted as a break even quality level for some general types of risk functions (Sections 1.6 and 2.8); (f) it provides a valid procedure for shifting from normal to tightened or reduced inspection (Section 1.7); (g) randomization with respect to two EQL plans is equivalent to a linear interpolation between the average probabilities of acceptance (Section 1.8) and finally (h) it is shown in Section 1.9 that the concept of EQL can be easily extended to cover multivariate OCs also.

In the remaining sections of this chapter, the general theory of EQL applicable to any type of OC and PC is dealt with. The concept of error areas under the OC is introduced in Chapter 2. The EQL turns out to be the point at which the two risks of accepting a bad lot and rejecting a good lot when measured in terms of suitable areas under the OC are equal. The theory as developed in Chapters 1 and 2 is then applied to particular OCs (Poisson, binomial and normal single sampling OCs, Poisson double sampling OCs and multinomial OCs) in Chapters 3 to 7.

1.3 The concept of EQL and its robustness

1.3a: The definition and an interpretation:-

The concept of EQL (Equitable Quality Level) was originally introduced by Mitra and Subrahmanya (1968). However the concept, though not stated explicitly, is already contained in the concept of the point of equal error areas introduced by Subrahmanya (1966).

For a given OC $L(w,\theta)$ and a PC with c.d.f. $F(w)$, the Equitable Quality Level (EQL) — to be denoted by $W_{eF}(\theta)$ — is defined by the equation,

$$F(W_{eF}(\theta)) = E_F\, L(w,\theta) \tag{1.12}$$

The suffixes e and F in $W_{eF}(\theta)$ stand for the EQL and the PC respectively. θ as usual stands for the plan under consideration. When there is no cause for confusion, we may omit the symbols F and θ.

The above equation determines the EQL uniquely when F is continuous; otherwise it has to be determined by suitable conventions. However, we would be dealing with continuous PC only, eventhough the fundamental lemma 1.1 given below is proved for discrete PCs also.

The right side of (1.12) is the relative proportion of lots accepted on an average under the plan. The left side of the equation is the proportion of lots of quality equal to or better than the EQL, as obtained under the PC. We also have

$$1 - F(W_e) = E_F \left\{1 - L(w)\right\} \tag{1.13}$$

From (1.12) and (1.13) we get the following interesting interpretation for the EQL:

If the EQL coincides with the agreed point of demarcation between good and bad lots, the producer has no reasons to grumble against the plan, because (1.12) and (1.13) imply that the proportion of lots accepted on an average by the plan is exactly equal to the proportion of good lots submitted for inspection and that the proportion of lots rejected on an average is equal to the proportion of bad lots submitted.

Other interpretations are given in later sections and in Chapter 2.

1.3b: A fundamental lemma:-

We shall now establish the following result.

Lemma 1.1:-

For any type of OC and PC,

$$F(W_e) = E_F\, L(w) = E_G F\,(\xi - 0) \tag{1.14}$$

where ξ is the random variable given by

$$L(w) = P\left\{ \xi > w \right\} \tag{1.15}$$

and $G(\cdot)$ is the c.d.f. of ξ.

<u>Proof:</u> The existence of ξ as given by (1.15) has already been established in (1.4). Since each of the functions $L(\cdot)$ and $F(\cdot)$ lie between 0 and 1, both $EL(w)$ and $EF(\xi - 0)$ exist. The first part of the relation (1.14) follows from the definition of the EQL. The second part follows by noting that both w and ξ are random variables which are distributed independently of each other and hence

$$\begin{aligned} E_F\, L(w) &= E_F\, P\left\{\xi > w | w\right\} \\ &= \text{the unconditional probability, } \\ &\quad P\left\{\xi > w\right\} \\ &= P\left\{w < \xi\right\} \\ &= E_G\, P\left\{w < \xi \,|\, \xi\right\} \\ &= E_G\, F(\xi - 0) \end{aligned}$$

under the convention that F is continuous to the right when it stands for the c.d.f. of a discrete random variable.

Hald has given the second part of (1.14) [page 403 in Hald (1967b)] for a single sampling binomial OC with a mixed binomial PC. It should be observed that our relations (1.12) to (1.15) <u>and also the interpretation of the EQL</u> are valid for any PC and also under multiple sampling and composite OCs. As shown in Section 1.9, the results can be extended to multivariate OCs also. They remain valid even when either one or both of the random variables w and ξ are discrete. However, we would be dealing with continuous w s and ξs only.

1.3c: Two theorems on the EQL :-

The following two theorems show that under certain conditions on the PC, there exists a close relationship between the EQL and the mean $m(\xi)$. Their multivariate analogues are straight forward and present no special difficulties, as shown in Section 1.9.

Theorem 1.1:-

If the c.d.f. of the process characteristic $F(w)$ has a density $f(w) = F_1(w)$ with a continuous derivative $F_2(w)$ such that $|F_2(w)| \leq M$, then

$$F(W_e) = F(m) + R_1 \quad \text{and} \quad |R_1| \leq \frac{1}{2} MV \qquad (1.16)$$

where m and V are respectively the mean and the variance of the random variable ξ defined in (1.4) or (1.15).

<u>Theorem 1.2:-</u>

If $F(w)$ has a continuous density $f(w)$ such that $|f(w)| \leq M^*$, then

$$F(W_e) = F(m) + R_2 \quad \text{and} \quad |R_2| \leq \frac{1}{2} M^* D \tag{1.17}$$

where D is the mean deviation about the mean of ξ.

<u>Proof:</u> By lemma 1.1,

$$F(W_e) = E_G F(\xi)$$

To prove theorem 1.1, we expand $F(\xi)$ around $m = E\xi$ in a Taylor series to get

$$F(W_e) = E_G \left\{ F(m) + (\xi - m) F_1(m) + \frac{1}{2} (\xi - m)^2 F_2(y) \right\}. \tag{1.18}$$

where y represents a value between ξ and m. Hence

$$F(W_e) = F(m) + R_1$$

where

$$R_1 = \frac{1}{2} E_G (\xi - m)^2 F_2(y).$$

Therefore

$$|R_1| \leq \frac{1}{2} MV, \text{ using } |F_2(w)| \leq M.$$

To prove theorem 1.2, we write as in (1.18),

$$F(W_e) = E_G\left\{F(m) + (\xi - m)f(y)\right\} = F(m) + R_2$$

where y is a value between ξ and m and

$$R_2 = E_G\ (\xi - m)f(y)$$

$$= \int_{x \geq m} (x - m)f(y)g(x)\,dx - \int_{x \leq m} (m - x)f(y)g(x)\,dx \tag{1.19}$$

where $g(\cdot)$ is the p.d.f. of ξ.

$f(\cdot)$ being a density is never negative. It is bounded under the assumption of the theorem: $0 \leq f(\cdot) \leq M^*$. Further, both the above integrands are positive. Therefore,

$$|R_2| \leq M^* \int_{x \geq m} (x - m)\ g(x)\,dx = \frac{1}{2} M^* E|\xi - m|,$$

because m is the mean of ξ. Hence

$$|R_2| \leq \frac{1}{2} M^* D.$$

This proves the theorem.

1.3d: Some observations on Theorems 1.1 and 1.2:-

The following observations are made with respect to the EQL and the above theorems.

(1) While W_e as defined by (1.12) always exists because of the continuity conditions of the theorem, we have tacitly assumed the existence of m, V and D, because for almost all the OCs used in practice, they do exist.

(2) The case of weighted and unweighted OCs:- When w lies between 0 and 1, for a uniform PC with $f(w) = 1$, the remainder term $R_1 = 0$ and $W_{eF} = m$. m can then be interpreted as the average probability of acceptance and also as the EQL with respect to a uniform prior distribution for w. However, in all cases including Poisson where w can go to infinity, m is referred to as 'the area under the unweighted OC' - provided that (1.8) holds - because it can b obtained formally by integrating $L(w)$ with respect to w as compared to $F(W_{eF})$ or $E_F\, L(w)$ which is obtained by integrating $L(w)$ with respect to $F(w)$. The later will be referred to as the 'area under the weighted OC' or the 'weighted area under the OC' with respect to the given PC.

(3) W_{eF} depends both on the PC and the plan θ whereas m, V. and D are independent of PC. However they are functions of θ and the above theorems are of practical interest only when $F(m)$ and $F(W_{eF})$ are asymptotically equivalent, that is, when D or V tends to zero as the sample size n increases to infinity. (Note that always $D \leq \sqrt{V}$). We find that for almost all the OCs of interest, m, V and other higher moments of ξ are of increasing orders of $n^{-1/2}$. Let us then assume that such is the case and make further comments.

(4) It is easy to see that for the convergence of $F(W_{eF})$ to $F(m)$ at a specified value $m = m^*$, the continuity of the process density $f(w)$ at $w = m^*$ alone suffices, because the boundedness of $f(w)$ in the neighbourhood of $w = m^*$ follows from this continuity.

(5) Robustness of the EQL:- Theorems 1.1 and 1.2 suggest that if the process characteristic obeys the conditions of the theorems, then the average probability of acceptance $F(W_{eF})$ in large samples is almost insensitive to slight changes in the PC and hence may be approximated by $F(m)$, m being the area under the corresponding unweighted OC. This is what we may call as the 'robust' property of the EQL.

It may be noted that in the case of Poisson, binomial and normal single sampling OCs — the three most widely used OCs m bears very simple relations to the elements of the plan θ and can be calculated easily. (Chapters 3, 4 and 5).

1.4 Accurate evaluation of the EQL

1.4a: If the PC has well behaved derivatives of higher order and if the j-th central moment of ξ, μ_j, $j \geq 2$, is $O(n^{-j/2})$, n being the total sample size, then the average probability of acceptance and also the EQL can be evaluated to any desired degree of accuracy.

Following the proof of theorem 1.1, we have

$$F(W_e) = E_F\, L(w)$$

$$= F(m) + \frac{\mu_2}{2!} F_2(m) + \dots + \frac{\mu_{r-1}}{(r-1)!} F_{r-1}(m) + O(n^{-r/2}) \qquad (1.20)$$

where $F_j(\cdot)$ is the j-th derivative, $\frac{d^j F(w)}{dw^j}$ Expanding $F(W_e)$ around m we get

$$F(W_e) = F(m) + (W_e - m)F_1(m) + \frac{(W_e - m)^2}{2!} F_2(m) + \ldots \tag{1.21}$$

Equating the two series and solving for W_e in terms of m, we obtain after rearranging the terms in order of magnitude in $\sqrt{n}$,

$$\begin{aligned} W_e = m &+ \left(\frac{\bar{\mu}_2}{2} \frac{F_2(m)}{F_1(m)}\right) + \left(\frac{\mu_3}{6} \frac{F_3(m)}{F_1(m)}\right) \\ &+ \left(\frac{\bar{\mu}_4}{24} \frac{F_4(m)}{F_1(m)} - \frac{1}{8} \mu_2^2 \left\{\frac{F_2(m)}{F_1(m)}\right\}^3 \right) \\ &+ \left(\frac{\mu_5}{120} \frac{F_5(m)}{F_1(m)} - \frac{1}{12} \frac{\mu_2\mu_3 \{F_2(m)\}^2 F_3(m)}{\{F_1(m)\}^3} \right. \\ &+ 0(n^{-3}). \end{aligned} \tag{1.22}$$

<u>1.4b:</u>- From theorem 1.1,

$$F(W_e) = F(m) + E_G \; (\xi - m)^2 \; F_2(y) \tag{1.23}$$

If $F(\cdot)$ is convex, F_2 is always positive and hence $F(W_e) > F(m)$. If $1 - F(\cdot)$ is convex, $F(W_e) < F(m)$. In otherwords, when $F(\cdot)$ is convex m underestimates the EQL and when $1 - F(\cdot)$ is convex, m overestimates the EQL.

1.4c: Approximations to the c.d.f. of a compound distribution :-

Notice that $E_F L(w)$ in (1.12) can be identified as the c.d.f. of a compound distribution (RMM tables, 1966) and that the series (1.20) can be used to provide approximations to the c.d.f. of the compound distribution. (Mitra and Subrahmanya (1968) and Khatri and Mitra, 1969). It is therefore of independent interest to study the behaviour of D and V which occur in the remainder terms of Theorems 1.1 and 1.2.

1.5 The EQL and the IQL (point of control)

Theorems 1.1 and 1.2 together with (1.10) imply that all the three points, the EQL, the IQL and m tend to be one and the same point as n increases to infinity. Even for moderately large values of n, they are sufficiently close so that the interpretations of the EQL and the IQL are interchangeable: the IQL which is defined as the point at which the probability of acceptance is half, gains added significance under the EQL interpretation as given by (1.12) and (1.13). The plans based on the IQL (Philips SSS in Willemze and Fuijt 1955) seem to have been introduced originally on empirical grounds i.e., that the plans work in practice and from the point of view of easy computability. (Hamacker (1954) and (1959); Hald, 1960). For large samples, they turn out to be plans under which the average probability of acceptance is equal to the proportion of submitted lots that are of quality better than or equal to the IQL.

Hamacker has discussed how an IQL value can be chosen and a suitable IQL plan be selected in practice for the case of inspection by attributes and count of defects i.e., the binomial and the Poisson situations. He has also given some empirical relations connecting the lot size, the sample size and the IQL. (Hamacker (1954) and 1959).

1.6 The EQL as a break even quality level and the Bayesian plans

Either when the sampling is done with replacement or when the sample sizes are small compared to the lot size, the effect of lot size on the probability of acceptance can be ignored and therefore, following the usual practice, we have considered OCs as independent of lot size. However, the acceptance of a bad lot or the rejection of a good lot is much more serious (riskier) when the lot size is large as compared with when it is small. It becomes necessary to take lot size into consideration before selecting a suitable plan. The idea is to connect the sample size and the lot size through risk functions. This can be carried out either through empirical studies (Hamacker's papers) or through working out mathematically the optimum plans. (Hald's papers where references to other authors may also be found).

In the later case one starts with a reasonable regret function R which depends on certain cost parameters, the lo size N and also the elements of the plan $\theta = (\theta_1, \theta_2 \dots)$. Thus $R = R(N, \theta)$. A regret function normally implies a break even value for w, that is, if $w \leq w_0$, acceptance is preferre and if $w > w_0$, rejection is preferred. w_0 is called the break – even value. The Bayesian plan is obtained by minimizin R with respect to the elements θ. This will result in certain relationships among the elements of θ and also between θ and N. A restricted Bayesian plan is obtained by minimizing R with respect to θ under some suitably chosen condi tions as demanded by a practical situation, for example, the EQL condition (1.12). These conditions may establish a relationship among the elements θ which need not be the same as the corresponding one under the Bayesian plan. Under certair conditions, Hald (1964)(1967a) and (1968) has shown that if the right relationship among the elements θ as obtaining under the Bayesian plan is not used, then as the lot size (or the sample size) increases, the efficiency of a restricted Bayesian plan tends to zero as compared with that of the Bayesian plan. However it is possible to introduce restrictions on OC which will change the Bayesian solution for small lots in a desired direction and at the same time preserve the properties of the Bayesian solution for large lots.

It is shown in Hald (1964),(1967a) etc., that under certain general types of regret functions, the right relationship for the Poisson and the binomial single sampling plans (n,c) should be such that $c/n \to w_0$, the break-even value, as $n \to \infty$, ($N \to \infty$ but $n/N \to 0$). It is shown in Chapters 3 and 4 that the areas under the unweighted Poisson and binomial single sampling OCs i.e., the values of m are $(c+1)/n$ and $(c+1)/(n+1)$ respectively. In otherwords m or EQL in these cases tends to w_0. Similar results can be proved for other OCs also.

There are two methods of showing that $m \to w_0$: one can use the representation of either (i) $1-L(w)$ as the c.d.f. of the hypothetical random variable ξ or (ii) $L(w,\theta)$ as the conditional c.d.f. of the decision variable(s), given w.

The former method is given in Sections 2.7 (Theorem 2.7) and 2.8. A broad outline of the later is given below.

Consider a single sampling plan (n, t) where the lot is rejected or accepted according as the decision variable η_n exceeds or does not exceed t. Here n is the sample size and t the acceptance number. It is supposed that

$$E(\eta_n | w) = w \qquad (1.24)$$

and that conditional on w, the variance of η_n is of the order of n^{-1}. The regret function is of the form

$$\ell = \begin{cases} R_1(w_0 - w) & \text{if } w \leq w_0 \\ R_2(w - w_0) & \text{if } w > w_0 \end{cases} \qquad (1.25)$$

where $R_1(w)$ and $R_2(w)$ are increasing functions of w. w_0 is the break-even value. It is economical to accept the lot when $w \leq w_0$ and to reject the lot when $w > w_0$. It may therefore be expected that for an optimum plan, the procedure of acceptance $\eta_n \leq t$ should tend to the exact procedure $w \leq w_0$; or in otherwords-because of (1.24)- the optimum valu of t, t_0 say, should tend to w_0 as n increases to infini

Assuming some regularity conditions for $L(w)$, $R(w)$ and the process density $f(w)$, Hald (1967a) has indeed proved mathematically that

$$t_0 = w_0 + O(\frac{1}{\sqrt{n}}) \qquad (1.26)$$

His methods can be used to obtain t_0 to any desired degree of accuracy. This result has been extended to multiple sampling schemes in Hald and Keiding (1969) and Johansen (1969).

Starting with (1.26), it is very easy to show that the IQL, $W_0 \to w_0$. For an optimum plan, we have

$$\begin{aligned} L(w) &= P\left\{ \eta_n \leq t_0 \mid w \right\} \\ & P\left\{ \eta_n \leq w_0 \mid w \right\}, \text{ using } (1.26) \\ &= P\left\{ \eta_n \leq E(\eta_n \mid w = w_0) \mid w \right\}, \text{ using } (1.24) \end{aligned}$$

The value of the OC at the point $w = w_0$ is thus approximately equal to the probability that the conditional random variable $\eta_n \mid w = w_0$ is less than are equal to its mean. Since under our assumptions, this mean is equal to the median upto the order of $n^{-(1/2)}$, we get

$$\begin{aligned} L(w_0) &= \frac{1}{2} + O(\frac{1}{\sqrt{n}}) \\ &= L(W_0) + O(\frac{1}{\sqrt{n}}) \end{aligned} \qquad (1.27)$$

Hence the IQL, $W_0 \to w_0$. Because, the IQL, the EQL and m are asymptotically equal, it follows that both the EQL and m also tend to w_0.

Using the properties of the hypothetical random variabl ξ, we would be arriving at the same conclusions not only for single sampling but also for double and multiple sampling plans. (Theorem 2.7 in Section 2.7 and Section 2.8).

Summing up, we observe that the EQL for large samples can be interpreted as the break — even quality level and tha the EQL relationship (1.12) is asymptotically a Bayesian relationship.

1.7 Tightened and reduced inspection with respect to the EQL

When a plan with a certain OC is adopted for normal inspection, it is implicit that the maintenance of certain quality standard by the producer is expected. Whether the producer maintains this standard consistently can be judged b a knowledge of proportion of lots that are not accepted which in an EQL plan directly estimates the proportion of bad lots submitted (1.13). When there is a considerable increase in the number of bad lots submitted for inspection, there will be a corresponding increase in the number of lots not accepted by the plan and the consumer will be put into a great inconvenience. It would be necessary to warn the producer by resorting to tightened inspection. This can be done simply by shifting to a plan with a lower EQL (or m), thus effectively reducing the average probability of acceptance well below the level of proportion of good lots submitted.

An example is given below for the sake of illustration.

Suppose that a plan is adopted with EQL = 5 °/. with the understanding that 90 °/. of the lots submitted will be of quality equal to or better than 5 °/. . (In the Poisson case, the plan may be taken as $n = 100$, $c = 4$ so that $m = (c+1)/n = 5$ °/.). $F(w)$ is the proportion of lots submitted for inspection that are of quality better than or equal to w. Initially the proportion of good lots submitted is given by

$$F_{(o)}(.05) = 90 \text{ °/.} \quad .$$

The plan will on an average accept 90 °/. of the lots and reject 10 °/. of the lots. If t_s is the number of lots submitted and t_a the number of lots accepted, then the proportion of good lots F (5 °/.) is estimated by t_a/t_s with the standard deviation $\sqrt{t_a(t_s - t_a)/t_s^3}$ and it is easy to carry out a binomial test of significance to find out whether F(5 °/.) is considerably less than 90°/. or not. Let us suppose that after the plan is in use for some time, a deterioration sets in so that the PC shifts from $F_{(0)}$ to $F_{(1)}$:

$$F_{(1)}(.05) = 75 \text{ °/.} \quad \text{and} \quad F_{(1)}(.03) = 60 \text{ °/.}$$

Because of the robustness of the EQL, the proportion of lots accepted is nearly equal to the proportion of good lots submitted and the producer may not worry. However the actual proportion of lots accepted is about 75 °/. — much less than the expected 90 °/. — and the consumer will be in shortage of good lots. He shifts to a plan with EQL = 3 °/. (In the Poisson case, the tightened plan can be $n = 100$, $c = 2$ so that $m = (c+1)/n = 3$ °/.). This plan will accept only 60 °/. of the lots eventhough 75 °/. the lots submitted are good according to the original specification. Also more good lots the region 3 °/. $< w <$ 5 °/. are rejected now and the producer is forced to improve the standard.

When there is a considerable improvement in the quality of incoming lots, the producer may be encouraged by relaxing (or reducing) inspection. This can be done by shifting to a plan with a higher EQL (or m), thus effectively increasing the average probability of acceptance well above the relative proportion of good lots submitted for inspection.

Thus the concept of EQL provides a rational and easy procedure for shifting from normal to tightened or reduced inspection. It may also be noted that the shifting can be done by desired amounts and if necessary in gradual stages (Section 1.8).

1.8 The effect of randomization on the EQL

It may happen sometimes that OCs which satisfy exactly the given specifications lead to non-integral values for the sample size or the acceptance number and it becomes impossible to construct such plans without the use of randomization. This is not, in general, a serious problem, because in most of the practical cases, one can be satisfied with nearby plans and also because randomization between two plans can be carried out, if necessary, without much difficulty.

Suppose that W_{eo} is the stipulated value for the EQL and that there are two plans θ_1 and θ_2 with OCs L_1 and L_2 and EQLs W_{e1} and W_{e2} respectively. Suppose further that $W_{e1} \leq W_{eo} \leq W_{e2}$. Choose a number a such that

$$F(W_{eo}) = aF(W_{e1}) + (1-a)F(W_{e2}), \quad 0 \leq a \leq 1 \qquad (1.28)$$

For each lot, before starting the sampling inspection, choose one or the other of the two plans with probabilities a and $1-a$ respectively. The OC for such a procedure is given by

$$L(w,\theta_1,\theta_2 \,;\, a) = a\, L_1(w,\, \theta_1) + (1-a)L_2(w,\, \theta_2) \qquad (1.29)$$

Hence

$$\begin{aligned} E_F\, L(w;\, a) &= a\, E_F\, L_1(w) + (1-a)\, E_F\, L_2(w) \\ &= a\, F\,(W_{e1}) + (1-a)F(W_{e2}) \\ &= F(W_{eo}),\ \text{exactly.} \end{aligned}$$

The composite OC is not in general of the same structure as the constituent OCs: if L_1 and L_2 are Poisson single sampling OCs, it is not, in general, possible to represent $L(w;\, a)$ as a Poisson single sampling OC; in other words, $L(w;\, a)$ cannot be considered as the c.d.f. of the decision variable, namely, the number of defects in the sample

However as stated in Section 1.1c , $1 - L(w; a)$ can be represented as the c.d.f. of a hypothetical random variable. Theorems 1.1 and 1.2 are applicable so that $F(W_{eo})$ can be approximated by $F(m)$ where m is the area under the unweighted OC, $L(w; a)$ i.e., $m = am_1 + (1-a)m_2$. m_1 and m_2 being the areas under the unweighted OCs of the constituent plans.

Thus, randomization with respect to plans is equivalent to a linear combination, not of the EQLs, but of the average probabilities of acceptance. It is also approximately equivalent to a linear interpolation between the areas under the unweighted OCs.

Note that if L_1 and L_2 have the same EQL, the composite OC will also have the same EQL. It may also be observed that if L_1 and L_2 correspond to normal and tightened (reduced) inspection, then by varying a between 0 and 1 in (1.29), one can obtain gradations in shifting from normal to tightened (reduced) inspection.

1.9 The EQL for multivariate OCs

1.9a: The conceptual set-up:-

The concept of EQL along with its good interpretation can be extended to multivariate OCs also. Eventhough the numerical value of IQL is close to that of EQL in the univariate case, the concept of IQL does not allow its extension to multivariate OCs.

In cases where a lot has to satisfy the quality require ments with respect to more than one quality characteristic, t lot quality measurement is a vector, w say (cf. Section 1.1b (iii)). Suppose that w is a k-dimensional vector

$$w = (w^{(1)}, w^{(2)}, \ldots, w^{(k)})$$

where $w^{(i)}$ refers to the measurement of lot qualitv with respect to the i-th quality characteristic. Suppose further that an increase in the value of any one or more of $w^{(i)}$s means a deterioration in lot quality. A good lot is defined by $w^{(i)} \leq W^{(i)}$, $i = 1,2,\ldots, k$, where $W^{(i)}$s are specifie Let us denote the joint c.d.f. of a k-variate prior distribution of w by F(w). Then the proportion of good lots submitted for inspection is $F(\underset{\sim}{W})$, where $\underset{\sim}{W} = (W^{(1)}, W^{(2)}, \ldots, W^{(k)})$.

1.9b: <u>The hypothetical random vector $\underset{\sim}{\xi}$:-</u>

The vector θ will as in earlier sections denote the plan or alternatively the elements of the plan. The probability of accepting a lot of quality $\underset{\sim}{w}$ under the operation of the plan θ is denoted by the k-variate OC $L(w^{(1)}, w^{(2)}, \ldots, w^{(k)}, \theta)$, or $L(\underset{\sim}{w})$ for brevity. Given the above conceptual set-up, it would be reasonable to assume that

(a) $L(\underset{\sim}{w})$ is a non-increasing function of each of the variables $w^{(i)}$, $i = 1, 2, \ldots, k$;

(b) $L(\underset{\sim}{w})$ takes the value 1 when each $w^{(i)}$ is at its lowest value $A^{(i)}$; and that

(c) $L(\underset{\sim}{w})$ is zero when at least one of $w^{(i)}$s is at its highest value $B^{(i)}$.

We make an additional assumption that

(d) $L(\underset{\sim}{w})$ is continuous with respect to each $w^{(i)}$.

Conditions (a) to (d) imply the existence of a k-dimensional random vector $\underset{\sim}{\xi} = (\xi^{(1)}, \xi^{(2)}, \ldots, \xi^{(k)})$ such that

$L(\underset{\sim}{w})$ = the joint probability

$$P\left\{ \xi^{(1)} > w^{(1)}, \xi^{(2)} > w^{(2)}, \ldots, \xi^{(k)} > w^{(k)} \right\} \tag{1.30}$$

The mean of the random variable $\xi^{(i)}$ is denoted by $m^{(i)}$ and the mean deviation about the mean of $\xi^{(i)}$ by $D^{(i)}$. The covariance between $\xi^{(i)}$ and $\xi^{(j)}$ is denoted by V_{ij}. The joint c.d.f. of $\underset{\sim}{\xi}$ may be denoted by $G(\underset{\sim}{\cdot})$. Thus

$$G(\underset{\sim}{w}) = P\left\{\xi^{(i)} \leq w^{(i)},\ i = 1,2,\ldots,k\right\} \leq 1 - L(\underset{\sim}{w}) \tag{1.31}$$

1.9c: <u>The EQL-vector and its robustness</u>:-

Given an OC $L(\underset{\sim}{w}, \theta)$ and a PC with c.d.f. F, the equation

$$F(\underset{\sim}{w}) = E_F\, L(\underset{\sim}{w}, \theta) \tag{1.32}$$

does not, in general, lead to a unique determination of $\underset{\sim}{w}$ even when F is continuous with respect to each of its arguments. Any particular set of values,

$$\underset{\sim}{W_{eF}} = (W_{eF}^{(1)},\ W_{eF}^{(2)}, \ldots,\ W_{eF}^{(k)})$$

satisfying the above equation may be called an EQL-vector for the plan θ with respect to F.

$$F(\underset{\sim}{W_{eF}}) = E_F\, L(\underset{\sim}{w}) \tag{1.33}$$

and hence

$$1 - F(W_{eF}) = E_F\left\{1 - L(\underset{\sim}{w})\right\} \tag{1.34}$$

It is now easy to see that the interpretation of the EQL as given in Section 1.3a for the univariate case is valid for k-variate OCs also. The fundamental lemma 1.1 given in Section 1.3b takes the form

$$F(\underset{\sim}{W}_{eF}) = E_F\, L(\underset{\sim}{w}) = E_G\, F(\underset{\sim}{\xi} - \underset{\sim}{0}) \tag{1.35}$$

where F and G stand for the k-variate c.d.f.s of $\underset{\sim}{w}$ and $\underset{\sim}{\xi}$ respectively.

Results similar to Theorems 1.1 and 1.2 can also be proved.

<u>Theorem 1.3:</u>

If the c.d.f. of the process characteristic, $(\underset{\sim}{w}) = F(w^{(1)}, w^{(2)}, \ldots, w^{(k)})$ has continuous partial derivatives of second order such that

$$\left.\frac{\partial^2 F}{\partial w^{(i)}\ \partial w^{(j)}}\right| \leq M_{ij},\ i,\ j = 1,2,\ldots,k, \quad \text{then}$$

$$F(\underset{\sim}{W}_e) = F(\underset{\sim}{m}) + R_1$$

and $|R_1| \leq \frac{1}{2} \sum_{j=1}^{k} \sum_{i=1}^{k} M_{ij} \sqrt{V_{ii} V_{jj}}$ (1.36)

where $\underset{\sim}{m}$ is the mean vector; $\underset{\sim}{m} = (m^{(1)}, m^{(2)}, \ldots, m^{(k)})$

Theorem 1.4:

If $F(\underset{\sim}{w})$ has continuous first order partial derivatives such that $\frac{\partial F}{\partial w^{(i)}} < M_i^*$, $i = 1,2,\ldots, k$, then

$$F(\underset{\sim}{W_e}) = F(\underset{\sim}{m}) + R_2$$

where

$$|R_2| \leq \frac{1}{2} \sum_{i=1}^{k} M_i^* D^{(i)} \qquad (1.37)$$

Proof: Under the conditions of theorem 1.3, expanding $F(\underset{\sim}{\xi})$ in (1.35) around the point $\underset{\sim}{m}$ in a Taylor series, we get

$$R_1 = \frac{1}{2} E_G \sum_{j=1}^{k} \sum_{i=1}^{k} (\xi^{(i)} - m^{(i)})(\xi^{(j)} - m^{(j)}) \left(\frac{\partial^2 F}{\partial w^{(i)} \partial w^{(j)}}\right) \text{at } \underset{\sim}{w} =$$

where $\underset{\sim}{y} = (y^{(1)}, \ldots, y^{(k)})$, $y^{(i)}$ being a value between $\xi^{(i)}$ and $m^{(i)}$. The result (1.36) follows by observing that

$$|E_G(\xi^{(i)} - m^{(i)})(\xi^{(j)} - m^{(j)})| \leq \sqrt{V_{ii} V_{jj}},$$

$$i, j = 1,2,\ldots, k$$

Under the conditions of theorem 1.4, we have

$$R_2 = E_G \sum_{i=1}^{k} (\xi^{(i)} - m^{(i)}) \left(\frac{\partial F}{\partial w^{(i)}}\right) \text{ at } \underset{\sim}{w} = \underset{\sim}{y}$$

Theorem 1.4 is now proved by using the same arguments as those employed in proving Theorem 1.2.

It may be remarked that if $D^{(i)}$s or V_{ij} s decrease to zero as the sample size increases, $F(\underset{\sim}{W_o})$ can be approximated by $F(\underset{\sim}{m})$.

It is of theoretical interest to know that when the PC is uniform either

(i) in the unit (hyper-) cube with the c.d.f. given by

$$F(\underset{\sim}{w}) = w^{(1)} w^{(2)} \dots w^{(k)}, \quad 0 \leq w^{(i)} \leq 1, \quad i = 1,2,\dots, k \tag{1.38}$$

or (ii) in the simplex

$$T_k : \left\{ \underset{\sim}{w} ; 0 \leq w^{(i)} \leq 1, \; i = 1,2,\dots, k \text{ and } \sum_{i=1}^{k} w^{(i)} \leq 1 \right\}$$

with the c.d.f. given by

$$F(\underset{\sim}{w}) = k!\, w^{(1)} w^{(2)} \cdots w^{(k)} \quad \text{for } \underset{\sim}{w} \in T_k \qquad (1.39)$$

the following relation holds exactly.

$$W_e^{(1)} W_e^{(2)} \cdots W_e^{(k)} = E\,(\xi^{(1)} \xi^{(2)} \cdots \xi^{(k)}) \qquad (1.40)$$

The above equation reduces to

$$W_e^{(1)} W_e^{(2)} \cdots W_e^{(k)} = n^{(1)} n^{(2)} \cdots n^{(k)} \qquad (1.41)$$

when $\xi^{(i)}$ s are independent.

1.9d: The EQL vector as an extension of the EQL in the univariate case:-

If the acceptability of a lot with respect to the i-th and j-th characteristics is decided on the basis of independent samples, $\xi^{(i)}$ and $\xi^{(j)}$ will be independent random variables. In case a lot is finally accepted if and only if each of the k inspectors pass the lot after carrying out sampling inspection independently with respect to k different characteristics, $\xi^{(i)}$ s are all mutually independent and the OC splits up into k factors

$$L(w^{(1)}, \ldots, w^{(k)}) = L^{(1)}(w^{(1)}) L^{(2)}(w^{(2)}) \cdots L^{(k)}(w^{(k)}) \qquad (1.42)$$

where $L^{(i)}(w^{(i)})$ is the (marginal) OC corresponding to the i-th characteristic (or i-th inspector).

If $w^{(i)}$s are independent, $F(\underset{\sim}{w})$ also splits up into factors

$$F(w^{(1)},\ldots w^{(k)}) = F^{(1)}(w^{(1)})F^{(2)}(w^{(2)})\ldots F^{(k)}(w^{(k)}) \tag{1.43}$$

where $F^{(i)}(w^{(i)})$ is the c.d.f. of the marginal distribution of $w^{(i)}$.

Now an extremely simple but important result is given in the form of a theorem.

Theorem 1.5:

Suppose that a lot has to meet quality requirements with respect to k characteristics and that it is accepted if and only if it is accepted under each of k independent univariate EQL plans - the i-th plan accepting or rejecting the lot with respect to the i-th characteristic only. Suppose further that the condition (1.43) holds. Then, for the ultimate k-variate plan which finally decides the acceptance or otherwise of the lot, there exists a k-dimensional EQL vector such that its i-th component is exactly equal to the EQL of the i-th univariate plan.

Proof:- Let the given value $W^{(i)}$ be made the EQL of the i-th univariate plan $\theta_e^{(i)}$ with respect to the prior distribution of the i-th characteristic. Then

$$F^{(i)}(W^{(i)}) = E_{F^{(i)}} L^{(i)}(W^{(i)}, \theta_e^{(i)}), \; i = 1,2,\ldots, k \tag{1.44}$$

Multiplying the k equations and using (1.42) and (1.43), we get

$$F(\underset{\sim}{W}) = E_F L(\underset{\sim}{W}, \theta_e) \tag{1.45}$$

where

$$\underset{\sim}{W} = (W^{(1)}, W^{(2)}, \ldots, W^{(k)}),$$

and
$$\theta_e = (\theta_e^{(1)}, \theta_e^{(2)}, \ldots, \theta_e^{(k)}) \tag{1.46}$$

Now, by difinition (1.33), $\underset{\sim}{W}$ is a k-dimensional EQL-vector of the k-variate θ_e and hence the theorem is proved.

The content on of the theorem is that the concept of EQ can be extended from the univariate to multivariate OCs. In order to appreciate this point more clearly, two additional observations are made in the following.

(1) The theorem 1.5 is not true for the IQL:

The IQL is defined to be that point at which both the probabilities of acceptance and rejection are equal to 50 %. If the i-th plan $\theta_0^{(i)}$ has the given value $W^{(i)}$ as its IQL,

$$L^{(i)}(W^{(i)}, \theta_0^{(i)}) = 2^{-1}, \quad i = 1,2,\ldots, k \tag{1.47}$$

Multiplying the k-equations, the final probability of acceptance for the k-variate plan

$\theta_0 = (\theta^{(1)}, \theta^{(2)},\ldots, \theta_0^{(k)})$ is obtained as

$$L(\underset{\sim}{W}, \theta_0) = 2^{-k} \tag{1.48}$$

Because of theorems 1.1 to 1.4, in large samples, the k - variate plan θ_0 will be close to the corresponding k variate plan θ_0. However $\underset{\sim}{W}$ in (1.48) cannot be viewed as an IQL-vector of the k- variate plan θ_0.

On the otherhand, if we start with plans $\theta_1^{(i)}$ satisfying

$$L^{(i)}(W^{(i)}, \theta_1^{(i)}) = 2^{-1/k}, \quad i = 1,2,\ldots, k \tag{1.49}$$

we obtain for the k- variate plan $\theta_1 = (\theta_1^{(1)},\ldots, \theta_1^{(k)})$,

$$L(\underset{\sim}{W}, \theta_1) = 2^{-1} \tag{1.50}$$

$W = (W^{(1)}, W^{(2)}, \ldots, W^{(k)})$ is an IQL - vector of the k-variate plan θ_1, but non of its components $W^{(i)}$ can be interpreted as the IQL of the univariate plan $\theta_1^{(i)}$

(2) Because of theorems 1.3 and 1.4, we may remove the conditions of independence in theorem 1.5 provided that the words 'exactly equal to the EQL' are replaced by 'approximately equal to the EQL'.

Multinomial OC - that is, an OC under the condition of a multinomial distribution for the decision variables - i considered in the last chapter as an example for the multivariate OC with dependent $\xi^{(i)}$s.

CHAPTER 2

THE ERROR-AREAS UNDER THE OC CURVES

2.1 The error- areas

For a given plan θ and a univariate OC, $L(w,\theta)$ and a PC with c.d.f. $F(w)$, $0 \leq w \leq B \leq \infty$, the error- area of the first kind at the point $w = W$ is defined to be - as in Subrahmanya (1966) -

$$J_{1F}(W, \theta) = \int_{w \leq W} \left\{1 - L(w, \theta)\right\} dF(w) \tag{2.1}$$

and the error - area of the second kind at the point $w = W$ is defined to be

$$J_{2F}(W, \theta) = \int_{w > W} L(w, \theta)\, dF(w) \tag{2.2}$$

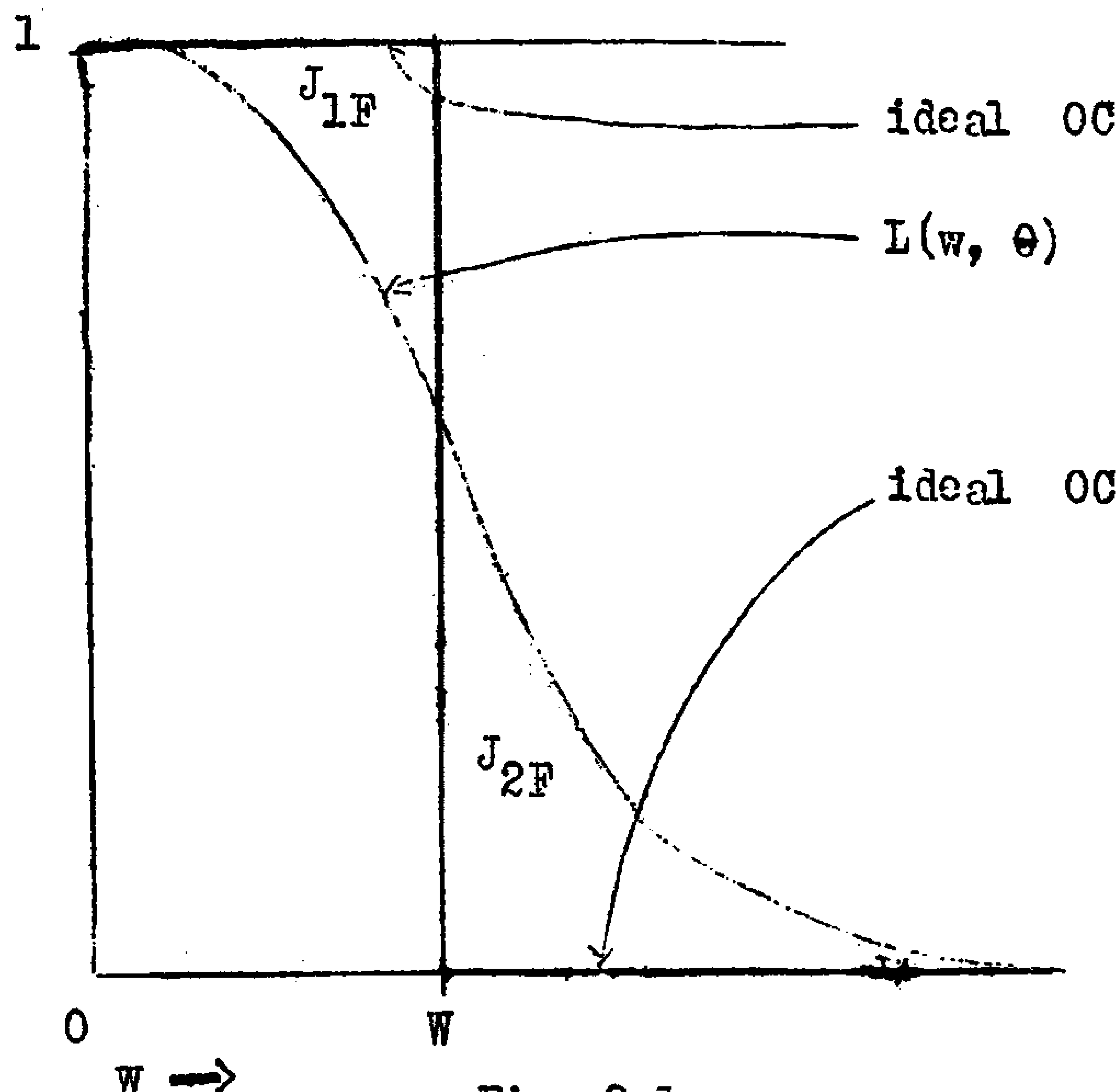


Fig. 2.1

$J_{1F}(W, \theta)$ is the average probability of not accepting lots of quality better than or equal to W and $J_{2F}(W, \theta)$ is the average probability of accepting lots of quality worse than W. If W is the point of demarcation between good and bad lots or an economic break - even quality level, $J_{1F}(W, \theta)$ and $J_{2F}(W, \theta)$ can be taken respectively as measures of producer's and consumer's risks at the quality level W.

The total error- area at $w = W$ is the sum of two kind of error- areas at that point. It is given by

$$J_F(W, \theta) = J_{1F}(W, \theta) + J_{2F}(W, \theta) \qquad (2.3)$$

The ideal OC at $w = W$ is that which accepts with probability 1 all lots with $w \leq W$ and rejects with probability 1 all lots with $w > W$. (See Fig. 2.1). The total error- area at W, namely $J_F(W, \theta)$, is the deviation of the OC $L(w, \theta)$ from the ideal OC at W. If W is the value that demarcates good lots from bad, $J_F(W, \theta)$ can be considered as a measure of the decision loss due to sampling as compared with the ideal situation when all good lots are accepted and all bad lots rejected.

When necessary, the error- areas defined in (2.1), (2.2) and (2.3) will be referred to as the 'weighted error- areas under the OC' or as 'the 'error- areas under the weighted OC' at the point $w = W$ with respect to the given PC.

The total weighted area under the OC is, of course, the average probability of acceptance. It is denoted by $A_F(\theta)$. We have

$$A_F(\theta) = E_F\, L(w,\theta) = \int_0^B L(w,\theta)\,dF(w) = J_{2F}(0,\theta) \qquad (2.4)$$

The difference between the two kinds of error- areas at the point W is given by

$$J_{1F}(W, \theta) - J_{2F}(W,\theta) = F(W) - A_F(\theta) \qquad (2.5)$$

This is an important relation which follows directly from the definitions. If one of the error-areas is known, the other can be determined easily by this relationship.

It may be observed that for a given plan θ, as w increases from its lowest value to the highest,

$L(w, \theta)$ decreases from 1 to zero

$J_{2F}(w, \theta)$ decreases from $A_F(\theta)$ to zero, and

$J_{1F}(w, \theta)$ increases from zero to $1 - A_F(\theta)$ (2.6)

When there is no cause for confusion, the symbol θ may be omitted and the notations $L(w, \theta)$, $J_{1F}(w, \theta)$ etc., shortened to $L(w)$, $J_{1F}(w)$ etc. Similarly the suffix F may also be dropped for the sake of convenience.

2.2 The EQL and the IQL in relation to error-areas

2.2a: The EQL:-

If the prior distribution of w is continuous, the EQL $W_{0F}(\theta)$ — $W_0(\theta)$ — or W_0 for brevity — exists uniquely and is given by (1.12). Using (2.4),

$$F(W_0) = A_F(\theta) \qquad (2.7)$$

Now the relation (2.5) takes the form

$$J_1(W, \theta) - J_2(W, \theta) = F(W) - F(W_e) \qquad (2.8)$$

for any W in the interval $0 \leq W \leq B$.

It follows by putting $W = W_e$ in (2.8) that the EQL is actually the 'point of equal error- areas' i.e., the point at which the two kinds of error- areas are equal.

Since F(w) is an increasing function of w, the following inequality connecting the two error- areas is obtained

$$J_1(W,\theta) \lesseqgtr J_2(W,\theta) \text{ according as } W \lesseqgtr W_e(\theta) \qquad (2.9)$$

It follows that for a given plan θ, the EQL is also the point at which the bigger of the two error- areas $J_1(w, \theta)$ and $J_2(w, \theta)$ considered as functions of w is a minimum with respect to w.

For the sake of easy reference, the above two interpretations of the EQL in terms of the error- areas may be formulated in the form of a

Theorem 2.1:-

For a given plan θ and a continuous prior distributio
for w, the EQL W_e satisfies

$$J_1(W_e, \theta) = J_2(W_e, \theta) \tag{2.10}$$

$$= \min_{w} \max \{ J_1(w,\theta),\ J_2(w,\theta) \} \tag{2.11}$$

For lots of quality better than W_e, the second kind of error-areas is larger than the first and for lots of quality worse than W_e, the first kind of error-area is bigger than the second. The two error-areas are equal for lots of quali W_e. This supports the idea of taking J_1 and J_2 as measur of producer's and consumer's risks when W_e is the point of demarcation between good and bad lots or the break — even quality level.

The following comments are made in relation to the assumption of continuity of the prior distribution in Theorem 2.1.

Suppose the prior distribution of w is discrete. Let w take the values w_r with probabilities $f(w_r)$, $r = 0,1,2.$ Without loss of generality let $w_0 < w_1 < w_2 < \ldots$. For an value w in the interval $w_r \leq w < w_{r+1}$, we have

$$F(w) = F(w_r);\ J_1(w) = J_1(w_r) \text{ and } J_2(w) = J_2(w_r) \tag{2.12}$$

Let w_e and w_{e-1} be two consecutive mass points such that

$$F(w_e) \leq A_F < F(w_{e+1}) \tag{2.13}$$

If the equality holds then any w in the interval $w_e \leq w < w_{e+1}$ is an EQL value and consequently satisfies (2.10) and (2.11). The theorem can therefore be generalized to read 'for a given θ, if there exists an EQL value satisfying (2.7), then it will also satisfy (2.10) and (2.11). Such a generalization is of little interest, because, when the prior distribution is discrete, theorems 1.1 and 1.2 do not apply and therefore we cannot gaurantee that the EQL is close to m, the mean of the hypothetical random variable ξ, however large the sample size may be.

2.2b: The IQL:

As noted in connection with lemma 1.1, we always consider $L(w, \theta)$ to be a continuous function of w. Therefore the IQL $W_0(\theta)$ - or W_0 for brevity - exists uniquely and is given by

$$L(W_0, \theta) = \frac{1}{2} \tag{2.14}$$

The relationship between the IQL and the error- areas is formulated in the following theorem.

Theorem 2.2:-

For a given plan θ, the IQL W_0 satisfies

$$J(W_0, \theta) = \min_{w} J(w, \theta) \tag{2.15}$$

where $J(w, \theta)$ is the sum of two kinds of error- areas at w.

Proof:- If $J(w)$ is a minimum at $w = W$, then for an arbitrary $h > 0$,

$$J(W+h) - J(W) = \int_{W}^{W+h} \left\{ 1 - 2L(w) \right\} dF(w) \geq 0, \quad \text{and}$$

$$J(W) - J(W-h) = \int_{W-h}^{W} \left\{ 1 - 2L(w) \right\} dF(w) < 0 \tag{2.16}$$

Since $L(w)$ decreases as w increases,

$$1 - 2L(w) \lesseqgtr 0 \quad \text{according as} \quad w \lesseqgtr W_0 \tag{2.17}$$

It follows that W_0 satisfies the inequalities (2.16), irrespective of whether the prior distribution is continuous or discrete. Hence the theorem.

Thus the IQL emerges as the point at which the sum of two error-areas is a minimum with respect to w.

It is not surprising to find that the IQL which is independent of PC should admit an interpretation in terms of error-areas related to PC', because it is the point at which the conditional probability of acceptance (or rejection) given that the lot is of quality W_0 is 50 °/. . The unconditional probability (-density) will be different for different PCs.

When the prior distribution is discrete, W_0 is not the only point satisfying (2.15). Let the two consecutive mass points w_i and w_{i+1} be such that $w_i \leq W_0 < w_{i+1}$. Since $J(w) = J(w_i)$ for all w in the interval $w_i \leq w < w_{i+1}$, it follows that any value of w in the interval gives the minimum value of $J(w)$. If W_0 does not coincide with any of the mass points, i.e., w_i is strictly less than W_0, then the unconditional probability of acceptance at W_0 will be zero. To avoid this, one can define

$$W_0' = \sup_{w} \left\{ w: L(w) \geq \frac{1}{2} ; \quad f(w) > 0 \right\} \qquad (2.18)$$

W_0', of course, minimises $J(w)$, but $L(W_0')$ may not be equal to 50 °/. . In the case of a two point prior distribution with mass points at W_1 (AQL) and W_2 (LTPD), $W_1 < W_2$, any reasonable plan will have $L(W_1) > \frac{1}{2} > L(W_2)$. The above defi nition implies $W_0' = W_1$. W_1 being an AQL value, $L(W_1)$ will be much more than 50 °/° .

2.2c:- To summarize, the interpretations of the EQL and the IQL in terms of error- areas essentially depend on the continuity of the prior distribution. In this case, the con- cept behind the EQL is one of splitting the risk equally between the producer and the consumer. The total risk at the EQL will be different from the minimum risk possible. Howev the difference is small in large samples because of theorems 1.1, 1.2 (given in Section 1.3) and 2.4 (to be given in Section 2.4). The concept behind the IQL is one of minimizin the total risk. At the IQL, the risk of acceptance, as measu red by the error- area of the second kind, will be different from the risk of rejection, as measured by the error- area of the first kind. However, the difference is small in large samples (Theorems 1.1, 1.2 and 2.4).

2.2d:- It is sometimes thought that the concept behind the IQL is one of splitting the risk equally between the consume and the producer. In a certain sense, it is true; because, given that a lot is of quality better than W_0, it has more chance of being accepted than rejected and given that the lot quality is worse than W_0, it has more chance of being rejected than accepted whereas a lot of quality W_0 has as much chance of being accepted as rejected. When an IQL value is chosen so as to lie between an acceptable and a rejectable quality levels (AQL and LTPD), it gives a fairly good idea of the shape of the OC. The use of IQL as an indexing parameter to a system of sampling plans is then intuitively felt and pragmatically justified. (Hamacker (1954) and (1955); Hald (1960), (1964) and 1967c).

If the mean of the hypothetical random variable, m is used to index a system of sampling plans, it may prove helpful in selecting a suitable plan both when the prior distribution is continuous and also when it is discrete — the former because of the closeness of m to the EQL and the later because of its nearness to the IQL.

2.3 The unweighted error-areas

2.3a: The definition:-

The 'unweighted error-areas under the OC' or 'the error-areas under the unweighted OC' at the point $w = W$

are denoted by $D_1(W)$ and $D_2(W)$ and their sum by $D(W)$. They are defined by

$$D_1(W, \theta) = \int_0^W \left\{1 - L(w, \theta)\right\} dw \qquad (2.19$$

$$D_2(W, \theta) = \int_W^B L(w, \theta) dw$$

and $D(W, \theta) = D_1(W, \theta) + D_2(W, \theta)$

The area under the unweighted OC is given by

$$D_2(0, \theta) = \int_0^B L(w, \theta) dw \qquad (2.20$$

$D_1(W)$, $D_2(W)$ and $D_2(0)$ are thus obtained by formally subs tituting $F(w) = w$ in the respective formulae - (2.1), (2.2) and (2.4) - for $J_1(W)$, $J_2(W)$ and A_F.

If $0 \leq w \leq 1$, the area $D_2(0)$ and also the error-areas under the unweighted OC refer to a uniform prior dis tribution. If the range of w exceeds unity, they cannot b related to any particular c.d.f. and if w goes to infinity it cannot even be said that they exist and are finite. We will however assume that $D_2(0)$ exists and is finite. It i found that all OCs of practical interest do obey this assumption provided that the lot quality measurement is suitably chosen. (In the case of normal OCs, lot quality should be measured by proportion of defectives and not by the normal mean, Section 5.1). Consequently $D_2(w)$ exists and is finite for any value of w. It is clear that $D_1(w)$ is finite for finite values of w; but -- unlike $J_1(w)$ in (2.6) which can increase upto $1 - A_F$ only -- $D_1(w)$ can become infinite as w tends to infinity. However this is only an exception: all other properties of the weighted error-areas, as discussed and derived in the last two sections in relation to a uniform prior distribution, hold true for the unweighted error-areas also, irrespective of whether $F(w) = w$ can be interpreted as a c.d.f. or not. Note that, because of (2.4) and (2.7), the unweighted area $D_2(0)$ corresponds to A_F as well as to the EQL, W_{eF}.

2.3b: <u>The unweighted error-areas as moments of ξ</u>:-

If $wL(w) \to 0$ as $w \to B$ - a condition which we will assume - then the unweighted error-areas can be represented as suitable complete and incomplete moments of the hypothetical random variable ξ defined in (1.4).

Integrating by parts, the relations (2.19) yield

$$D_1(W) = W[1 - L(W)] - \int_0^W g(w)dw$$

and $$D_2(W) = -WL(W) + \int_W^B g(w)dw$$

where $g(\cdot)$ is the p.d.f. of ξ (cf. (1.6)). If $G(\cdot)$ is the c.d.f. of ξ, we may write (2.21)

<u>Lemma 2.1:</u>

$$D_1(W) = E_G\, \epsilon_W\, (W - \xi)$$

$$D_2(W) = E_G\, (1 - \epsilon_W)(\xi - W)$$

and hence

$$D(W) = E_G\, |\xi - W|$$

where

$$\epsilon_w = \begin{cases} 1 & \text{if } \xi \leq w \\ 0 & \text{if } \xi > w \end{cases} \qquad (2.22)$$

It should be noted that

$$E_G \epsilon_W k = k\left\{1 - L(w)\right\} \text{ and } E_G(1 - \epsilon_W)k = k\, L(w) \qquad (2.23)$$

where k is a constant with respect to ξ.

The area under the unweighted OC, $D_2(0)$ is seen to be the mean of ξ which has already been denoted by m, i.e.

$$D_2(0) = \int_0^B L(w)dw = m = E\xi \qquad (2.24)$$

Putting $W = m$ in (2.21), we get

$$\begin{aligned} D_1(m) &= E_G\ \epsilon_m(m - \xi) \\ &= E_G\ (1 - \epsilon_m)(\xi - m) = D_2(m) \end{aligned} \qquad (2.25)$$

Thus m turns out to be the 'point of equal error-areas' for the unweighted OC. Further the total error-area under the unweighted OC at the point of equal error-areas is given by the mean deviation about the mean of ξ.

$$D(m) = E_G\ | \xi - m | \qquad (2.26)$$

It is of interest to know that $D(m) = D$ has been used already in Theorem 1.2.

<u>2.3c:</u> We have the relation

$$D_1(W) - D_2(W) = W - m \qquad (2.27)$$

and also the theorems 2.1 and 2.2, i.e.,

$$D_1(m) = D_2(m) \tag{2.28}$$

$$= \min_{w} \max \left\{ D_1(w),\ D_2(w) \right\}. \tag{2.29}$$

and $$D(W_o) = \min_{w} \ D(w) \tag{2.30}$$

where W_o is the IQL i.e., $L(W_o) = 2^{-1}$.

<u>Lemma 2.2:</u>-

For any two points W_1 and W_2 in the interval $[0,B]$

(a) $(W_1-W_2)\left\{1-L(W_2)\right\} \leq D_1(W_1)-D_1(W_2) \leq (W_1-W_2)\left\{1-L(W_1)\right\}$

(b) $(W_2-W_1)\ L(W_2) \leq D_2(W_1)-D_2(W_2) \leq (W_2-W_1)L(W_1)$

and hence

(c) $(W_1-W_2)\left\{1-2L(W_2)\right\} \leq D(W_1)-D(W_2) \leq (W_1-W_2)\left\{1-2L(W_1)\right\}$

<u>Proof:</u> By definition (2.19),

$$D_1(W_1) - D_1(W_2) = \int_{W_2}^{W_1} \left\{1 - L(w)\right\} dw$$

$1-L(w)$ is an increasing function of w. When $W_2 < W_1$, the integral is positive and the maximum and the minimum values of $1-L(w)$ in the interval $W_2 \leq w \leq W_1$ are $1-L(W_1)$ and $1-L(W_2)$ respectively. When $W_1 < W_2$, the integral is negative and the maximum and the minimum values of $-\{1 - L(w)\}$ in the interval $W_1 \leq w \leq W_2$ are $-\{1 - L(W_1)\}$ and $-\{1-L(W_2)\}$ respectively. This proves part (a) of the lemma. Parts (b) and (c) are proved similarly. Hence the lemma.

2.3d: The relation between unweighted error-areas at m and W_0:-

The following theorem connects the unweighted error-areas at m with those at the IQL, W_0.

Theorem 2.3:

If the j-th cumulant of ξ, λ_j is of the order of n^{1-j}, $j \geq 2$, n being the total sample size, then (2.32)

(a) $D_1(m) = D_2(m) = 0(n^{-1/2})$

(b) $D_1(m) - D_1(W_0) = \frac{1}{12}\lambda_3 V^{-1} + 0(n^{-2})$

$D_2(m) - D_2(W_0) = -\frac{1}{12}\lambda_3 V^{-1} + 0(n^{-2})$

and $D(m) - D(W_0) = \frac{1}{36}\lambda_3^2 V^{-2} g(W_0) + 0(n^{-3})$

and (c) $D_1(W_0) - D_2(W_0) = -\frac{1}{6}\lambda_3 V^{-1} + 0(n^{-2})$

where m is the mean, V the variance, i.e., the second cumulant and $g(\cdot)$ the p.d.f. of the random variable ξ.

Proof:-

(a) From (2.25) and (2.26). $2D_1(m) = 2D_2(m)$ = the mea deviation about the mean of ξ and hence is less than or equal to $\sqrt{V}$. But V is assumed to be $O(n^{-1})$. This proves (a).

(b) Using the definitions (2.19),

$$D_1(m) - D_1(W_o) = \int_{W_o}^{m} \left\{ 1 - L(w) \right\} dw = \int_{W_o}^{m} G(w)dw \qquad (2.33$$

where $G(\cdot)$ is the c.d.f. of ξ (cf. (1.6)). Similarly

$$D_2(m) - D_2(W_o) = (W_o - m) + \int_{W_o}^{m} G(w)dw,$$

and

$$D(m) - D(W_o) = (W_o - m) + 2\int_{W_o}^{m} G(w)dw$$

Expanding $G(w)$ around W_o and integrating,

$$\int_{W_o}^{m} G(w)dw = \frac{m - W_o}{2} + \frac{(m - W_o)^2}{2!} g(W_o) + \ldots \qquad (2.34$$

if we note that $G(W_o) = 1 - L(W_o) = \frac{1}{2}$. Replacing $m - W_o$ the series (1.10) leads to (b) of the theorem.

(c) The relation (2.27) together with the series (1.10) proves (c).

This completes the proof of the theorem.

2.4 A robust property of the weighted error-areas

2.4a:- It was shown in Section 1.3 that under certain conditions on PC, $F(W_{eF})$ can be approximated by $F(m)$, m being the mean of the random variable ξ and W_{eF} the EQL. In the present section, similar results are derived to connect the error-areas under the weighted OC with those under the corresponding unweighted OC. It turns out that in large samples $J_F(W_{eF})/f(W_{eF})$ is fairly insensitive to small changes in the PC and hence may be approximated by $D(m)$, the mean deviation about the mean of ξ (Theorem 2.4).

2.4b: The fundamental lemma for weighted error-areas:-

Lemma 2.3:-

For any W in the interval $0 \leq W \leq B$

$$J_1(W) = E_G\, e_W \left\{ F(W) - F(\xi - 0) \right\} \qquad (2.35.1)$$

$$J_2(W) = E_G (1 - e_W) \left\{ F(\xi - 0) - F(W) \right\} \qquad (2.35.2)$$

and hence

$$J(W) = E_G \mid F(\xi - 0) - F(W) \mid \tag{2.35.3}$$

where $F(\cdot)$ is the c.d.f. of the prior distribution of the lot quality w, $G(\cdot)$ the c.d.f. of the random variable ξ defined in (1.4) and ϵ_W is given by (2.22) i.e., e_W is 1 or 0 according as $\xi \leq w$ or $> w$.

<u>Proof:</u> By (2.1) and (1.6),

$$J_1(W) = \int_{w \leq W} G(w)\,dF(w)$$

$$= \text{the unconditional probability,}$$

$$P\left\{ \xi \leq w \leq W \right\}$$

where ξ and w are considered as independent random varia bles. Hence

$$J_1(W) = E_G\, \epsilon_W\, P\left\{ \xi \leq w \leq W \mid \xi \right\}$$

which leads to the desired relation for $J_1(W)$. The relatio for $J_2(W)$, namely (2.35.2), is proved similarly. The relation (2.35.3) is obtained by adding (2.35.1) and (2.35.2). This proves the lemma.

It may be noted that by putting $W = 0$ in (2.35.2), the fundamental lemma 1.1 given in Section 1.3b is obtained and that the above lemma 2.3 reduces to the lemma 2.1 if we formally substitute $F(w) = w$. $J_1 = D_1$; etc.

2.4c: Miscellaneous results for the weighted error- areas:

Lemma 2.4:

For any two points W_1 and W_2 in the interval [0, B]

(a) $$\left\{F(W_1) - F(W_2)\right\}\left\{1 - L(W_2)\right\} \leq J_1(W_1) - J_1(W_2) \leq \left\{F(W_1) - F(W_2)\right\}\left\{1 - L(W_1)\right\}$$

(b) $$\left\{F(W_2) - F(W_1)\right\} L(W_2) \leq J_2(W_1) - J_2(W_2) \leq \left\{F(W_2) - F(W_1)\right\} L(W_1)$$

and hence

(c) $$\left\{F(W_1) - F(W_2)\right\}\left\{1 - 2L(W_2)\right\} \leq J(W_1) - J(W_2) \leq \left\{F(W_1) - F(W_2)\right\}\left\{1 - 2L(W_1)\right\}$$

Proof: By the definition (2.1),

$$J_1(W_1) - J_1(W_2) = \int_{W_2}^{W_1} \left\{1 - L(w)\right\} dF(w)$$

and

$$J_2(W_1) - J_2(W_2) = \int_{W_1}^{W_2} L(w)dF(w)$$

Both $F(w)$ and $1-L(w)$ are increasing functions of w. The lemma follows by replacing the integrands by their maximum and minimum values.

The following lemma connects the weighted error-areas at $w = W$ with the corresponding unweighted error-areas.

<u>Lemma 2.5:-</u>

If $F(w)$ has well behaved derivatives of higher orders,

$$J_1(W) - D_1(W)f(W) = - S_1(W)$$

and (2.37)

$$J_2(W) - D_2(W)f(W) = S_2(W)$$

where

$$S_1(W) = E_G \epsilon_W \left[\frac{(\xi - W)^2}{2!} F_2(W) + \frac{(\xi - W)^3}{3!} F_3(W) + \cdots \right],$$

$$S_2(W) = E_G(1 - \epsilon_W) \left[\frac{(\xi - W)^2}{2!} F_2(W) + \frac{(\xi - W)^3}{3!} F_3(W) + \cdots \right],$$

(2.38)

$G(\cdot)$ is the c.d.f. of ξ, $f(\cdot)$ the process density and $F_j(w)$ is the j-th derivative of $F(w)$. Further, ϵ_W is 1 or 0 according as $\xi \leq w$ or $> w$.

Proof:- The lemma is proved by expanding $F(\xi)$ in (2.35) around the point W in a Taylor series.

In order to connect the weighted error-areas at $w = W_1$ with the unweighted error-areas at a different point $w = W_2$ we may proceed as follows:

$$J_1(W_1) - D_1(W_2)f(W_1) = J_1(W_1) - D_1(W_1)f(W_1) + \left\{D_1(W_1) - D_1(W_2)\right\} f(W_1),$$

or

$$J_1(W_1) - D_1(W_2)f(W_2) = J_1(W_2) - D_1(W_2)f(W_2) + J_1(W_1) - J_1(W_2)$$

Using lemmas 2.5, 2.4 and 2.2, two sets of inequalities are obtained for J_1 and similarly for J_2 and J. Hence,

Lemma 2.6:- For J_1 we have

$$f(W_1)(W_1 - W_2)\left\{1 - L(W_2)\right\} - S_1(W_1) \leq J_1(W_1) - D_1(W_2)f(W_1) \leq (W_1)(W_1 - W_2)\left\{1 - L(W_1)\right\} - S_1(W_1)$$

and

$$\left\{F(W_1) - F(W_2)\right\}\left\{1 - L(W_2)\right\} - S_1(W_2) \leq J_1(W_1) - D_1(W_2)f(W_2) \leq \left\{F(W_1) - F(W_2)\right\}\left\{1 - L(W_1)\right\} - S_1(W_2)$$

Similarly for J_2, we have (2.40)

$$f(W_1)(W_2-W_1)L(W_2) + S_2(W_1) \leq J_2(W_1) - D_2(W_2)f(W_1) \leq f(W_1)(w_2-W_1)L(W_1) + S_2(W_1)$$

and

$$\left\{F(W_2) - F(W_1)\right\} L(W_2) + S_2(W_2) \leq J_2(W_1) - D_2(W_2)f(W_2) \leq \left\{F(W_2)-F(W_1)\right\} L(W_1) + S_2(W_2).$$

The inequalities for the total error-area J are obtained by adding the corresponding inequalities for J_1 and J_2.

(2.41)

If the c.d.f. $F(w)$ has well behaved derivatives of higher orders and if the j-th central moment of ξ is $O(n^{-j/2})$, n being the total sample size - a condition which is obeyed by most of the OCs arising in statistical practice - then the weighted error-areas at any given point W may be calculated to a desired degree of accuracy by expanding $F(\xi)$ in (2.35) around the point $\xi = m$ in a Taylor series:

Lemma 2.7:-

$$J_1(W) = \{F(W) - F(m)\}\{1 - L(W)\} - E_G\, \epsilon_W \left[(\xi - m)F_1(m) + \frac{(\xi - m)^2}{2!} F_2(m) + \dots \right] \tag{2.42.1}$$

and

$$J_2(W) = \{F(m) - F(W)\} L(W) + E_G(1 - \epsilon_W) \left[(\xi - m)F_1(m) + \frac{(\xi - m)^2}{2!} F_2(m) + \dots \right] \tag{2.42.2}$$

where ϵ_W, $F_j(w)$ etc., are as defined in lemma 2.5.

It may be noted that the series (1.20) for the expansion of $F(W_{eF})$ given in Section 1.4 is obtained by putting $W = 0$ i (2.42.2).

2.4d: The case of risk functions:

Lemma 2.3 can be generalized to risk functions define by

$$r_{1F}(W, \theta) = \int_{w \leq W} \{1 - L(w,\theta)\} \lambda'(w) f(w)\, dw$$

and

$$r_{2F}(W, \theta) = \int_{w > W} L(w, \theta) \lambda''(w) f(w)\, dw$$

where the functions $\lambda'(w)$ and $\lambda''(w)$ are non-negative, bounded and integrable.

$\lambda'(w)$ and $\lambda''(w)$ may be called 'loss-functions' if $\lambda'(w)$ decreases in the interval $0 \leq w \leq w_0$ and $\lambda''(w)$ increases in the interval $w_0 \leq w \leq B$ where w_0 is an economic break-even quality level.

Let (2.44)

$$\int_{w \leq W} \lambda'(w) f(w)\,dw = F'(w)$$

and

$$\int_{w \leq W} \lambda''(w) f(w)\,dw = F''(w)$$

Writing

$$r_{1F}(W, \theta) = \int_{w \leq W} \int_{x \leq W} g(x)\,\lambda\,(w) f(w)\,dx\,dw$$

where $g(\cdot)$ is the p.d.f. of the random variable ξ and changing the order of integration, we get as in (2.35.1) of lemma 2.3

$$r_{1F}(W) = E_G\, e_W \left\{ F'(W) - F'(\xi) \right\}$$

and similarly (2.45)

$$r_{2F}(W) = E_G (1 - e_W) \left\{ F''(\xi) - F''(W) \right\}$$

If the functions $\lambda'(w)$ and $\lambda''(w)$ and also the process density $f(w)$ have higher order derivatives, $F'(\xi)$ and $F''(\xi)$ in (2.45) can be expanded in Taylor series to obtain results similar to Lemmas 2.5 and 2.7. For instance

$$r_{1F}(W) = D_1(W)\lambda'(W)f(W) - E_G\, e_W \left[\frac{(\xi - W)^2}{2!} F_2'(W) + \frac{(\xi - W)^3}{3!} F_3'(W) + \ldots \right]$$

and

$$r_{2F}(W) = D_2(W)\lambda''(W)f(W) + E_G(1 - e_W) \left[\frac{(\xi - W)^2}{2!} F_2''(W) + \frac{(\xi - W)^3}{3!} F_3''(W) + \ldots \right] \tag{2.46}$$

where $F_j'(w)$ and $F_j''(w)$ are the j-th derivatives of $F'(w)$ and $F''(w)$ respectively.

2.4e: The robust property of the weighted error-areas:-

We will now collect all of our assumptions and present the final result in the following theorem.

Theorem 2.4:-

Given an OC with the representation

$$L(w, \theta) = P\left\{\xi_\theta > w, \text{ given } w\right\}$$

assume that

(i) the unweighted error-areas are finite for finite values of w;

(ii) the j-th cumulant of ξ_θ is of the order of n^{1-j} $j \geq 2$, n being the total sample size

(iii) the c.d.f. $F(w)$ has density $f(w) = F_1(w)$ with a continuous bounded derivative $F_2(w)$; and that

(iv) the first order derivatives of $\lambda'(w)$ and $\lambda''(w)$ are continuous and bounded.

(a) Under assumptions (i) - (iii),

$$J_{iF}(W_1) = D_i(W_2)f(W_3) + 0(n^{-1}), \quad i = 1,2, \qquad (2.47)$$

where W_3 is either W_1 or W_2 while W_1 and W_2 are any two of the three points W_{eF} (the EQL), W_o(the IQL) and m (the mean of ξ_θ).

(b) If, in addition to (i)-(iii), $F(w)$ has higher order derivatives,

$$J_1(m) = D_1(m)f(m) - E_G\, \epsilon_m \left[\frac{(\xi - m)^2}{2!} F_2(m) + \frac{(\xi - m)^3}{3!} F_3(m) + \ldots\right] \qquad (2.48)$$

and

$$J_2(m) = D_2(m)f(m) + E_G(1 - \epsilon_m) \left[\frac{(\xi - m)^2}{2!} F_2(m) + \frac{(\xi - m)^3}{3!} F_3(m) + \ldots\right] \qquad (2.49)$$

(c) Under assumptions (i) - (iv),

$$r_{1F}(m) = D_1(m)\, \lambda'(m)f(m) + 0(n^{-1})$$

and

$$r_{2F}(m) = D_2(m)\, \lambda''(m)f(m) + 0(n^{-1})$$

Proof:

(a) If W_1 and W_2 are any two of the three points W_{eF}, W_0 and m, the terms on both sides of the inequalities (2.39) and (2.40) of lemma 2.6 are atmost of the order of η because of theorem 1.1 and the relations (1.10) and (1.11). This proves (a).

(b) follows from lemma 2.7 and

(c) is a consequence of (2.46).

Hence the theorem.

It may be remarked that for the convergence of $J_{1F}(W_{eF})$ to $D_1(m)f(m)$ at a specified value $m = m^*$, the continuity of $F_1(w)$ at $w = m^*$ alone suffices. Similar remarks hold for the convergence of $r_{1F}(m)$ to $D_1(m)\ \lambda'(m)f($ and of $r_{2F}(m)$ to $D_2(m)\ \lambda'(m)f(m)$.

2.5 Optimality conditions

In Section 2.2, the error-areas were considered as fun tions of w for a given plan θ and it was shown how they are related to the EQL and the IQL of that plan. Such a study gives a valuable insight into the structure of OC curves. What may be more important from a practical point of view is to find a plan θ so as to satisfy certain optimum properties given the value of the economic break - even quality level or the point of demarcation between good and bad lots. It happens that certain optimality conditions in terms of error areas directly lead to an EQL relationship between the elements of the plan θ while some others lead to interesting interpretations.

To fix ideas, single sampling plans with two elements are considered. $\theta = (n, c)$ where n is the sample size and c an acceptance number, that is, the lot is accepted when and only when the observed value of the decision variable η_n does not exceed c. In other words,

$$L(w, n, c) = P\left\{\eta_n \leq c, \text{ given } w\right\} \tag{2.50}$$

Any reasonable OC should satisfy the following inequalities: (2.51)

$$L(w, n, c') \gtreqless L(w,n,c) \quad \text{according as} \quad c' \gtreqless c$$

$$L(w, n', c) \gtreqless L(w,n,c) \quad \text{according as} \quad n' \lesseqgtr n$$

and, of course, those mentioned in (2.6) i.e.,

$$L(w', n, c) \gtreqless L(w,n,c) \quad \text{according as} \quad w' \lesseqgtr w$$

The same inequalities are valid for $J_2(w,n,c)$. The inequalities are reversed for $J_1(w,n,c)$.

$L(w,n,c)$ is always considered a continuous function o: w. It may or may not be continuous with respect to c depe: ding on whether η_n is a continuous or a discrete random variable. In the later case, we shall suppose without loss d generality that η_n takes the values c_0, c_1, c_2 ... arrang in increasing order of magnitude i.e., $c_0 < c_1 < c_2 \ldots$ so that $L(w,n,c_1) < L(w,n,c_2) < \ldots$.

The possibility of interpreting the error-areas as measures of producer's and consumer's risks and the total er: area as a measure of the decision loss due to sampling together with the fact that the two kinds of error-areas behave in opposite ways with respect to each of their arguments suggest the desirability of searching for a plan which will control both the risks simultaneously. In this context the following optimality conditions are of interest. (2.52)

(A') The two error-areas at the given point $w = W$ are equal.

(A) The bigger of the two error-areas at the given point W is a minimum with respect to the elements of the plan θ.

(B) The total error-area at W is a minimum with respect to θ.

Neither of the requirements (A'), (A) or (B) can determine the plan uniquely. Therefore the problem (in each case) can be considered as one of seeking a relationship between the elements of the plan θ.

The above optimality conditions are dealt with in the next two sections.

2.6 Optimality condition (A)

The problem of determining a relationship between the elements of a plan $\theta = (\theta_1, \theta_2, \dots \theta_k)$ under the conditions (A') or (A) of (2.52) can be formulated as follows.

Given W and the values of all the elements except one, θ_k (say), solve for θ_k in each of the following two cases: (2.53)

(A') $J_1(W, \theta) = J_2(W, \theta)$

(A) $\max [J_1(W, \theta), J_2(W, \theta)]$

$$\leq \max_{\theta'_k} [J_1(W, \theta'_{(k)}), J_2(W, \theta'_{(k)})]$$

where $\theta'_{(r)} = (\theta_1, \dots, \theta_{r-1}, \theta'_r, \theta_r \dots \theta_k)$

Slightly more general conditions (A'_∂) and (A_∂) are obtained by taking ∂J_2, $\partial > 0$, instead of J_2 in (2.53) where ∂ is any given positive number.

The solution is given in Theorem 2.5 below.

Theorem 2.5:-

For both weighted and unweighted error-areas, provided that the OC, $L(w, \theta)$ is monotonic with respect to w and the element θ_k,

(a) conditions (A'_∂) and A_∂ are equivalent,

(b) if $\partial = 1$ and if a solution exists, the relationship between the elements of the plan θ is an EQL relationship with the given value W as the EQL; further

(c) under the conditions of theorems 2.3 and 2.4, the relationship in the case of weighted error-areas is asymptotically equivalent to the corresponding relationship for the unweighted error-areas; and consequently each of the error-areas at W can be made as small as we please by taking the total sample size sufficiently large.

(d) if $\partial > 0$ and if a solution exists, then

$$\min_{w} \max \left\{ J_1(w, \theta), \ \partial\, J_2(w, \theta) \right\}$$
$$= J_1(W, \theta) = \partial\, J_2(W, \theta)$$
$$= \min_{\theta_k'} \ \max \left\{ J_1(W, \theta_{(k)}'), \ \partial J_2(W, \theta_{(k)}') \right\} \quad (2.54)$$

It may be remarked that while the problems treated in Theorems 2.1 and 2.5(b) are conceptually different, they give rise to the same relationship between the elements of the plan θ. However, the above theorem is not trivial and requires a proof.

Proof:

The theorem will be proved first for the single sampling plans (n, c) and $\partial = 1$. The generalization to other cases are straightforward and offers no difficulty.

Condition (A'):-

If there exists a plan satisfying (A'), it should have W as its EQL. This is clear from (2.8). It may be noted that such exact solutions always exist either when the decision variable η_n is continuous or when randomized plans are admitted; otherwise there may or may not exist any exact solutions and one may have to be satisfied with approximate solutions.

Condition (A):

Suppose there exists a plan (n, c) for which W is the EQL. Keeping n fixed, let c be changed to c' so that the plan (n, c') has its own EQL at W', say.

$$F(W) = J_2(o, n, c) \quad \text{and} \quad F(W') = J_2(o, n, c')$$

From (2.51), it follows that $W \leq W'$ when $c \leq c'$ and $W \geq W'$ when $c \geq c'$. Applying (2.9) to the plan (n, c'),

$$J_1(W, n, c') \begin{cases} \leq J_2(W,n,c') & \text{if } c \leq c' \\ \geq J_2(W,n,c') & \text{if } c \geq c' \end{cases}$$

Using (2.51),

$$\max\, [J_1(W,n,c'),\ J_2(W,n,c')]$$

$$= \begin{cases} J_2(W,n,c') \geq J_2(W,n,c) & \text{if } c \leq c' \\ J_1(W,n,c') \geq J_1(W,n,c) & \text{if } c \leq c' \end{cases} \qquad (2.55$$

Since $J_1(W,n,c) = J_2(W,n,c)$, the minimum of the left side of the above relation is attained when $c = c'$.

Similarly, keeping c fixed, if n is varied to n', we get

$$\max[\, J_1(W,n',c),\ J_2(W,n',c)\,] \geq J_1(W,n,c), \tag{2.56}$$

Therefore, the minimum is again attained when $n = n'$.

Since we are only seeking a relationship between n and c, the question of varying both n and c simultaneously does not arise.

If η_n is discrete and if randomized plans are not admitted, there may or may not exist any plan for which W is the EQL. In the later case however, for a fixed n, two plans $(n,\ c_e)$ and (n, c_{e+1}), $c_e < c_{e+1}$, with their respective EQLs at W_e and W_{e+1} can be found such that $W_e < W < W_{e+1}$. It can now be shown that

$$\max[\, J_1(W,n,c'), J_2(W,n,c')\,] \geq \begin{cases} J_2(W,n,c_e) & \text{if } c' \leq c_e \\ J_2(W,n,c_{e+1}) & \text{if } c' \geq c_{e+1} \end{cases} \tag{2.57}$$

If $J_1(W,n,c_{e+1}) \neq J_2(W,n,c_e)$, the optimum plan is (n,c_e) or (n,c_{e+1}) according as $J_2(W,n,c_e) \lessgtr J_1(W,n,c_{e+1})$. If $J_1(W,n,c_{e+1}) = J_2(W,n,c_e)$, one cannot choose between the two plans. Similar arguments apply when n is varied, keeping c fixed.

Subject to the above observations, we have thus proved theorem 2.5(a) and 2.5(b) for weighted error-areas. It is easy to verify (and therefore the proof is omitted) that these results are true for unweighted error-areas also.

(c) is a direct consequence of Theorems 2.3 and 2.4. It implies that if $n \to \infty$ and c varies such that $m(\theta)$ — the mean of the random variable ξ_θ — is kept fixed at the given value W, then each of the weighted error-areas $J_1(W,\theta)$ and $J_2(W,\theta)$ approaches $\frac{1}{2} D(W,\theta)$ with $O(n^{-1})$ whereas $D(W,\theta)$ itself tends to zero with $O(n^{-1/2})$ (2.58)

When the relationship $m(\theta) = W$ is violated, one of the error-areas at W may become small while the other large as $n \to \infty$.

Conditions (A'_∂) and (A_∂):-

For a given plan θ, let $W_{e\partial}(\theta)$ — or $W_{e\partial}$ for brevity — be defined by

$$J_1(W_{e\partial}, \theta) = \partial J (W_{e\partial}, \theta), \qquad \partial > 0 \tag{2.59}$$

$W_{e\partial}$ reduces to the EQL, W_e when $\partial = 1$. Because $J_1(w)$ is a non-decreasing and $J_2(w)$ a non-increasing function of w, we have for any W'

$$J_1(W',\theta) \lesseqgtr \partial\, J_2(W',\theta) \quad \text{according as} \quad W' \lesseqgtr W_{e\partial}(\theta) \qquad (2.60)$$

and hence

$$J_1(W_{e\partial}, \theta) = \partial\, J_2(W_{e\partial}, \theta)$$

$$= \min_{w} \max [\, J_1(w,\theta),\ \partial J_2(w,\theta)\,] \qquad (2.61)$$

It is easy to see that

$$W_{e\partial}(n,c) \lessgtr W_{e\partial}(n,c') \quad \text{according as} \quad c \lessgtr c'$$

and (2.62)

$$W_{e\partial}(n,c) \lessgtr W_{e\partial}(n',c) \quad \text{according as} \quad n \lessgtr n'$$

Using these and proceeding as in the case of $\partial = 1$, the part (d) of the theorem is obtained.

Finally we note that the proof so far has depended only on the monotonicity of $L(w,n,c)$ and Theorems 2.3 and 2.4. The later are true for any (univariate) plan. It follows that the above results are valid for any univariate plan θ provided only that the OC $L(w,\theta)$ is monotonic with respect to w and the element θ_k whose value is to be determined, given the values of other elements - a condition assumed by the theorem.

This completes the proof of Theorem 2.5.

2.7 Optimality condition (B)

2.7a: A mathematical formulation of the problem (B) of (2.52) for the single sampling plans (n,c) can be given as follows: (2.63)

given W and n determine c so as to satisfy

$$(B) \quad J(W,n,c) = \min_{c'} J(w,n,c')$$

or more generally,

$$(B_\partial) \quad J_1(W,n,c) + \partial J_2(W,n,c)$$
$$= \min_{c'} [\ J_1(W,n,c') + \partial J_2'(W,n,c')\ ]$$

Prompted by the identity of the solutions in Theorems 2.1 and 2.5(b) and also by the similarity of the problem (B) with that of Theorem 2.2, one may be tempted to believe that the condition (B) leads to an IQL relationship (i.e., $\min_W J(w,n,c) = J(W,n,c) = \min_{c'} J(W,n,c')$). Such, in general, is not the case, as shown in Theorem 2.6. In fact - unlike (2.15) of Theorem 2.2 - the solution of (B) depends on the prior distribution.

2.7b:- The following notations are required. If the decision variable η_n is discrete, let

$$b(w,n,c_r) = P\left\{\eta_n = c_r, \text{ given } \right\} = L(w,n,c_r) - L(w,n,c_{r-1}) \tag{2.64}$$

$b(w,n,c_r)$ is the conditional probability of the inspector observing the value c_r in his sample, given w. Let $\bar{b}_b(W,n,c_r)$ denote the average probability of observing c_r for lots of quality worse than W and $\bar{b}_g(w,n,c_r)$ for lots of quality better than or equal to W. Finally let $\bar{b}(n,c_r)$ be the average(unconditional) probability of observing c_r. (2.65)

$$\bar{b}_b(W,n,c_r) = P\left\{\eta_n = c_r, \ w > W\right\} = \int_{w>W} b(w,n,c_r)\,dF(w)$$

$$\bar{b}_g(W,n,c_r) = P\left\{\eta_n = c_r, w \leq W\right\} = \int_{w\leq W} b(w,n,c_r)\,dF(w)$$

and

$$\bar{b}(n,c_r) = \bar{b}_g(W,n,c_r) + \bar{b}_b(W,n,c_r) = \int b(w,n,c_r)\,dF(w)$$

If η_n is continuous, the above expressions are interpreted in terms of densities. It may be noted that no assumptions have been made about the continuity or otherwise of the prior distribution, $F(w)$.

2.7c:- The solution to the problems (B) and (B_∂) is contained in

Theorem 2.6:-

Let (n, c) be the plan satisfying (B_∂).

(a) If η_n is discrete, then a necessary relationship between n and c is given by (2.66

$$\bar{b}_g(W,n,c_{i+1}) \leq (1+\partial)^{-1}\,\bar{b}(n,c_{i+1}) \leq \partial \bar{b}_b(W,n,c_{i+1})$$

and

$$\bar{b}_g(W,n,c_i) > (1+\partial)^{-1}\,\bar{b}(n,c_i) > \partial\,\bar{b}_b(W,n,c_i)$$

where c_i and c_{i+1} are the two consecutive mass points of η_n such that $c_i = c < c_{i+1}$ and b_g etc., are as defined in (2.65)

(b) If η_n is continuous, then the relationship is given by

$$b_g(W,n,c) = (1+\partial)^{-1}\,b(n,c) = \partial\,b_b(W,n,c) \qquad (2.67$$

where $\bar{b}_g$ etc., are the densities corresponding to those defined in (2.65)

(c) The results (a) and (b) are true for unweighted error-areas also.

Proof:

Write for the sake of convenience

$$H(W,n,c) = J_1(W,n,c) + \partial J_2(W,n,c)$$

(a) Suppose η_n is discrete and that randomized plans are not admitted. For a given value of n and W, let $c_i = c_i(n,W)$ be chosen so as to satisfy (B_∂). Then, for $r = 1,2,\ldots$ (2.68)

$$H(W,n,c_{i+r}) - H(W,n,c_i) \geq 0$$

and

$$H(W,n,c_i) - H(W,n,c_{i-r}) < 0$$

Using the definitions (2.1) and (2.2) and also (2.6), these inequalities take the form (2.69)

$$\int_{w\leq W}\left\{L(w,n,c_{i+r}) - L(w,n,c_i)\right\} dF(w) \leq \partial \int_{w>W}\left\{L(w,n,c_{i+r}) - L(w,n,c_i)\right\} dF(w)$$

and

$$\int_{w\leq W}\left\{L(w,n,c_i) - L(w,n,c_{i-r})\right\} dF(w) > \int_{w>W}\left\{L(w,n,c_i) - L(w,n,c_{i-r})\right\} dF(w)$$

Applying (2.64) and (2.65), we obtain, for $r = 1,2,\ldots$

$$\sum_{s=1}^{r} \bar{b}_g(W,n,c_{i+s}) \leq \partial \sum_{s=1}^{r} \bar{b}_b(W,n,c_{i+s})$$

and (2.70)

$$\sum_{s=0}^{r-1} \bar{b}_g(W,n,c_{i-s}) > \partial \sum_{s=0}^{r-1} \bar{b}_b(W,n,c_{i-s})$$

(2.70) provide a set of necessary and sufficient conditions for c_i to be an optimum value. Putting $r=1$, a set of necessary conditions as given in the theorems are obtained. They are given in that form (i.e., (2.66)) because it is easier to work with them and also because in some cases of interest (i.e., the cases where $H(c)$ has only one minimum with respect to c), they imply (2.70) and thus become sufficient conditions as well. (Sections 3.8c and 4.8c).

Suppose η_n is discrete and that randomization with respect to c is allowed. For a fixed value of n and W let c be chosen so as to satisfy (B_∂). Let c_i and c_{i+1} be two consecutive mass points of η_n such that $c_i < c < c_{i+}$ c can then be represented as

$$c = a\, c_i + (1-a)\, c_{i+1}, \quad 0 < a < 1.$$

Write

$$H'(c) = aH(W,n,c_i) + (1-a)H(W,n,c_{i+1})$$

Note that $H'(c)$ is the risk function corresponding to the composite OC, $L(w,n,c;\ a)$ defined in (1.29) i.e., to the procedure of selecting the plan (n,c_i) with probability a and the plan (n,c_{i+1}) with probability $1-a$.

c must satisfy the inequality

$$H'(c') - H'(c) \geq 0 \quad \text{for any } c' \tag{2.71}$$

Since $H'(c')$ always lies between $H(W,n,c_r)$ and $H(W,n,c_{r+1})$, $c_r \leq c' \leq c_{r+1}$, it is enough if c satisfies (2.70) and in addition (2.71) with c' in the interval (c_i,c_{i+1}). Let c' be given by

$$c' = a'c_i + (1-a')c_{i+1}, \quad 0 \leq a' \leq 1$$

(2.71) reduces to

$$(a' - a)\ [H(W,n,c_{i+1}) - H(W,n,c_i)\ ] \geq 0$$

By taking c' to be less or greater than c, $a'-a$ can be made positive or negative. It follows that the above inequality is satisfied if and only if

$$H(W,n,c_i) = H(W,n,c_{i+1}) \quad \text{and consequently}$$
$$= H(W,n,c')$$

for any c' in the interval $c_i \leq c' \leq c_{i+1}$.

If $H(W,n,c_r) \neq H(W,n,c_{r+1})$ for any r, the optimum value of c coincides with one of the mass points of η_n. If the equality holds for $r = i$, say, the optimum value can be c_i, c_{i+1} or any value in between. In either case it shou satisfy (2.66) of the theorem.

This proves part (a) of the theorem.

(b) If η_n is continuous, keeping n fixed and equating to zero the derivative of $H(W,n,c)$ with respect to c, w get

$$\int_{w \leq W} \frac{d}{dc} L(w,n,c)\,dF(w) = a \int_{w > W} \frac{d}{dc} L(w,n,c)\,dF(w)$$

Identifying the integrand with the conditional density of η_n at c and using the notations in (2.64) and (2.65), we arrive at part (b) of the theorem.

(c) It is easy to see that the above results are true for unweighted error-areas also. (Proof is omitted). However b_g etc., may not admit interpretations in terms of probabiliti they should be understood as short hand symbols for integrals obtained by replacing $F(w)$ by w in (2.65):

$$\bar{b}_g(W,n,c) = \int_{w \leq W} b(w,n,c)\,dw \;; \quad \text{etc.},$$

This completes the proof of Theorem 2.6.

2.7d: Remarks on Theorem 2.6:-

(1) Suppose $\partial = 1$, η_n is discrete and c_i is the optimum value. $J(W,n,c)$ equals $1-F(W)$ and $F(W)$ respectively when c is taken to be the highest and lowest mass points of η_n. From (2.68),

$$0 \leq J(W,n,c_i) \leq \min[\,1-F(W),\ F(W)\,] \tag{2.72}$$

This is as it should be, because W being the break - even value, 0, $1-F(W)$ and $F(W)$ are the decision losses corresponding to the three cases without sampling i.e., the cases where (i) all lots are classified correctly, (ii) all lots are accepted, and (iii) all lots are rejected.

(2) An interpretation of condition (B):- When $\partial = 1$, using (2.65), the conditions (2.70) can be written as

$$P\left\{w \leq W,\ c_{i+r} \geq \eta_n \geq c_{i+1}\right\}$$

$$\leq P\left\{w > W,\ c_{i+r} \geq \eta_n \geq c_{i+1}\right\}$$

and (2.7

$$P\left\{w \leq W,\ c_i \geq \eta_n \geq c_{i-r+1}\right\}$$

$$> P\left\{w > W,\ c_i \geq \eta_n \geq c_{i-r+1}\right\}$$

for $r = 1, 2, \ldots$. In particular, the conditions (2.66) of the theorem become

$$P\left\{w \leq W,\ \eta_n = c_{i+1}\right\} \leq P\left\{w > W,\ \eta_n = c_{i+1}\right\}$$

and (2.74

$$P\left\{w \leq W,\ \eta_n = c_i\right\} > P\left\{w > W,\ \eta_n = c_i\right\}$$

W is the value that demarcates the good lots from bad whereas the acceptance number c_i is the value that demarcate the acceptability or otherwise of a lot. A lot which results in the observation of the value c_i in the sample (and hence gets accepted) can give a feeling of the 'worst' lot that is accepted; similarly a lot which results in the observation of c_{i+1} in the sample (and hence gets rejected) can give a feelin of the 'best' lot that is rejected. Relations (2.73) imply (via Bayes' theorem?) that it is more likely that the accepted lots are good and the rejected bad. The relations (2.74) tell us that it is also more likely that the 'worst' among the accepted is actually a good lot and that the 'best' among the rejected is really a bad one.

(3) <u>The case of a 'two-point' prior distribution</u>:- If the prior distribution of w is discrete and w takes two values W_1 and W_2 with probabilities f_1 and $1-f_1$ respectively, then for a discrete η_n, the conditions (2.66) of the theorem reduce to

$$f_1 b(W_1,n,c_{i+1}) \leq \partial(1-f_1)b(W_2,n,c_{i+1})$$

and

$$f_1 b(W_1,n,c_i) > \partial(1-f_1)b(W_2,n,c_i)$$

provided, of course, $W_1 < W < W_2$. This is a generalization of the result given by Hald (1967c) for binomial OCs.

(4) <u>The case of a hypergeometric OC</u>:- In the case of a single sampling hypergeometric OC, let w be the proportion of defectives in the submitted lot of size N. For a given lot size, the plan is uniquely determined by the two elements n and c. $b(w,n,c)$ and also the OC $L(w,n,c)$ depend on N. In fact, $L(w,n,c)$ is well defined only for the $N+1$ values of w, $0, N^{-1}, 2N^{-1}, \ldots 1$. At other values of w, it may be left undefined or it may be formally defined as a hypergeometric

function. However, this does not effect the results given in the theorem. If w takes the value rN^{-1} with probability f_r, $r = 0,1,\ldots N$, the optimum value of c, given W, N and n should satisfy (2.66) i.e.,

$$\sum_{r=0}^{[NW]} b(rN^{-1}, n, c+1)\, f_r \leq \partial \sum_{r=[NW]+1}^{N} b(rN^{-1}, n, c+1) f_r$$

and (2.76)

$$\sum_{r=0}^{[NW]} b(rN^{-1}, n, c) f_r > \partial \sum_{r=[NW]+1}^{N} b(rN^{-1}, n, c) f_r$$

(5) The case of a multiple sampling OC:- A necessary relationship between the elements of a multistage sampling plan satisfying the condition (B_∂) is still given by Theorem 2.6 provided that c is taken as the acceptance number at the ultimate stage of sampling and η_n the conditional decision variable given that no decision as to the final disposal of the lot has been taken in the previous stages.

(6) In some cases of interest, the condition (B) in the case of unweighted error-areas gives rise to plans for which the given point W is the IQL. It then follows from lemma 2 that subject to the assumptions of Theorem 2.4, the relationship between the elements of the plan obeying (B) in the case of weighted error-areas is asymptotically an IQL (and hence

an EQL) relationship.

2.7e: The following theorem deals with the asymptotic properties and is valid for any plan.

Theorem 2.7:-

Subject to the assumptions of Theorem 2.4, suppose that W and the values of all the elements except one are given.

(a) If $J_1(W, \theta) + \partial J_2(W,\theta)$, $\partial > 0$ is a minimum for the plan θ, then

$$L(W,\theta) \to (1+\partial)^{-1}, \text{ as } n \to \infty \qquad (2.77)$$

In particular, when $\partial = 1$, the relationship between the elements is asymptotically an IQL (or EQL) relationship.

(b) Let $r_{1F}(W,\theta)$ and $r_{2F}(W,\theta)$ be defined by (2.43)

If the risk function $r_{1F}(W,\theta) + r_{2F}(W,\theta)$

is a minimum for a plan θ, then also the relationship between the elements of the plan θ is asymptotically an IQL relationship.

Proof: First (2.77) is proved in the case of unweighted error-areas. The rest of the theorem follows from Theorem 2.4

The mean m and the variance V of the random variable ξ depend on the elements θ. Alternatively the risk function

$$H = D_1(W, \theta) + D_2(W, \theta)$$

can be considered as a function of m, V and possibly some other independent parameters. Keeping all other parameters fixed, we try to find the optimum value of m by equating to zero the derivative of H with respect to m.

$$\int_{w \leq W} \frac{d}{dm} \left\{ 1 - L\,(w,m) \right\} dw + \partial \int_{w > W} \frac{d}{dm} L(w,m) dw = 0 \qquad (2.78)$$

$1-L(w)$ is the c.d.f. of the random variable ξ. Under the given assumptions, it can be approximated by the c.d.f. of a normal distribution with mean m and variance V. In other words, when n is large, it depends on m only through $(w-m)V^{-1/2}$. Hence,

$$\frac{d}{dm} L(w,m) \longrightarrow \frac{d}{dw} L(w,m)$$

Substituting this in (2.78), we get

$$-L(W) + \partial \left\{ 1 - L(W) \right\} \longrightarrow 0$$

This proves (2.77) and hence the theorem.

2.8 Determination of lot size and the Bayesian plans

2.8a: The relationship between the sample size(s) and the lot size N depends on how the chosen risks or cost functions are made to depend on N. This problem has received considerable attention in recent years. Significant contributions have been made by many authors (Hamacker, Hald, Kousgaard, Thyregod, Van der Warden etc., to mention only a few). Their cost models are relevant and generally applicable in the context of the theory developed in previous sections. No attempt will be made in the following pages to develop a general theory of cost functions. However some salient features of the asymptotic relationship between the lot size and the sample size are noted below.

2.8b:- If a plan stipulates that all rejected lots be submitted to hundred per cent inspection, then the average amount of inspection provides a simple measure of the cost of a sampling plan. (Dodge and Romig, 1959). For a single sampling plan (n,c), the average amount of inspection - on per unit basis - at the given quality level w is

$$I(w,N,n,c) = \frac{n}{N} + (1-\frac{n}{N})\left\{1-L(w,n,c)\right\} \qquad (2.79)$$

The expected value of $I(w)$. with respect to a prior distribution of w is given by

$$\bar{I}\ (N,n,c) = 1 - A_F(n,c) + \frac{n}{N} A_F(n,c) \tag{2.80}$$

More generally, for a k-stage sampling plan θ,

$$\bar{I}\ (N,\theta) = 1 - A_F(\theta) + N^{-1}\left[n_1 A_F(\theta) + \sum_{r=2}^{k} n_r A_{rF}(\theta)\right] \tag{2.81}$$

where $A_F(\theta)$ is the overall average probability of acceptanc and $A_{rF}(\theta)$ is the (unconditional) average probability of acceptance at r-th or any of the succeeding stages. Note tha n_r is the sample size at the r-th stage and that $A_F(\theta) > A_{2F}(\theta) > \dots > A_{kF}(\theta)$.

It is a case of fairly common occurrence in industry that lots are formed - without regard to any specific rules - as and when items come out of a production process, so that, on an average, the proportion of good lots is equal to the proportion of good items passed out of the assembly line. Even if the prior distribution of the incoming quality includes N as a parameter, $A_F(\theta)$ - being the average probability of acceptance - should be considered as independent of at least for large lots. (Otherwise, the probability of acceptance under a given plan θ could possibly tend to zero or one as $N \to \infty$).

If the elements θ have been determined already on the basis of certain desired specifications, $\bar{I}(N,\theta)$ in (2.81) decreases and tends to $1 - A_F(\theta)$ as $N \to \infty$. If the specifications imply an exact or an asymptotic EQL relationship between the elements of the plan θ, $A_F(\theta) \doteq F(W_{eF})$ and hence the average amount of inspection for large lots is equal to the proportion of bad lots submitted for inspection.

Suppose the stipulation demands an EQL relationship at a specified value m_e but leaves the total sample size n free to be determined on the basis of lot size. Then, under the conditions of Theorem 1.1 or 1.2, $A_F(\theta) - F(m_e) = O(n^{-1})$ or $O(n^{-1/2})$. As $n \to \infty$, $N \to \infty$ but $nN^{-1} \to 0$, the optimum value of n minimizing $\bar{I}$ is $n_0 \propto N^{1/2}$ or $N^{2/3}$.

<u>2.8c:-</u> It may be noted that $\bar{I}$ in (2.81) is less than $1 + nN^{-1}$, n being the total sample size. If the cost of sampling n items is kn^{α}, $k > 0$, $\alpha > 0$, then the average cost is atmost $1 + n^{\alpha}N^{-1}$. We shall assume for the purpose of an asymptotic theory that the cost of (sampling and screening) inspection is $kn^{\alpha} N^{-1}$.

Suppose m_0 is the specified value of the break even quality level i.e., the level below which rejection is costli than acceptance and above which acceptance is costlier than rejection. A simple measure of the cost of wrong decision is provided by $\alpha_1 J_1(m_0, \theta) + \alpha_2 J_2(m_0, \theta)$. The first term refer to the loss due to a wrong acceptance and the second due to a wrong rejection of the lot. This together with the inspectio costs leads (but for a constant factor) to the total regret,

$$R = n^{\alpha} N^{-1} + \gamma \left\{ J_1(m_0, \theta) + \partial\, J_2(m_0, \theta) \right\},$$

$$\gamma > 0, \; \partial > 0 \qquad (2.82)$$

If R is minimized subject to an EQL restriction i.e., $J_1(m_0, \theta) = J_2(m_0, \theta)$, each of the error-areas is of the orde of $n^{-1/2}$ (Theorem 2.4) and hence the optimum value of n is $n_0 \propto N^{2/(2\alpha+1)}$

If R is minimized without any restriction on θ (to obtain a Bayesian solution), it is clear that the optimum value of θ should satisfy the condition (B_∂) of Theorem 2.7 as a first step. When $\partial = 1$, we get asymptotically an IQL (or EQL) relationship and $n_0 \propto N^{2/(2\alpha+1)}$.

Thus the EQL relationship is asymptotically a Bayesian relationship with respect to the regret R and $\partial = 1$. In otherwords, the EQL can be interpreted as a break-even quality level. The same holds true for a more general type of regret function given by

$$R' = n^{\alpha}N^{-1} + \gamma\left\{r_1(m_0, \theta) + r_2(m_0, \theta)\right\} \tag{2.83}$$

where

$$r_1(W,\theta) = \int_{w \leq W} \left\{1 - L(w, \theta)\right\} \lambda'(w)f(w)dw$$

$$r_2(W,\theta) = \int_{w > W} L(w,\theta)\, \lambda''(w)f(w)dw$$

and $\lambda'(w)$ and $\lambda''(w)$ are non-negative loss functions which increase with $|w - m_0|$. Under the conditions of Theorems 2.4 and 2.7, we get asymptotically an EQL relationship i.e., $m_0 = m(\theta)$. Further, from Theorems 2.3 and 2.4(c), for the optimum plan θ, we have (2.84)

$$r_1(m_0, \theta) = D_1(m_0, \theta)\, \lambda'(m_0)f(m_0) + O(n^{-1})$$

$$r_2(m_0, \theta) = D_2(m_0, \theta)\, \bar{\lambda}''(m_0)f(m_0) + O(n^{-1})$$

and $D_i(m_0, \theta) = O(n^{-1/2})$, $i = 1,2$.

Again the optimum value of n is given by $n_0 \propto N^{2/(2\alpha+1)}$.

If the loss functions contain $|w - m_0|^\beta$ as a factor, $\beta \geq 0$, their first $\beta - 1$ derivatives vanish at $w = m_0$. If β^{th} derivatives are bounded in the neighbourhood of m_0, then from (2.46), we find that

$$r_i(m_0, \theta) = O(n^{-(\beta+1)/2}), \quad i = 1,2 \qquad (2.85)$$

It follows that $n_0 \propto N^{2/(2\alpha+\beta+1)}$.

2.88:- The above problem has been treated in detail for single sampling plans by Hald (1967a). His method in our notations consists in considering the OC $L(w,n,c)$ as the c.d.f. of the decision variable η and expanding it in an Edgeworth series. This method is suited for calculating the optimum values of n and c to any desired degree of accuracy. The method has been generalized to multiple sampling plans in Johansen (1969) and Hald and Keiding (1969). In our approach $1 - L(w, \theta) = G(w)$ is considered as the c.d.f. of the hypothetical random variable ξ and therefore there is no need to make a distinction in theory between single and multiple sampling plans.

2.9 The $\{m_e, D_e\}$ system of sampling plans

Sampling plans θ_e can be constructed so as to satisfy the specifications (2.86)

$$m(\theta_e) = m_e$$
$$D(m_e, \theta_e) = D_e$$

where m_e and D_e are the specified quantities. Such plans were originally constructed on an empirical basis for normal, binomial and Poisson single sampling OCs in Subrahmanya (1966) and for Poisson double sampling plans in Subrahmanya (1968). The discussions carried out so far show that the specification parameters m_e and D_e admit good interpretations and have some other interesting and useful properties.

m_e and D_e bear simple relations to the elements of the plan and therefore the $\{m_e, D_e\}$ system of sampling plans can be constructed without much difficulty in the case of single sampling inspection schemes (as shown in Chapters 3, 4 and 5).

m_e is the point of equal error-areas. It can be taken as an IQL. D_e measures the steepness of the OC at m_e. For a specified value of m_e, lesser and lesser values of D_e

correspond to steeper and steeper OCs. Therefore, given a set of OC.curves, the parameter D_e can be used to help us pick and choose a suitable OC. (Subrahmanya (1966) and 1968).

Tightened (or reduced) sampling can be resorted to by shifting to a plan with lower (or higher) values of m_e and D_e.

Randomization with respect to any two plans of the system $\{m_e, D_e\}$ is equal to a linear interpolation between the values of m_e, whereas randomization between two plans having the same value of m_e is equivalent to a linear inte polation between the values of D_e. If θ_e stands for the procedure of choosing plans θ_{e1} and θ_{e2} with probabilitie a and 1-a respectively,

$$m(\theta_e) = am(\theta_{e1}) + (1-a)m(\theta_{e2}) \quad (2.87$$

and if $m(\theta_{e1}) = m(\theta_{e2}) = m_e$,

$$D(m_e, \theta_e) = aD(m_e, \theta_{e1}) + (1-a)D(m_e, \theta_{e2}) \quad (2.88$$

m_e and D_e do not depend on the process characterist: but in certain circumstances, m_e can be taken as the EQL a: D_e as a measure of the decision loss. This is due to the fact that W_{eF} and also $J_F(W_{eF})/ f(W_{eF})$ are fairly insensitive to small changes in the PC (Theorems 1.1, 1.2 and 2.4)

If the prior distribution is known with sufficient accuracy, plans θ_{eF} can be constructed to meet the specifications (2.89)

$$W_{eF}(\theta_{eF}) = W_e$$

$$J_F(W_e, \theta_{eF}) = J_e$$

where W_e and J_e are the given quantities. A plan in the $\{m_e, D_e\}$ system with $m_e = W_e$ and $D_e = J_e/f(W_e)$ can be taken as an approximation to the exact plan θ_{eF}. In that case, the difference between the values of the average probability of acceptance under the plans θ_e and θ_{eF} is given by $F(W_{eF}(\theta_e)) - F(m_e)$ and can be estimated by $M^*D_e/2$ (Theorem 1.2) or more accurately by $MV/2$ (Theorem 1.1), whereas the difference between the total risks $J_F(m_e, \theta) - D(m_e,\theta)f(m_e)$ is of the order of n^{-1} (Theorem 2.4). It would be necessary to construct the exact plan only when the approximations are poor.

Under certain conditions, m_e can be interpreted as an economic break — even quality level when the prior distribution is continuous (Section 2.8) and also in some cases when it is discrete. (Section 1.6 and Hald, 1967a). In otherwords, the restriction $m(\theta_c) = m_e$ is asymptotically a Bayesian relationship with respect to fairly general type of cost models; and the minimum cost attainable is a simple function of $D(m)$. (Theorems 2.4(c) and (2.84)) Indexing the plans wit respect to m and $D(m)$ can therefore be useful even in those cases where the actual values of D_e are not specific

If detailed tables of error-areas are available, plans can be chosen so as to satisfy the stipulations

$$\text{(i)}\ J_1(W_1) = J_2(W_2); \quad J_1(W_1) = \alpha_1$$

$$\text{(ii)}\ J_1(W_1) = \alpha_1 ; \quad J_2(W_2) = \alpha_2$$

etc., where α_1, α_2, W_1 and W_2 are the specified quantities.

The theory of EQL and the error-areas as given in Chapters 1 and 2 is applied to Poisson, binomial and normal single sampling OCs. in Chapters 3, 4 and 5 and Poisson double sampling OCs in Chapter 6. The Poisson single samp ing OCs are treated in greater detail. In each case, a brief consideration of weighted error-areas follows a detail discussion of the properties of unweighted error-areas and t $\{m_e, D_e\}$ system of sampling plans. Multinomial OCs are

C H A P T E R 3

POISSON SINGLE SAMPLING OC CURVES

3.1 The Poisson OC and the error-areas

3.1a: The sampling procedure:-

A random sample of size n is drawn from the lot submitted for inspection (with equal probability and usually without replacement) and the total number of defects in the sample d_n is noted. d_n is the decision variable. The lot is accepted if d_n is less than or equal to the acceptance number c; the lot is not accepted if d_n exceeds c. The parameters n and c are the elements of the plan. The plan is uniquely determined by n and c. The plan itself can be denoted by (n, c).

The lot quality is measured by λ, the average number of defects per item in the lot, $0 \leq \lambda \leq \infty$. Given λ, d_n follows a Poisson distribution with parameter λn. It may be noted that the Poisson distribution can be used (as an approximation to the hypergeometric, via binomial) even when d_n stands for the number of defectives in the sample. The approximation is sufficiently close for practical purposes i the lot size N is large compared to n (say $n/N < .1$), th sample size itself is large ($n > 10$, say) and the proportion of defectives in the lot, p is small (say $p < .1$). (Cowden, 1960).

OC curves under the condition of a Poisson distributi for the decision variable are termed the Poisson OC curves

3.1b The Poisson OC and the hypothetical random variable ξ :-

Given λ, the conditional probability of getting exact r defects in a sample of size n is given by

$$P\left\{d_n = r, \text{ given } \lambda\right\} = \frac{e^{-\lambda n}(\lambda n)^r}{r!}, \quad r = 0, 1, 2 \ldots$$

$$= b(\lambda n, r), \text{ say} \qquad (3.1)$$

The OC of the plan (n,c) is given by

$$L(\lambda n, c) = P\left\{d_n \leq c, \text{ given } \lambda\right\} \qquad 0 \leq \lambda \leq \infty$$

$$= \sum_{r=0}^{c} b(\lambda n, r) = \int_{\lambda n}^{\infty} \frac{e^{-x} x^c}{c!} dx$$

$$= P\left\{\xi_{n,c} > \lambda\right\} \qquad (3.2)$$

where $\xi_{n,c}$ is a random variable following a Gamma distribution with parameters n and $c+1$. Its c.d.f. and p.d.f. are denoted by $G(\cdot)$ and $g(\cdot)$ respectively. (3.3)

$$g(x,n,c) = \frac{e^{-nx} x^c n^{c+1}}{c!} = n\,b(xn,\ c)$$

and $G(x,n,c) = 1 - L(xn,\ c)$.

Note that $2n\xi_{n,c}$ is a χ^2-variable with $2(c+1)$ degrees of freedom.

When there is no cause for confusion, the symbols n and c may be dropped from the notations $\xi_{n,c}$, $g(x,n,c)$ etc.

For the random varible ξ, we have

the mean, $m(n,c)$ or $\bar{m} = \frac{c+1}{n}$ (3.4)

the variance, $V = \frac{c+1}{n^2}$ (3.5)

the j-th cumulant, $\chi_j = \frac{(j-1)!\,(c+1)}{n^j}$, $j = 1,2,\ldots$ (3.6)

and also the incomplete moments, (3.7)

$$E_G\, e_\lambda\, \xi^j = \frac{(c+j)!}{n^j\, c!}\,[\,1 - L(\lambda n,\ c+j)\,]$$

and $E_G\,(1 - e_\lambda)\xi^j = \frac{(c+j)!}{n^j c!}\, L(\lambda n,\ c+j)$

where ϵ_λ is 1 or 0 according as $\xi \leq \lambda$ or $\xi > \lambda$. It follows that

$$E_G\, \epsilon_m(m - \xi) = E_G(1 - \epsilon_m)(\xi - m)$$
$$= m \quad L(mn, c+1) - L(mn, c)$$
$$= mb\,(mn, c+1) \qquad (3.8)$$

Therefore the mean deviation about the mean of ξ is

$$D = E_G\, |\xi - m| = 2mb(mn, c+1) = 2mb(mn, c) \qquad (3.9)$$

Noting

$$(c+1)b(\lambda n, c+1) = \lambda nb\,(\lambda n, c) \qquad (3.10)$$

it can also be shown that (3.11)

$$E_G(1 - \epsilon_m)(\xi - m)^2 = E_G\epsilon_m(\xi^2 - 2m\xi + m^2)$$
$$= n^{-2}\,[(c+1)(c+2)L(mn, c+2)$$
$$-2(c+1)^2L(mn, c+1) + (c+1)^2L(mn, c)\,]$$
$$= n^{-2}(c+1)L(mn, c+1)$$

and $$E_G\, \epsilon_m(\xi - m)^2 = V - E_G(1 - \epsilon_m)(\xi - m)^2$$
$$= n^{-2}(c+1)\,[1 - L(mn, c+1)]$$

Further, (3.12

$$E_G(1-\epsilon_m)(\xi-m)^3 = 2n^{-3}(c+1)\ [L(mn,c+1)+(c+1)b(mn,c+1)\ ]$$

$$E_G\ \epsilon_m(\xi-m)^3 = \mu_3 - E_G\ (1-\epsilon_m)(\xi-m)^3$$

$$E_G(1-\epsilon_m)(\xi-m)^4 = n^{-4}(c+1)^2\ [\,9L(mn,c+1)+6b(mn,c+1)\ ]$$

and $E_G\ \epsilon_m(\xi-m)^4 = \mu_4 - E_G(1-\epsilon_m)(\xi-m)^4$

where μ_3 and μ_4 are the third and fourth central moments of ξ i.e.,

$$\mu_3 = \frac{2(c+1)}{n^3} \quad \text{and} \quad \mu_4 = \frac{3(c+1)(c+3)}{n^4} \tag{3.13}$$

3.1c: The IQL:-

The point of control or the IQL, Λ_0 (n,c) is the median of ξ. Observe that $2n\ \Lambda_0$ (n,c) is a 50 °/. fractile of the χ^2-distribution with 2(c+1) degrees of freedom.

$$L(\Lambda_0 n,\ c) = \frac{1}{2} \tag{3.14}$$

Using (1.10) and (1.11),

$$L(mn.c) = \frac{1}{2} - \frac{1}{\sqrt{2\pi}}\left(\frac{1}{3\sqrt{c+1}} + \frac{1}{540(c+1)\sqrt{c+1}}\right) + O\left(\frac{1}{(c+1)^2\sqrt{c+1}}\right) \tag{3.15}$$

and

$$\Lambda_0\ (n,c) = m + \sqrt{\frac{m}{n}}\left[\frac{1}{3\sqrt{mn}} + \frac{8}{405\ mn\sqrt{mn}} + O\left(\frac{1}{m^2n^2\sqrt{mn}}\right)\ \right] \tag{3.16}$$

or

$$n\ \Lambda_0\ (n,c) = c + \frac{2}{3} + \frac{8}{405(c+1)} + 0(\frac{1}{(c+1)^2}) \qquad (3.17$$

The relation

$$n\ \Lambda_0\ (n,c) = c + \frac{2}{3} \qquad (3.18$$

gives $n\ \Lambda_0$ correct to four significant figures if $c \geq 10$.
The values for $c \leq 10$ are given in Table 3.7.

It may be observed that

$$\Lambda_0\ (n,\ c+1) > m(n,c) > \Lambda_0(n,c) \qquad (3.19$$

3.1d: <u>The error-areas</u>

The error-areas of the first and second kind at the po
$\lambda = \Lambda$ are defined by (3.20

$$D_1(\Lambda\ ,\ n,c) = \int_0^\Lambda \left\{1 - L(\lambda n,c)\right\} d\lambda = E_G\ \epsilon_\Lambda(\ \Lambda - \xi)$$

and

$$D_2(\Lambda,n,c) = \int_\Lambda^\infty L(\lambda n,c)\, d\lambda = E_G(1 - \epsilon_\Lambda)(\xi - \Lambda)$$

where ϵ_Λ is 1 or 0 according as $\xi \leq \Lambda$ or $> \Lambda$. Using (3

$$D_1(\ \Lambda,n,c) = \Lambda\left\{1 - L(\Lambda\ n,c)\right\} - m(n,c)\left\{1 - L(\Lambda\ n,c+1)\right\} \qquad (3.21$$

$$= \Lambda\left\{1 - L(\Lambda\ n,c-1)\right\} - m(n,c)\left\{1 - L(\ \Lambda n,c)\right\}, \qquad (3.21$$

$$c \geq 1$$

$$D_2(\Lambda,n,c) = m(n,c)L(\Lambda n,c+1) - \Lambda L(\Lambda n,c) \qquad (3.22.1)$$

$$= m(n,c)L(\Lambda n,c) - \Lambda L(\Lambda,n,c-1),\ c \geq 1 \qquad (3.22.2)$$

The total error-area at $\lambda = \Lambda$ is given by

$$D(\Lambda,n,c) = E|\xi - \Lambda|$$

$$= \Lambda\{1-2L(\Lambda n,c-1)\} - m(n,c)\{1-2L(\Lambda n,c+1)\} \qquad (3.23.1)$$

$$= \Lambda\{1-2L(\Lambda n,c-1)\} - m(n,c)\{1-2L(\Lambda n,c)\},\ c \geq 1 \qquad (3.23.2)$$

The second set of formulae (3.21.2), (3.22.2) and (3.23.2) follow from the first set because of (3.10). This fact leads to an elegant formula for the error-area of a plan obtained by randomizing between (n,c) and (n,c+1) (Section 3.4).

The relation between D_1 and D_2 is given by

$$D_1(\Lambda,n,c) - D_2(\Lambda,n,c) = \Lambda - m = \Lambda - \frac{c+1}{n} \qquad (3.24)$$

If any one of $D_1(\Lambda)$, $D_2(\Lambda)$ and $D(\Lambda)$ is known, the others can be determined easily using the above relationship. For example, if $D(\Lambda,n,c)$ is known, then

$$2D_1(\Lambda,n,c) = D(\Lambda,n,c) + \Lambda - n^{-1}(c+1)$$

and

$$2D_2(\Lambda,n,c) = D(\Lambda,n,c) + n^{-1}(c+1) - \Lambda \qquad (3.25)$$

The area under the (unweighted) OC and also the point of equal error-areas is $m = n^{-1}(c+1)$. The total error-area at m is the mean deviation about the mean of ξ as given in (3.9)

$$D(m,n,c) = 2mb(mn,c+1) = 2mb(mn,c) \tag{3.26}$$

The total error-area at the IQL Λ_0 (n,c) is given by

$$D(\Lambda_0,n,c) = 2mb(\Lambda_0 n,c+1) = 2\Lambda_0 b(\Lambda_0 n,c) \tag{3.27}$$

The formulae for the error-areas at the IQLs of neighbouring plans also take simple forms. If

$$\Lambda_0' = \Lambda_0(n,c-1) \quad \text{and} \quad \Lambda_0'' = \Lambda_0(n,c+1)$$

$$\text{i.e.,} \quad L(\Lambda_0' n,c-1) = \tfrac{1}{2} = L(\Lambda_0'' n,c+1) \tag{3.28}$$

$$\text{then} \quad D(\Lambda_0'',n,c) = 2\Lambda_0'' b(\Lambda_0'' n,c+1) \tag{3.29}$$

$$\text{and} \quad D(\Lambda_0',n,c) = 2mb(\Lambda_0' n,c) \tag{3.30}$$

Note that in the above, $m = m(n,c) = n^{-1}(c+1)$;

$\Lambda_0 = \Lambda_0\ (n,c) \doteq n^{-1}(c+2/3)$; $\Lambda_0 \doteq n^{-1}(c-1/3)$ and

$$\Lambda_0'' = n^{-1}\ (c+1+2/3) \tag{3.31}$$

From (3.23), it follows that

$$D(\Lambda_0(n,c),\ n,c) = D(\Lambda_0(n,c), n, c-1) \qquad (3.32)$$

3.1c: <u>The behaviour of the error-areas with respect to λ:-</u>

It is easily seen that for a given plan (n,c) as λ increases from zero to infinity, (3.33)

$L(\lambda n, c)$ decreases from 1 to zero

$D_2(\lambda, n, c)$ decreases from $n^{-1}(c+1)$ to zero,

and $D_1(\lambda, n, c)$ increases from zero to infinity.

The following properties will also be needed in Section 3.5

$\lambda^{-1} D_2(\lambda, n, c)$ decreases from ∞ to zero (3.34.1)

and $n^{-1} D_1(\lambda, n, c)$ increases from zero to 1 (3.34.2)

(3.34.1) is a direct consequence of (3.33). To prove (3.34.2), from (3.20),

$$\frac{d}{d\lambda} \frac{D_1(\lambda,n,c)}{\lambda} = -\frac{D_1(\lambda,n,c)}{\lambda^2} + \frac{1 - L(\lambda\, n, c)}{\lambda}$$

$$= \lambda^{-2} m\ [1 - L(\lambda n, c+1)\ ], \text{ using } (3.21.1)$$

$$\geq 0$$

Further, since $\frac{d}{d\lambda} D_1(\lambda,n,c) = 1 - L(\lambda\ n,c)$, it follows that

$$\lim_{\lambda \to 0} \frac{D_1(\lambda,n,c)}{\lambda} = 0 \quad \text{and} \quad \lim_{\lambda \to \infty} \frac{D_1(\lambda,n,c)}{\lambda} = 1$$

This proves (3.34.2)

From (3.24),

$$D_1(\lambda,n,c) \gtreqless D_2(\lambda,n,c) \quad \text{according as} \quad \lambda \gtreqless m \tag{3.35}$$

Therefore,

$$\min_{\lambda} \max [\, D_1(\lambda,n,c),\ D_2(\lambda,n,c)\,] = D_1(m,n,c) = D_2(m,n,c) \tag{3.36}$$

Thus m is the point at which the bigger of the two error-areas is a minimum with respect to λ (Theorem 2.1).

By differentiating $D(\lambda,n,c)$ with respect to λ, we fi[nd] that it is a minimum at $\lambda = \Lambda_0\ (n,c)$. (Theorem 2.2) i.e.,

$$D(\Lambda_0,n,c) \leq D(\lambda,n,c), \quad \text{for all} \quad \lambda \tag{3.37}$$

By differentiating $\lambda^{-1} D(\lambda,n,c)$ with respect to λ, we find that for a given plan (n,c), it is a minimum at $\lambda = \Lambda_0\,(n,c+1) = \Lambda_0''$ i.e.,

$$\frac{D(\Lambda_0'',n,c)}{\Lambda''} \leq \frac{D(\lambda,n,c)}{\lambda}, \quad \text{for all} \quad \lambda \tag{3.38}$$

It may be noted that for a given plan (n,c),

$$D(\lambda,n,c) \simeq \begin{cases} \lambda - m & \text{if } \lambda \text{ is large} \\ m - \lambda & \text{if } \lambda \text{ is small} \end{cases} \qquad (3.39)$$

3.1f: <u>Calculation of error-areas</u>:-

The Poisson probability $b(\lambda n,r)$ and also the OC $L(\lambda n,c)$ depend on λ and n only through their product λn. Therefore, $nD_1(\lambda,n,c)$ and $nD_2(\lambda,n,c)$ depend only on λn and c whereas mn and $\Lambda_0\, n$ depend on c only. This fact facilitates the presentation of tables concerning these functions.

$b(\lambda n,r)$ and $L(\lambda n,c)$ can be obtained from tables of individual and cumulative terms of Poisson distribution given in General Electric Company (1962). The error-areas at any desired value of λ can be calculated from the formulae given in this section. However the formulae for the error-areas at m, Λ_0, Λ_0^* and Λ_0'' are much simpler as can be seen from (3.26) - (3.30).

Tables 3.1 and 3.2 give $L(\lambda n,c)$, $nD_1(\lambda,n,c)$, $nD_2(\lambda,n,c)$, $nD(\lambda,n,c)$, $\lambda^{-1}D(\lambda,n,c)$ and also the ratio $D_2(\lambda,n,c)/D_1(\lambda,n,c)$ for various values of λn when c=1 and 4 respectively. The corresponding OC curves are drawn in Chart 3.1 whereas Chart 3.2 exhibits clearly the behaviour of the error-areas

TABLE 3.1

Values of Poisson OC and error-areas for $c = 1$ ($mn = 2$; $\Lambda_0 n = 1.678$ and $\Lambda_0'' n = 2.674$).

λn	$L(\lambda n, c)$	$nD_1(\lambda, n, c)$	$nD_2(\lambda, n, c)$	$nD(\lambda, n, c)$	$\frac{D_2(\lambda, n, c)}{D_1(\lambda, n, c)}$	$\frac{D(\lambda, n, c)}{\lambda}$
.0	1.000	.000	2.000	2.000	∞	∞
small	1.000	.000	2-λn	2-λn	∞	∞
.2	.983	.001	1.801	1.802	1800	9.0
.4	.938	.009	1.609	1.618	180	4.0
.6	.878	.027	1.427	1.454	53	2.4
.8	.809	.058	1.258	1.316	22	1.6
1.0	.736	.104	1.104	1.208	11	1.2
1.5	.558	.281	.781	1.062	2.8	.71
1.6	.525	.327	.727	1.054	2.2	.66
1.7	.493	.376	.676	1.052	1.8	.62
1.8	.463	.428	.628	1.056	1.5	.59
1.9	.434	.483	.583	1.066	1.2	.56
2.0	.406	.541	.541	1.082	1.0	.54
2.1	.380	.602	.502	1.104	1.2*	.53
2.2	.355	.665	.465	1.130	1.4*	.51
2.3	.331	.731	.431	1.162	1.7*	.51
2.4	.308	.799	.399	1.198	2.0*	.50
2.5	.287	.869	.369	1.238	2.4*	.50
2.6	.267	.942	.342	1.284	2.8*	.49
2.7	.249	1.016	.316	1.332	3.2*	.49
3.0	.199	1.249	.249	1.498	5*	.50
Large	.00	λn-2	.00	λn-2	∞*	1.00

* They are the values of $D_1(\lambda, n, c)/D_2(\lambda, n, c)$

TABLE 3.2

Values of Poisson OC and error-areas for c = 4
(mn= 5 ; $\Lambda_0 n = 4.671$ and $\Lambda_0'' n = 5.670$)

n	$L(\lambda n,c)$	$nD_1(\lambda,n,c)$	$nD_2(\lambda,n,c)$	$nD(\lambda,n,c)$	$\frac{D_2(\lambda,n,c)}{D_1(\lambda,n,c)}$	$\frac{D(\lambda,n,c)}{\lambda}$
.0	1.000	.000	5.000	5.000	∞	∞
all	1.000	.000	5 - λn	5 - λn	∞	∞
.0	.998	.001	4.001	4.002	4000	4.0
.0	.947	.023	3.023	3.046	130	1.5
.0	.815	.135	2.135	2.270	16	.76
.0	.629	.410	1.410	1.820	3.4	.46
.5	.532	.620	1.120	1.740	1.8	.39
.6	.513	.668	1.068	1.736	1.6	.38
.7	.495	.718	1.018	1.736	1.4	.37
.8	.476	.769	.969	1.738	1.3	.36
.9	.458	.822	.922	1.744	1.1	.36
.0	.441	.877	.877	1.754	1.0	.35
.5	.358	1.178	.678	1.856	1.7*	.34
.0	.285	1.518	.518	2.036	2.9*	.34
.5	.224	1.891	.391	2.282	4.8*	.35
.0	.173	2.293	.293	2.586	7.8*	.37
.5	.132	2.717	.217	2.934	13*	.39
.0	.100	3.159	.159	3.318	20*	.41
.0	.055	4.084	.084	4.168	49*	.46
rge	.00	λn - 5	.00	λn - 5	∞*	1.00

They are the values of $D_1(\lambda,n,c)/D_2(\lambda,n,c)$

TABLE 3.3

Values of $n\,D_2\,(\lambda, n, c)$

λn	c = 0	c = 1	c = 2	λn	c = 3	c = 4	c = 5
0.0	1.0000	2.0000	3.0000	1.0	3.0043	4.0007	5.0001
0.1	0.9048	1.9000	2.9000	1.2	2.8095	3.8018	4.8003
0.2	0.8187	1.8012	2.8001	1.4	2.6182	3.6039	4.6007
0.3	0.7408	1.7039	2.7003	1.6	2.4314	3.4077	4.4016
0.4	0.6703	1.6088	2.6008	1.8	2.2500	3.2136	4.2033
0.5	0.6065	1.5163	2.5019	2.0	2.0751	3.0225	4.0059
0.6	0.5488	1.4269	2.4038	2.2	1.9074	2.8349	3.8100
0.7	0.4966	1.3408	2.3066	2.4	1.7476	2.6517	3.6160
0.8	0.4493	1.2581	2.2107	2.6	1.5961	2.4735	3.4245
0.9	0.4066	1.1791	2.1162	2.8	1.4533	2.3010	3.2358
1.0	0.3679	1.1036	2.0233	3.0	1.3194	2.1345	3.0507
1.2	0.3012	0.9638	1.8433	3.5	1.0236	1.7490	2.6066
1.4	0.2466	0.8384	1.6719	4.0	0.7815	1.4103	2.1954
1.6	0.2019	0.7268	1.5102	4.5	0.5881	1.1202	1.8231
1.8	0.1653	0.6281	1.3688	5.0	0.4368	0.8773	1.4933
2.0	0.1353	0.5413	1.2180	5.5	0.3207	0.6782	1.2072
2.2	0.1108	0.4654	1.0881	6.0	0.2330	0.5181	0.9637
2.4	0.0907	0.3992	0.9689	6.5	0.1677	0.3913	0.7604
2.6	0.0743	0.3417	0.8601	7.0	0.1196	0.2926	0.5933
2.8	0.0608	0.2919	0.7613	7.5	0.0847	0.2167	0.4582
3.0	0.0498	0.2489	0.6721	8.0	0.0595	0.1591	0.3504
3.5	0.0302	0.1661	0.4869	8.5	0.0415	0.1159	0.2655
4.0	0.0183	0.1099	0.3480	9.0	0.0288	0.0838	0.1995
4.5	0.0111	0.0722	0.2458	9.5	0.0199	0.0601	0.1487
5.0	0.0067	0.0472	0.1718	10.0	0.0137	0.0429	0.1100

TABLE 3.3 (contd.)

λn	c = 6	c = 7	c = 8	c = 9	c = 10
0.0	7.0000	8.0000	9.0000	10.0000	11.0000
0.5	6.5000	7.5000	8.5000	9.5000	10.5000
1.0	6.0000	7.0000	8.0000	9.0000	10.0000
1.5	5.5002	6.5000	7.5000	8.5000	9.5000
2.0	5.0014	6.0003	7.0001	8.0000	9.0000
2.5	4.5057	5.5015	6.5004	7.5001	8.5000
3.0	4.0172	5.0053	6.0015	7.0004	8.0001
3.5	3.5413	4.5146	5.5047	6.5014	7.5004
4.0	3.0848	4.0336	5.0123	6.0041	7.0013
4.5	2.6542	3.5676	4.5273	5.5102	6.5036
5.0	2.2555	3.1221	4.0540	5.0222	6.0084
5.5	1.8932	3.7027	3.5970	4.5433	5.5180
6.0	1.5700	2.3140	3.1613	4.0773	5.0347
6.5	1.28[illegible]9	1.9597	2.7512	3.6286	4.5618
7.0	1.0430	1.6417	2.3708	3.2013	4.1028
7.5	0.8363	1.3609	2.0229	2.7993	3.6616
8.0	0.6637	1.1167	1.7092	2.4259	3.2417
8.5	0.5217	0.9073	1.4304	2.0833	2.8467
9.0	0.4063	0.7301	1.1858	1.7732	2.4792
9.5	0.3136	0.5823	0.9741	1.4959	2.1413
0.0	0.2401	0.4604	0.7932	1.2511	1.8341

CHART 3.1: POISSON SINGLE SAMPLING OCs.

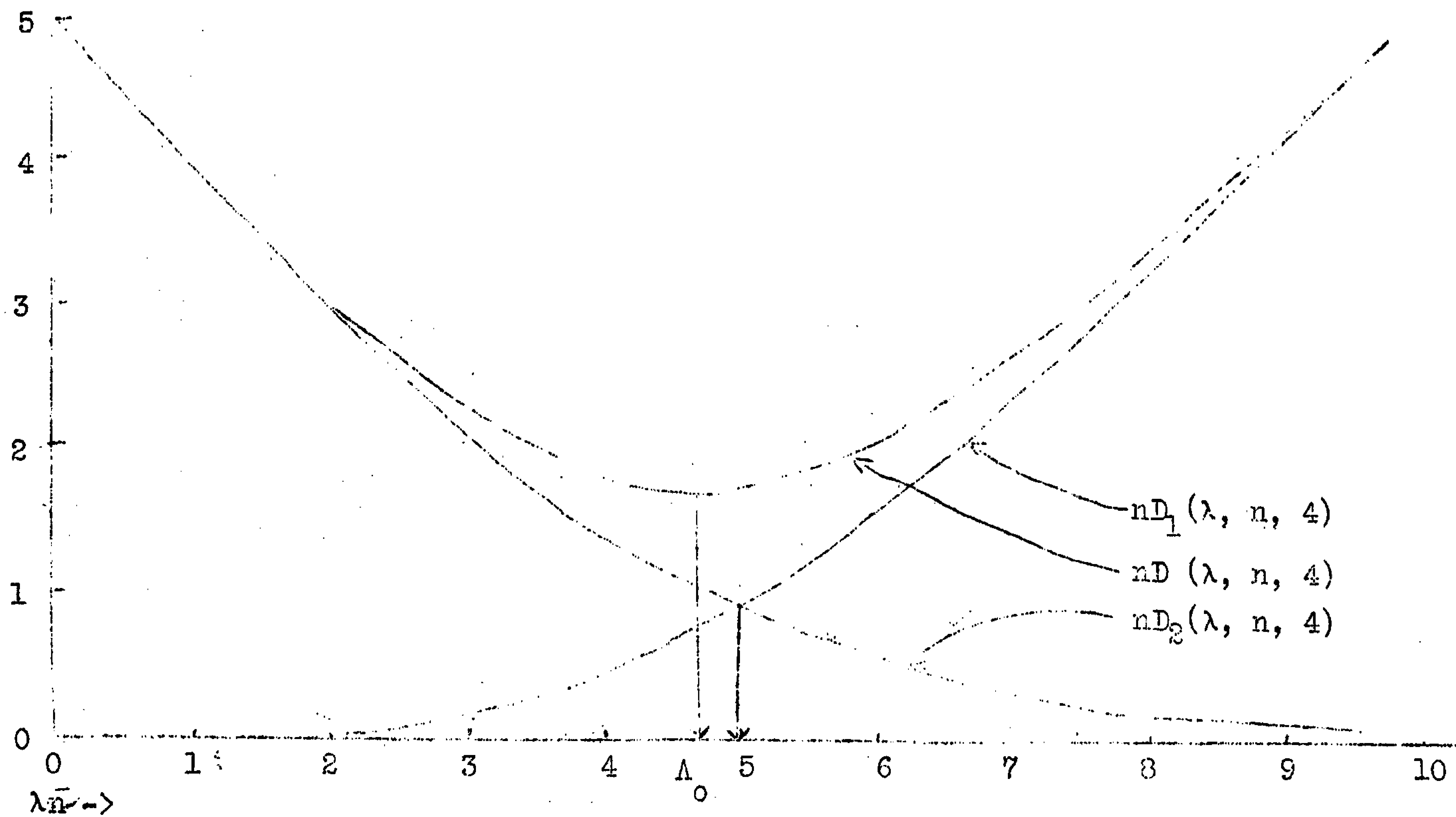


CHART 3.2: ERROR-AREAS OF POISSON SINGLE SAMPLING PLAN, 3 = 4.

3.2 Optimality conditions (A) and (B)

3.2a: The behaviour of the error-areas with respect to n and c:-

$L(\lambda n, c)$, $D_1(\lambda, n, c)$ and $D_2(\lambda, n, c)$ are monotonic functions of each of their arguments. The inequalities (2.51) are valid i.e., in addition to (3.33), we have (3.40)

$$L(\lambda n', c) \geq L(\lambda n, c) \quad \text{if} \quad n' \leq n$$

and

$$L(\lambda n, c') \leq L(\lambda n, c) \quad \text{if} \quad c' \leq c$$

The same inequalities are valid for $D_2(\lambda, n, c)$.
The inequalities are reversed for $D_1(\lambda, n, c)$.

As a consequence, Theorems 2.5, 2.6 and 2.7 which deal with the optimality conditions (A) and (B) are applicable.

3.2b: Condition (A):-

Given n and $\lambda = \Lambda$, the value of c – $c_e = c_e(\Lambda, n)$ say – for which the bigger of the two error-areas at Λ is a minimum should satisfy

$$\Lambda = m(n, c_e) = n^{-1}(c_e + 1) \quad \text{or} \quad c_e = n\Lambda - 1 \qquad (3.41)$$

This is a direct consequence of Theorem 2.5. All the observations made in connexion with the discreteness of c are valid and nothing further needs to be said about them.

3.2c: Condition (B):-

Theorems 2.6 and 2.7 are of course applicable; however something more is true - except for the limitations imposed by the discreteness of c, the condition (B) leads exactly to an IQL relationship.

Given n and $\lambda = \Lambda$, the value of $c = c_i = c_i(\Lambda, n)$ say for which the total error-area at Λ is a minimum is given by

$$L(\Lambda n, c_i+1) > \tfrac{1}{2} \geq L(\Lambda n, c_i)$$

i.e., $$\Lambda_0 (n, c_i+1) > \Lambda \geq \Lambda_0 (n, c_i) \qquad (3.42)$$

or approximately

$$c_i+1 + 2/3 > \Lambda n \geq c_i + 2/3 .$$

Proof: From (3.33),

$$L(\Lambda n, c_i+r) > \tfrac{1}{2} \geq L(\Lambda n, c_i) > L(\Lambda n, c_i-r), \quad r = 1, 2, \ldots$$

Using (3.23) and noting that $m(n, c+r) - m(n, c+r-1) = n^{-1}$, we get

$$D(\Lambda, n, c_i+r) - D(\Lambda, n, c_i) = n^{-1} \sum_{s=1}^{r} [2L(\Lambda n, c_i+s)-1] > 0$$

and

$$D(\Lambda,n,c_i-r) - D(\Lambda,n,c_i) = n^{-1}\sum_{s=0}^{r-1}[1-2L(\Lambda n,c_i-s)] \leq 0$$

which together prove that $D(\Lambda,n,c)$ is a minimum at $c = c_i$.

<u>Remark (1):</u>

If $\Lambda = \Lambda_0(n,c_i)$ exactly, then from (3.32) it follows that both c_i and c_i-1 are optimum values. If randomization with respect to c is admitted, any c in the interval $c_i \geq c \geq c_i - 1$ is also an optimum, because the total error-area at $\lambda = \Lambda$ under the composite OC $aL(\lambda n,c_i-1) + (1-a)L(\lambda n,c_i)$ (cf. (1.3) and (1.29)) is

$$aD(\Lambda, n, c_i-1) + (1-a)D(\Lambda,n,c_i)$$
$$= D(\Lambda, n,c_i)$$

where a and $(1-a)$ are the probabilities of choosing the plan (n,c_i-1) and (n,c_i) respectively and

$$c = a(c_i-1) + (1-a)c_i, \quad 0 \leq a \leq 1.$$

If c_0 is the optimum value of c under the condition (A), then from (3.41) and 3.19), $c_i > c > c_i-1$. It follows that c_0 also satisfies the condition (B).

Remark (2):

If c is an optimum under condition (B) for the given values Λ and n, it is also an optimum for n and any λ in the interval

$$\Lambda_0 \ (n,c) \leq \lambda < \Lambda_0 (n,c+1) = \Lambda_0'' \tag{3.43}$$

This follows from (3.42). The value of the minimum error-area will however depend on λ. The following table gives the Λn-intervals and also the (minimum) error-area at the end-points of the intervals for $c = 0(1)5$.

TABLE 3.4

Λ_0 n-intervals for which c is an optimum under condition (B)

c	$\Lambda_0 n \leq \lambda n < \Lambda_0'' n$	$nD(\Lambda_0 n, c)$	$nD(\Lambda_0'' n, c)$
0	0.693 - 1.678	0.693	1.05
1	1.678 - 2.674	1.05	1.32
2	2.674 - 3.672	1.32	1.54
3	3.672 - 4.671	1.54	1.73
4	4.671 - 5.670	1.73	1.90
5	5.670 - 6.670	1.90	2.06

$\Lambda_0 n$ and $\Lambda_0'' n$ are obtained from Table 3.7; $nD(\Lambda_0)$ and $nD(\Lambda_0'')$ are computed from (3.27) and (3.29) respectively. No that $\Lambda_0' - \Lambda_0 \doteq n^{-1}$

Thus we find that

Example (i):- if $n = 100$, the optimum value of c is for any given value of λ in the interval, $1.678\,\%. \leq \lambda <$ $2.674\,\%.$; and

Example (ii):- if $n = 50$, the optimum value of c is for any given value of λ in the interval, $?.342\,\%. \leq \lambda <$ $11.340\,\%.$.

If Λ is the value that demarcates good lots from bad, it may be necessary in practice to set up an indifference quality region (Wald 1947) $\Lambda_0 < \Lambda < \Lambda_0'$ such that lots with $\lambda \leq \Lambda_0$ are definitely good and lots with $\lambda \geq \Lambda_0'$ are definitel bad while one is indifferent to lots of quality $\Lambda_0 < \lambda < \Lambda_0''$ The above method of constructing sampling plans provides a conceptual background for such a region. Further, the difference between the extremities of a Λ - interval being n^{-1}, the indifference quality region dwindles as n increases. This is desirable, because a larger sample size (usually coupled with a larger lot size) is expected to give a better discriminatory power together with a reduction in the decision l

Observe that $m(n,c) = n^{-1}(c+1)$ always lies within the Λ-interval corresponding to c.

Remark (3):

If c is optimum under condition (B) for $\lambda = \Lambda$ and $n = n_0$, then it is also optimum for any n in the interval

$$n^-(\Lambda, c) = \frac{n_0 \Lambda_0 (n_0, c)}{\Lambda} \leq n < \frac{n_0 \Lambda_0 (n_0, c+1)}{\Lambda} = n^+(\Lambda, c) \tag{3.44}$$

The n-intervals can be obtained from Table 3.4 itself by considering $\Lambda_0 n$ as Λn^- and $\Lambda_0'' n$ as Λn^+. The error-areas at the end-points of the n-intervals are also obtained by observing that $nD(\Lambda_0, n, c) = n^- D(\Lambda, n^-, c)$ and $nD(\Lambda_0'', n, c) = n^+ D(\Lambda, n^+, c)$. For example, the n-intervals for the case of = 10 °/. and 5 °/. are given below.

TABLE 3.5

n-intervals for which c is an optimum under condition (B)

c	$\Lambda = 10$ °/. $n^- \leq n \leq n^+$		$\Lambda = 5$ °/. $n^- \leq n \leq n^+$	
0	7	16	14	33
1	17	26	34	53
2	27	36	54	73
3	37	46	74	93
4	47	56	94	113
5	57	66	114	133

In the above table, n^- has been rounded upwards and n^+ downwards to the nearest integer.

Observe also that n_e, the value of n for which $m(n_e, c) = n_e^{-1}(c+1) = A$, lies within the n-interval corresponding to c.

3.2d: Asymptotic properties of plans satisfying conditions (A) and (B):-

From (3.24) and (3.26),

$$D_1(m,n,c) = D_2(m,n,c) = \frac{me^{-mn}\ (mn)^{mn-1}}{\Gamma(mn)}, \quad mn = c+1$$

$$= \sqrt{\frac{m}{2\pi n}}\left[1 - \frac{1}{12mn}\right] + O(n^{-2}) \qquad (3.45)$$

using Sterling's approximation for $\Gamma(mn)$, Cramer (1946).

If $n \to \infty$, $c \to \infty$ such that $m = (c+1)/n$ remains constant at a given value (Condition (A)), then the error-areas at m tend to zero with the order of $1/\sqrt{n}$. Further,

$$\sqrt{n}\ D(m) \to 0.79788\ \sqrt{m} \qquad (3.46)$$

The following table shows clearly the behaviour of $\sqrt{n}\ D(m)$ when m is kept fixed at 25 °/. or 5 °/. .

TABLE 3.6

Values of $\sqrt{n}\, D(m)$ under the optimality condition (A)

$(c+1)/n = m = 25$ %

n	c + 1	$\sqrt{n}$ D(m)
4	1	.368
8	2	.383
12	3	.388
16	4	.391
20	5	.392
36	9	.3953
100	25	.3976
200	50	.3983
400	100	.3986
	∞	.3989

$(c+1)/n = m = 5$ %

n	c+1	$\sqrt{n}$ D(m)
20	1	.165
40	2	.171
60	3	.174
80	4	.175
100	5	.175
200	10	.1769
300	15	.1774
400	20	.1777
	∞	.1784

It is suggested that interpolation, where necessary, be carried out with respect to $\sqrt{n}\, D(m)$.

Plans satisfying the condition (B) are close to those under (A) and the same asymptotic properties hold. From (3.27

$$D(\Lambda_0, n, c) = \frac{2\, \Lambda_0\, e^{-\Lambda_0 n}\, (\Lambda_0 n)^{mn-1}}{\Gamma(mn)} \qquad (3.47)$$

and from (3.24) and (3.17),

$$D_1(\Lambda_0, n, c) - D_2(\Lambda_0, n, c) = \Lambda_0(n,c) - m(n,c)$$

$$= -\frac{1}{3n} + 0(n^{-2}) \qquad (3.48)$$

As $n \to \infty$, $c \to \infty$ such that Λ_0 (n,c) remains constant at a given value (Condition (B)), the difference between the error-areas at Λ_0 tends to zero with $0(n^{-1})$. The system (B) approaches the system (A) with $0(n^{-1})$ whereas each of the error-areas at Λ_0 approaches zero with $0(1 / \sqrt{n})$. Further, for any value of λ,

$$D_1(\lambda) < D_1(m) \to 0, \quad \text{if} \quad \lambda < m$$

and

$$D_2(\lambda) < D_2(m) \to 0, \quad \text{if} \quad \lambda > m$$

Thus, in both the systems, the OC tends to the ideal OC at Λ, the specified value, where all lots with $\lambda \leq \Lambda$ are accepted and all lots with $\lambda > \Lambda$ are rejected. (Fig. 2.1)

3.3 The $\{m_e, D_e\}$ system of sampling plans

3.3a: Formulae and the tables:-

Sampling plans can be constructed so as to satisfy the specifications

$$m(n,c) = m_e$$

and $$D(m_e,n,c) = D_e$$

where m_e and D_e are the specified numbers. The simple relations

$$nm_e = c+1 \tag{3.49}$$

and $$nD_e = 2nm_e b\,(m_e n, c+1) = 2nm_e \frac{e^{-nm_e}\,(nm_e)^{c+1}}{(c+1)!} \tag{3.50}$$

give the transformation from the set (n,c) to the set $\{m_e,\ D_e\}$ and vice-versa. Selection of m_e fixes the ratio $(c+1)/n$. We then have to select that plan for which D_e equals a given number (or very nearly so).

Note that nD_e depends only on c. Table 3.7 gives nD_e for $c = 0(1)50$. The table shows the values of some other measures also: g_e/n, L_e, $n\ \Lambda_0$, nAQL (.95) and nLTPD (.10).

The steepness of OC at a point λ can be measured by the absolute value of the slope,

$$g(\lambda,n,c) = -\frac{d}{d\lambda} L(\lambda n,c) = nb(\lambda n,c)$$

The slope at m_e is denoted by g_e. We have

$$\frac{g_e}{n} = b(m_e n, c) = \frac{D_e}{2m_e} = \frac{nD_e}{2(c+1)} \qquad (3.51)$$

For a given m_e, smaller values of D_e correspond to larger values of n and g_e and hence to steeper OCs. Values of g_e/n are shown in Col. (3) of Table 3.7.

L_e in column (4) refers to the value of OC at m_e. nA_0 in col. (6) stands for n times the IQL, as given in (3.17). AQL (.95) and LTPD (.10) are the quality levels fo which the probabilities of acceptance are 95 % and 10 % respectively, viz.,

$$L(AQL(.95)n,\ c) = 95\ \% \quad \text{and} \quad L(LTPD(.10)n,c) = 10\ \%$$

It may be noted that 2n AQL(.95) and 2n LTPD(.10) are obtained respectively as the upper 95 % and 10 % fractiles of a χ^2-distribution with 2(c+1) degrees of freedom. (Owen (1962) and Pearson and Hartley, 1957).

Chart 3.3 connects nm_e with nD_e .

3.3b: <u>Examples:-</u>

Example (i):- Suppose $m_e = 2$ % and $D_e = \frac{1}{2}$ % .
From Table 3.7 we obtain the following

n	c+1	D_e %
50	1	1.47
100	2	1.08
...	...	...
500	10	0.50

So (500,9) is the required plan.

Example (ii):- Suppose $n = 10$. Then

c	m_e %	D_e %	g_e
0	10	7.36	3.68
1	20	10.83	2.71

etc.

If we want D_e to be less than 7.36 % or an OC steeper than 3.68, we should have $n > 10$.

The $\{m_e, D_e\}$ system of sampling plans obey the condition (A) given in Section 3.2b. They are also very close to the plans satisfying condition (B). For a given m_e, D_e can be taken as small as we please by making n sufficiently large. (Section 3.2c).

A general discussion of tightened and reduced inspection has been given already in Section 1.7. Here two examples are given for the sake of illustration.

Example (iii):

Inspection	plan n	plan c	m_e °/°	D_e °/°	g_e
reduced	80	2	3.75	1.7	18
normal	200	5	5	1.0	32
tightened	200	3	2	0.8	39

Example (iv):-

Inspection	plan n	plan c	m_e °/°	D_e °/°	g_e
reduced	10	1	20	10.8	3
normal	50	2	6	2.7	11
tightened	50	1	4	2.2	14

Plans in example (iii) are taken from RMM Tables (1966) and those in (iv) are from MIL-STD (U.S. Defence Department, 1959).

3.3c: Philips SSS:-

The plans belonging to the Philips Standard Sampling System (Willemze and Fuijt, 1955) are based on the concept of A_0, the IQL. There are 25 single and 48 double sampling plans in the system. The single sampling plans are studied from the point of view of m_e and D_e. The results are given in Table 3.8. (The double sampling plans are dealt with in Table 6.2 given in Section 6.9).

TABLE 3.7

The $\{m_e, D_e\}$ system of Poisson single sampling plans.

c	$m_e n$.	nD_e	$g_e n^{-1}$	L_e	nAQL(.95)	$n\,\Lambda_0$	nLTPD(.10)
	(1)	(2)	(3)	(4)	(5)	(6)	(7)
0	1	0.7358	0.3679	0.3679	0.051	0.693	2.303
1	2	1.0827	.2707	.4060	0.355	1.678	3.890
2	3	1.3443	.2240	.4232	0.818	2.674	5.322
3	4	1.5629	.1954	.4335	1.366	3.672	6.681
4	5	1.7547	.1755	.4405	1.970	4.671	7.993
5	6	1.9275	.1606	.4457	2.613	5.670	9.275
6	7	2.0860	.1490	.4497	3.285	6.670	10.53
7	8	2.2334	.1396	.4530	3.981	7.669	11.77
8	9	2.3716	.1318	.4557	4.695	8.669	12.99
9	10	2.5022	.1251	.4579	5.425	9.669	14.21
10	11	2.6263	.1194	.4599	6.169	10.67	15.41
11	12	2.7448	.1144	.4616	6.924	11.67	16.60
12	13	2.8584	.1099	.4631	7.690	*	17.78
13	14	2.9677	.1060	.4644	8.464		18.96
14	15	3.0731	.1024	.4657	9.246		20.13
15	16	3.1750	.0992	.4667	10.04		21.29
16	17	3.2737	.0963	.4677	10.83		22.45
17	18	3.3695	.0936	.4686	11.63		23.61
18	19	3.4627	.0911	.4695	12.44		24.76
19	20	3.5534	.0888	.4703	13.25		25.90
20	21	3.6419	.0867	.4710	14.07		27.05
21	22	3.7283	.0847	.4716	14.89		28.18
22	23	3.8127	.0829	.4723	15.72		29.32
23	24	3.8953	.0812	.4728	16.55		30.45
24	25	3.9762	.0795	.4734	17.38		31.58

* $n\,\Lambda_0 = c + 2/3$ for $c \geq 10$

contd

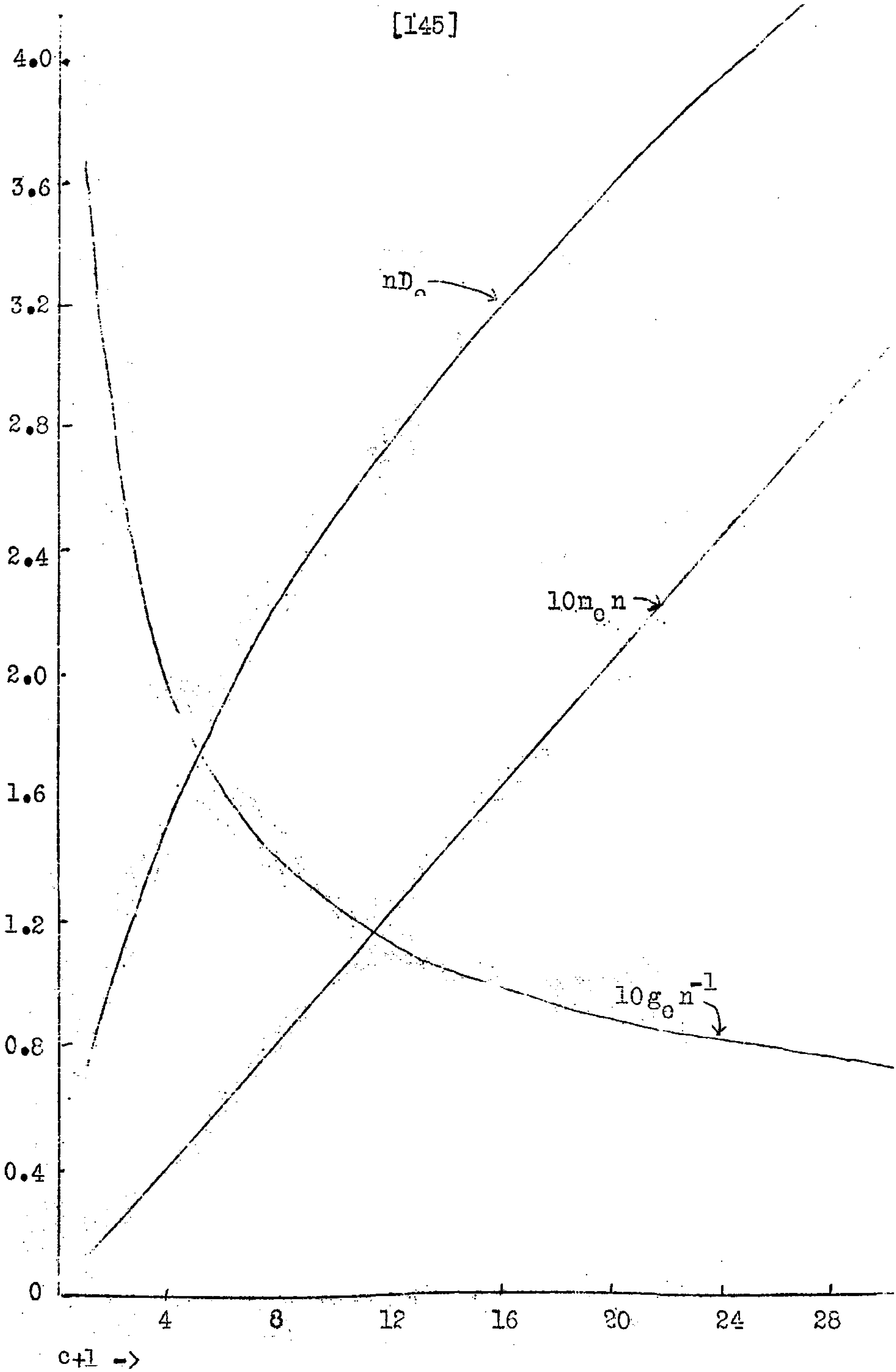
TABLE 3.7 (contd.)

c	$m_c n$	nD_c	$\xi_c n^{-1}$	L_c	nAQL(.95)	nLTPD(.10)
	(1)	(2)	(3)	(4)	(5)	(7)
25	26	4.0554	0.0780	0.4739	18.22	32.71
26	27	4.1332	.0765	.4744	19.06	33.84
27	28	4.2095	.0752	.4749	19.90	34.96
28	29	4.2844	.0739	.4753	20.75	36.08
29	30	4.3581	.0726	.4757	21.59	37.20
30	31	4.4305	.0715	.4761	22.44	38.32
31	32	4.5018	.0703	.4765	23.30	39.43
32	33	4.5720	.0693	.4768	24.15	40.54
33	34	4.6411	.0683	.4742	25.01	41.65
34	35	4.7091	.0673	.4775	25.87	42.76
35	36	4.7763	.0663	.4778	26.73	43.87
36	37	4.8424	.0654	.4781	27.59	44.98
37	38	4.9077	.0646	.4784	28.46	46.08
38	39	4.9722	.0637	.4787	29.33	47.19
39	40	5.0358	.0629	.4790	30.20	48.29
40	41	5.0986	.0622	.4792	31.07	49.39
41	42	5.1606	.0614	.4795	31.94	50.49
42	43	5.2219	.0607	.4797	32.81	51.59
43	44	5.2826	.0600	.4799	33.69	52.69
44	45	5.3425	.0594	.4802	34.56	53.78
45	46	5.4017	.0587	.4804	35.44	54.88
46	47	5.4604	.0581	.4806	36.32	55.97
47	48	5.5183	.0575	.4808	37.20	57.07
48	49	5.5757	.0569	.4810	38.08	58.15
49	50	5.6325	.0563	.4812	38.96	59.25
50	51	5.6887	.0558	.4813	39.85	60.34

TABLE 3.8

Philips Standard Sampling System - single sampling plans.

Sl. No.	IQL A_0 °/. Indexed value	IQL A_0 °/. Actual value	Plan n	Plan c	m °/.	D(m) °/.	g(m)	L(m) °/.
1	0.25	0.31	225	0	0.44	0.23	83	36.8
2	0.25	0.40	175	0	0.57	0.42	64	36.8
3	0.50	0.69	100	0	1.00	0.74	37	36.8
4	1.00	1.16	60	0	1.67	1.23	22	36.8
5	2.00	1.98	35	0	2.86	2.10	13	36.8
6	2.00	2.31	30	0	3.33	2.45	11	36.8
7	3.00	3.47	20	0	5.00	3.68	7	36.8
8	5.00	5.33	13	0	7.69	5.66	5	36.8
9	7.00	6.93	10	0	10.00	7.36	4	36.8
10	10.00	9.90	7	0	14.29	10.51	3	36.8
11	0.50	0.74	225	1	0.89	0.48	61	40.6
12	1.00	1.12	150	1	1.33	0.72	41	40.6
13	1.00	1.24	135	1	1.48	0.80	37	40.6
14	2.00	1.97	85	1	2.35	1.27	23	40.6
15	2.00	2.24	75	1	2.67	1.44	20	40.6
16	3.00	3.05	55	1	3.64	1.97	15	40.6
17	5.00	4.80	35	1	5.71	3.09	9	40.6
18	7.00	6.71	25	1	8.00	4.33	7	40.6
19	10.00	9.87	17	1	11.76	6.37	5	40.6
20	3.00	3.15	85	2	3.53	1.58	19	42.3
21	5.00	4.86	55	2	5.45	2.44	12	42.3
22	7.00	6.69	40	2	7.50	3.36	9	42.3
23	10.00	10.70	25	2	12.00	5.38	6	42.3
24	7.00	6.68	55	3	7.27	2.84	11	43.3
25	10.00	10.49	35	3	11.43	4.47	7	43.3


4.0
3.6
3.2
2.8
2.4
2.0
1.6
1.2
0.8
0.4
0
4
8
12
16
20
24
28
c+1 ->
nD_o
$10m_o n$
$10g_o n^{-1}$

3.4 Methods of randomization in the $\{m_e, D_e\}$ system.

The impossibility of constructing sampling plans with non-integral values of n and c so as to meet the stated stipulations exactly with respect to m_e and D_e is not, in general, a serious problem, because in most of the practical cases, one can be satisfied with nearby plans. However if one wishes to meet the specifications exactly, randomization with respect to two suitable plans of the system should be adopted. Two procedures of carrying out this are considered in this section.

Procedure (1):-

Two successive plans (n_1, c_1) and (n_2, c_2) are selected such that

$$n_1^{-1}(c_1 + 1) = n_2^{-1}(c_2 + 1) = m_e, \quad \text{the given value.}$$

Before inspecting a lot, plans (n_1, c_1) and (n_2, c_2) are chosen with probabilities a and $1-a$ respectively, $0 \leq a \leq 1$. Whatever may be the value of a, the area under the composite OC will be m_e. This is because the composite OC is given by

$$L(\lambda, n_1, c_1, n_2, c_2;\ a) = aL(\lambda n_1, c_1) + (1-a)L(\lambda n_2, c_2)$$

so that by integrating with respect to λ,

$$m(n_1,c_1, n_2,c_2; a) = am(n_1,c_1) + (1-a)m(n_2,c_2) = m_e$$

a should therefore be chosen such that

$$aD(m_e,n_1,c_1) + (1-a)D(m_e,n_2,c_2) = \text{a given number}$$

Since m_e is the point of equal error-areas for the plan (n_1,c_1) as well as for the plan (n_2,c_2), the procedure is equivalent to a linear interpolation between the D_e's of the constituent plans given in Table 3.7.

Examples:-

(i) $m_e = 15$ % and $D_e = 6$ %

(ii) $m_e = 15$ % and $D_e = 10$ %

From Table 3.7

Plan (n,c)	m_e %	D_e %	
$(n_1,c_1) = (20,2)$	15	6.7	with probability a
$(n_2,c_2) = (40,5)$	15	4.8	with probability 1-a

For $a = \frac{12}{19}$ and $1-a = \frac{7}{19}$, we get $D_e = 6$ % . However $D_e = 10$ % cannot be achieved.

Procedure (2):-

A sample of size n is selected and the number of defects in the sample d_n is observed.

The lot is accepted if $d_n \leq c$;

the lot is rejected if $d_n > c+1$;

if $d_n = c+1$, the lot is accepted with probability $1-a$ and rejected with probability a.

This procedure amounts to choosing the plan (n,c) with probability a and the plan $(n,c+1)$ with probability $1-a$.

The composite OC is given by

$$L(\lambda n, c;\ a) = aL(\lambda n, c) + (1-a)L(\lambda n,\ c+1)$$

Therefore

$$m_c = m(n,c;\ a) = a\,\frac{c+1}{n} + (1-a)\,\frac{c+2}{n} = \frac{c+2-a}{n},\quad 0 \leq a \leq 1 \tag{3.52}$$

The total error-area at λ under the composite OC is

$$D(\lambda,n,c;\ a) = aD(\lambda,n,c) + (1-a)D(\lambda,n,c+1)$$

Using (3.23.1) for $D(\lambda,n,c)$ and (3.23.2) for $D(\lambda,n,c+1)$, we get

$$D(\lambda,n,c;\ a) = \lambda\,[1-2L(\lambda n,c)] - m(n,c;a)\,[1-2L(\lambda n,c+1)] \qquad (3.53)$$

which is identical in form to the corresponding formula (3.23) for the non-randomized plan (n,c).

Putting $\lambda = m_e$ in (3.53),

$$D(m_e,n,c;\ a) = 2m_e\ b(m_e n,c+1)$$

Hence

$$nD_e = 2nm_e\ b(m_e n,c+1) = 2nm_e\,\frac{e^{-nm_e}\,(nm_e)^{c+1}}{(c+1)!} \qquad (3.54)$$

in complete analogy with the corresponding formula (3.50) for the non-randomized plan. Note that nm_e is no more equal to c+1; it is given by (3.52) i.e., $nm_e = c+2-a$.

Table 3.9 is an extension of Table 3.7. It gives nD_e for non-integral values of nm_e when $c \leq 10$.

Examples:-

(i) $m_e = 15$ °/. and $D_e = 6$ °/.

(ii) $m_e = 15$ °/. and $D_e = 10$ °/.

Note that $m_e = 15$ °/. $= \frac{2.4}{16} = \frac{2.7}{18} = \frac{3}{20} = \frac{3.3}{22}$ etc. From Table 3.9 we have

n	nm_e	m_e %	D_e %
24	3.6	15	6.4
26	3.9	15	6.0
28	4.2	15	5.8

So $n = 26$ and $nm_e = 3.9$ is the combination that satisfies (i). Since $c+2-a = 3.9$ we have $c=2$ and $a=0.1$. This means selecting the plan (26,2) with probability 0.1 and (26,3) with probability 0.9

Again from Table 3.9

n	nm_e	m_e %	D_e %
8	1.2	15	10.8
10	1.5	15	10.0
12	1.8	15	8.9

So the combination $n = 10$ and $nm_e = 1.5$ satisfies (ii). This means selecting the plans (10,0) and (10,1) with probabilities 0.5 and 0.5.

<u>Remark (1)</u>:-

In procedure (1), one uses plans with different sample sizes and acceptance numbers: one has to choose between them every time a lot is submitted for inspection. In procedure (2) sample size is always the same; one has to choose between the two acceptance numbers c and c+1 only when exactly c+1 defects are observed in the sample. Thus from the operational point of view procedure (2) is superior to procedure (1)

Remark (2):-

By procedure (1), it is not possible to obtain a D_e larger than $D(m_o, n_o, c_o)$ where $m_e = (c_o + 1)/n_o$, expressed in its lowest terms. Thus for $m_e = 15\ \%\ = 3/20$, we cannot have a $D_e > D(15\ \%, 20, 2) = 6.7\ \%$; for $m_e = 7\ \% = 7/100 = 14/200 = \ldots$, the maximum value attainable is only $D(7\ \%, 100, 6) = 2.1\ \%$. So procedure (2) turns out to be better than procedure (1).

TABLE 3.9

Values of $m_e n$ and nD_e for composite Poisson single sampling plans obtained by randomizing between the plans (n,c) and (n,c+1)

$m_e n = a(c+1) + (1-a)(c+2)$ and $nD_e = 2m_e n\ b(m_e n, n+1)$

c	$m_e n$	$n D_e$	c	$m_e n$	$n D_e$	c	$m_e n$	$n D_e$
0	1.00	0.7358	1	2.0	1.0827	3	4.0	1.5629
	1.05	0.7716		2.1	1.1341		4.1	1.6000
	1.10	0.8055		2,2	1.1798		4.2	1.6332
	1.15	0.8375		2.3	1.2198		4.3	1.6622
	1.20	0.8674		2.4	1.2541		4.4	1.6873
	1.25	0.8953		2.5	1.2826		4.5	1.7083
	1.30	0.9212		2.6	1.3054		4.6	1.7253
	1.35	0.9449		2.7	1.3228		4.7	1.7383
	1.40	0.9667		2.8	1.3349		4.8	1.7475
	1.45	0.9864		2.9	1.3420		4.9	1.7529
	1.50	1.0041	2	3.0	1.3443	4	5.0	1.7547
	1.55	1.0199		3.1	1.3868		5.5	1.8854
	1.60	1.0337		3.2	1.4247	5	6.0	1.9275
	1.65	1.0457		3.3	1.4580		6.5	2.0473
	1.70	1.0559		3.4	1.4866	6	7.0	2.0860
	1.75	1.0644		3.5	1.5105		7.5	2.1973
	1.80	1.0711		3.6	1.5298	7	8.0	2.2334
	1.85	1.0763		3.7	1.5445		8.5	2.3376
	1.90	1.0799		3.8	1.5549	8	9.0	2.3716
	1.95	1.0820		3.9	1.5609		9.5	2.4700

3.5 Adjusting plans of the $\{m_e , D_e\}$ system when the inspection technique is not exact

A particular plan can meet the stated specifications only when the inspection technique is exact. By an inspectio technique is meant a practical procedure of finding out, manually or with the aid of mechanical devices, how far an item conforms to the requirements of quality. In practice an inspection technique need not be exact: the number of defects in a sample as revealed by an inspection need not be the same as the actual number of defects in that sample. The defects found out by the inspection will be called 'assumed defects' as compared to the true or actual defects. When counting defects with regard to a characteristic is costly or time-consuming, one may count defects with respect to a related characteristic i.e., deliberately adopt a technique which is not exact and then make adjustments. Even when an inspection technique is intended to be exact, a defect may go unnoticed or a count may be made of a defect when none exists. A diffe rence between the number of assumed and actual defects does effect the OC curve of a plan.

A technique of inspection which finds on an average mor defects than there are in the sample may be called a 'strict technique'. The technique which finds - on an average - fewer defects than there are in the sample may be called a 'liberal technique'.

The number of actual defects in a sample of size n is a Poisson with parameter λn, whereas the number of assumed defects is considered to be a Poisson with parameter $h\lambda n$, h being a positive number. In other words, the probability of counting r defects in a sample of size n is given by $b(h\lambda n, r)$, $h > 0$, $0 \leq \lambda \leq \infty$ and $r = 0,1,\ldots$. Specification of good and bad lots are made on the basis of λ, the average number of actual defects per item in the lot. $h\lambda$ is the average number of assumed defects per item in the lot i.e., defects that will be found in the lot when subjected to the particular inspection technique. The technique is strict, exact or liberal according as h is greater than, equal to or less than one.

We are able to give the following model which leads to a liberal technique of inspection. (Rao in Patil, 1965).

Prob. of counting a defect when it exists $= h$

Prob. of not counting the defect when it exists $= 1-h$

Prob. of counting a defect when none exists $= 0$

and the probability of not counting a defect when none exists $= 1$

Under the model the probability of counting r defect given λ, is

$$\sum_{s=0}^{\infty} b(\lambda n, r+s)\binom{r+s}{s} h^r (1-h)^s$$

$$= b(h\lambda n, r), \qquad 0 < h \leq 1.$$

Suppose the plan (n,c) is chosen so as to satisfy $m(n,c) = (c+1)/n = m_e$ and $D(m_e,n,c) = D_e$ under the assumption that the technique of inspection is exact. It is of interest to know to what extent the point of equal error-areas and also the error-areas at m_e are effected when the technique fails to be exact.

For the plan (n,c), the OC under the exact technique is

$$L(\lambda n,c) = \sum_{0}^{c} b(\lambda n,r)$$

Under the inspection technique-h, the probability of accepting a lot of quality λ is equal to the probability of getting not more than c assumed defects. It is given by

$$L^h(\lambda n,c) = \sum_{0}^{c} b(h\lambda n, c) = L(h\lambda n,c) \qquad (3.55)$$

Here and elsewhere in this section, h is used as a superscript in the notations L^h, m^h, D_1^h etc

The area under the OC L^h is given by

$$m^h(n,c) = \int_0^\infty L(h\lambda n,c)\, d\lambda = h^{-1}\, m(n,c) = \frac{c+1}{nh} \qquad (3.56)$$

The error-areas at $\lambda = \Lambda$ are given by

$$hD_i^h(\Lambda, n, c) = D_i(h\Lambda, n, c), \qquad i = 1,2. \qquad (3.57)$$

The error-areas under L^h at the target value m_e are no more equal to one another:

$$D_1^h(m_e,n,c) - D_2^h(m_e,n,c) = h^{-1}[D_1(hm_e,n,c) - D_2(hm_e,n,c)]$$

$$= (1-\tfrac{1}{h})m_e, \quad \text{using (3.24)} \qquad (3.58)$$

Also

$$D_i^h(m_e,n,c) - D_i(m_e,n,c) = h^{-1}D_i(hm_e,n,c) - D_i(m_e,n,c),$$

$$i = 1,2. \qquad (3.59)$$

Using the inequalities (3.34), we find

$$D_1^h(m_e) = h^{-1}D_1(hm_e) \lesseqgtr D_1(m_e) = D_2(m_e) \lesseqgtr h^{-1}D_2(hm_e)$$
$$= D_2^h(m_e)$$

according as $h \gtrless 1$. (3.60)

As expected, the effect of a liberal technique $(h < 1)$ is to decrease the first kind of error-area (a measure of pro ducer's risk) and to increase the second kind of error-area (a measure of consumer's risk) at the target value m_e while the effect of a strict technique $(h > 1)$ is exactly the opposite.

Further, from (3.58), (3.59) and (3.34), it follows tha the two kinds of error-areas at m_e differ more and more fro one another and also from the target value $D_e/2$ as h devia tes more and more from 1.

With regards to the total error-areas at m_e we have

$$D^h(m_e) = h^{-1}D(hm_e) = m_e(hm_e)^{-1} D(hm_e) \qquad (3.61)$$

From (3.38) we see that as h increases $D^h(m_e)$ decreases an reaches its minimum at $h = h_o$ given by

$$h_o m_e = \Lambda_o (n,c+1) = \Lambda_o' \qquad (3.62)$$

and then increases. Since $m_e < A_c''$ (3.19), h_o is always greater than unity. It follows that there are two values $h = 1 < h_o$ and $h = h_o' > h_o$ at which $D^h(m_e)$ equals the target value D_e. (See Chart 3.4)

$$D^h(m_e) \begin{cases} > D_e & \text{if } h < 1 \text{ or } h > h_o' \\ = D_e & \text{if } h = 1 \text{ or } h = h_o' \\ < D_e & \text{if } 1 < h < h_o' \end{cases} \qquad (3.63)$$

The above conclusions are illustrated with the help of two examples:

(i) $n = 20$ and $c = 1$ so that $m_e = 10$ % and $D_e = 5.4$ %

and (ii) $n = 500$ and $c = 9$ so that $m_e = 2$ % and $D_e = \frac{1}{2}$ %

For selected values of h, the error-areas under L^h at m_e are calculated. The results are presented in Tables 3.10 for example (i) and 3.11 for example (ii). The error-areas shown in Table 3.10 are also exhibited in Chart 3.4.

TABLE 3.10

The effect of technique - h on the error-areas of the plan

$n = 20$ and $c = 1$:,

$m_e = m(n,c) = 10$ °/. and $D_e = D(m_e,n,c) = 5.4$ °/.

h	at the target value 10 °/. D_1^h (10 °/.) °/.	D_2^h (10 °/.) °/.	D^h. (10 °/.) °/.	m^h °/.	$D^h(m^h)$ °/.
0.50	1.04	11.04	12.07	20.00	10.83
0.60	1.26	7.92	9.18	16.67	9.02
0.70	1.70	5.99	7.69	14.29	7.73
0.75	1.87	5.21	7.08	13.33	7.22
0.80	2.04	4.54	6.59	12.50	6.77
0.90	2.38	3.49	5.87	11.11	6.02
0.95	2.54	3.07	5.62	10.53	5.70
1.00	2.71	2.71	5.41	10.00	5.41
1.10	3.02	2.12	5.14	9.09	4.92
1.20	3.33	1.66	4.99	8.33	4.51
1.25	3.48	1.48	4.96	8.00	4.33
1.30	3.62	1.31	4.94	7.69	4.16
1.40	3.90	1.04	4.94	7.14	3.87
1.50	4.16	0.83	4.99	6.67	3.61
1.60	4.41	0.66	5.07	6.25	3.38
1.70	4.65	0.53	5.18	5.88	3.18
1.80	4.87	0.43	5.29	5.56	3.01
1.90	5.08	0.34	5.42	5.26	2.85
2.00	5.27	0.27	5.55	5.00	2.71

TABLE 3.11

The effect of technique - h on the error-areas of the plan
$n = 500$ and $c = 9$;

$m_e = m(n,c) = 2\ \%$ and $D_e = D(m_e,n,c) = 0.5\ \%$

h	at the target value 2 % $D_1^h(2\ \%)$ %	$D_2^h(2\ \%)$ %	$D^h(2\ \%)$ %	m^h %	$D^h(m^h)$ %
$\frac{1}{2}$	0.01	2.00	2.01	4.00	1.00
$\frac{3}{4}$	0.08	0.75	0.83	1.50	0.67
$\frac{4}{5}$	0.16	0.61	0.76	2.50	0.67
$\frac{9}{10}$	0.17	0.39	0.57	2.22	0.56
1	0.25	0.25	0.50	2.00	0.50
$\frac{6}{5}$	0.43	0.09	0.52	1.67	0.42
$\frac{3}{2}$	0.68	0.02	0.70	1.33	0.33
$\frac{9}{5}$	0.89	0.00	0.89	1.11	0.28
2	1.00	0.00	1.00	1.00	0.25

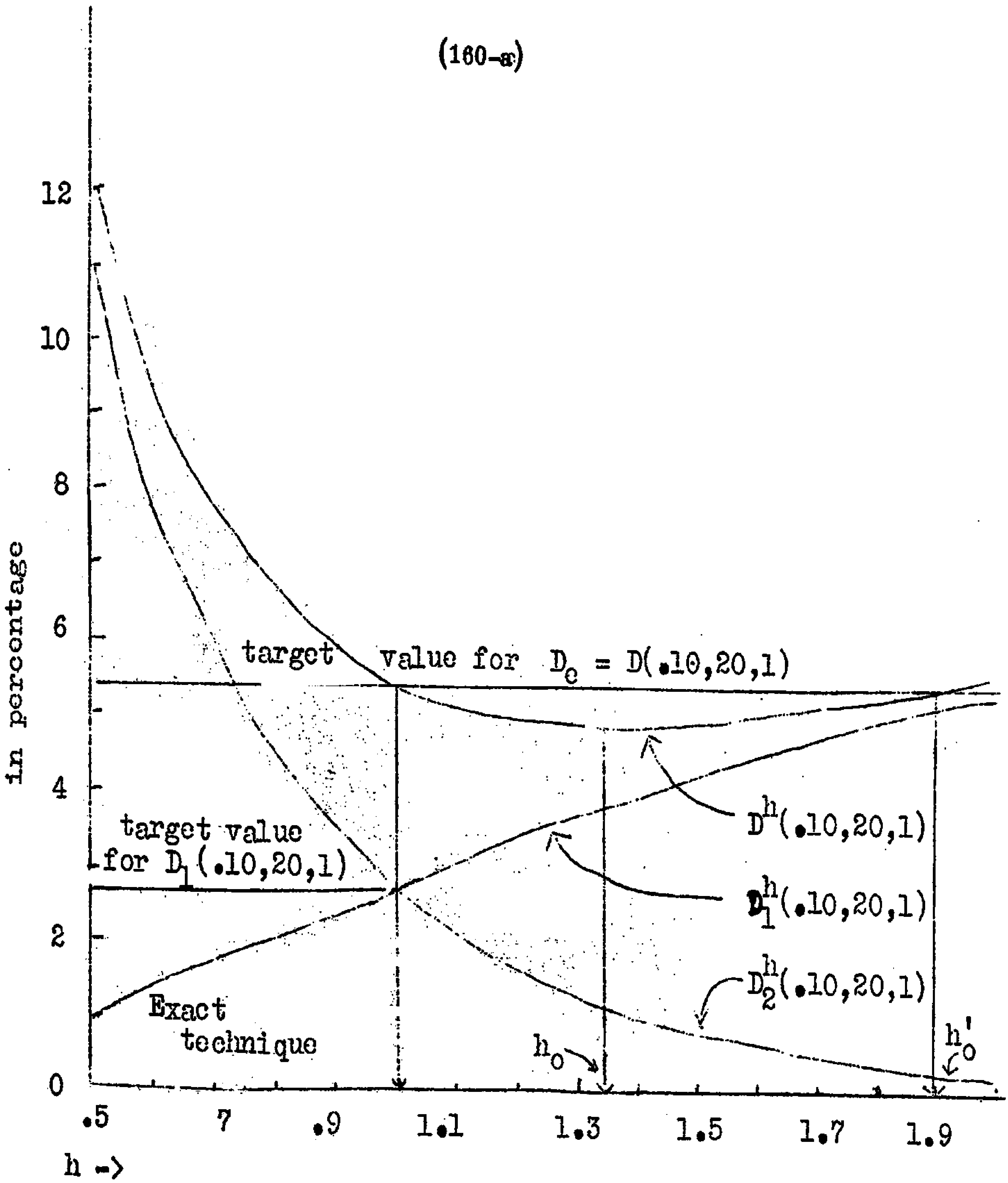


CHART 3.4: EFFECT OF TECHNIQUE-h ON THE PLAN (20,1)

The effect of h on the error-areas at m_e are negligible for small values of $|h-1|$ (< 0.1 say) If h is known or can be determined by experiments, plans can be adjusted so as to suit the initial specifications of m_e and D_e. In fac the plan $(h^{-1}n,c)$ under the technique-h is equal to the pla (n,c) under the exact technique. Thus

(25,1) with $h = 0.8$ is equal to (20,1) with $h = 1$;

(400,9) with $h = 5/4$ is equal to (500,9) with $h = 1$; etc.

For a liberal technique ($h < 1$) the sample size has to be increased by a factor h^{-1}. So theoretically, modified plans (ex: (25,1) with 0.8 instead of (20,1) with $h = 1$) can be adopted even when exact technique is possible provided that the savings due to the use of liberal technique are considerably larger than the additional cost due to an increase in sample size

Reduced (or tightened) inspection can be installed by shifting to a different plan with the same technique of inspection (Sections 1.7 and 3.3) or by keeping the same plan but using a liberal (or strict technique):

Inspection	plan n	c	technique h =	m^h %	$D^h(m^h)$ %
Reduced	16	1	1.00	12.5	6.8
	20	1	0.80		
Normal	20	1	1.00	10.0	5.4
Tightened	20	1	1.25	8.0	4.3
	25	1	1.00		

If h^{-1} is not an integer, it can be rounded up (m and D(m) will be slightly less than the intended values) or down (m and D(m) will be slightly more than the intended values) to the nearest integer.

Finally, it may be observed that the method of adjusting a plan by changing n to $h^{-1}n$ is valid for weighted error-areas also.

3.6 Some other points connected with the error-areas

3.6a: The point m_∂ and the condition (A_∂):-

In analogy with the point of equal error-areas m, we may define the point $m_\partial = m_\partial(n,c)$ by the equation $D_1(m_\partial,n,c) = \partial D_2(m_\partial,n,c)$, $\partial > 0$. As in Theorem 2.5(d) we find that

$$\min_{\lambda} \max [D_1(\lambda,n,c), \; \partial \, D_2(\lambda,n,c)]$$

$$= D_1(m_\partial,n,c) = \partial D_2(m_\partial,n,c)$$

$$= \min_{c'} \max [D_1(m_\partial,n,c'), \; \partial D_2(m_\partial,n,c')] \qquad (3.64)$$

It follows that given n and $\lambda = \Lambda$, the optimum value of c for which the bigger of $D_1(\Lambda, n,c)$ and $\partial D_2(\Lambda,n,c)$ is a minimum with respect to c (Condition (A_∂) of Section 2.6) should satisfy the relation: $m_\partial(n,c) = \Lambda$

3.6b: <u>The point $\Lambda_{o\partial}$ and the condition (B_∂):</u>

The point $_{o\partial} = \Lambda_{o\partial}(n,c)$ is defined by the equation

$$L(\Lambda_{o\partial}\, n,c) = (1+\partial)^{-1}, \quad \partial > 0 \qquad (3.65)$$

Note that $\Lambda_{o\partial} > \Lambda_{o\partial}$ if $\partial > \partial'$ and $\Lambda_{o1} = \Lambda_o$, the IQL $2n \quad \Lambda_{o\partial}$ is the upper $(1+\partial)^{-1}$ - fractile of a χ^2-distribution with $2(c+1)$ degrees of freedom.

For a given plan (n,c), $D_1(\lambda,n,c) + \partial D_2(\lambda,n,c)$ considered as a function of λ is a minimum at $\lambda = \Lambda_{o\partial}$. This property holds in the case of weighted error-areas also. (3.66

$\Lambda_{o\partial}$ can be considered as an AQL or a LTPD value according as $\partial < 1$ or > 1. For example, when $\partial = 1/19$, $\Lambda_{o\partial} =$ AQL (.95) and when $\partial = 9$, $\Lambda_{o\partial}$ = LTPD (.10). Thus for a given plan (n,c), AQL (.95), is the quality level at which $19\, D_1(\lambda) + D_2(\lambda)$ is a minimum and LTPD (.10) is the level at which $D_1(\lambda) + 9\, D_2(\lambda)$ is a minimum.

The error-areas at $\Lambda_{o\partial}$ are given by the following simple formulae (3.67)

$$D_1(\Lambda_{o\partial}, n, c) = mb(\Lambda_{o\partial}\, n, c+1) + \partial(1+\partial)^{-1}(\Lambda_{o\partial} - m)$$

$$D_2(\Lambda_{o\partial}, n, c) = mb\,(\Lambda_{o\partial}\, n, c+1) + (1+\partial)^{-1}(m - \Lambda_{o\partial})$$

and therefore

$$D_1(\Lambda_{o\partial}, n, c) + \partial D_2(\Lambda_{o\partial}, n, c) = m(1+\partial)b(\Lambda_{o\partial}\, n, c+1).$$

Given n and $\lambda = \Lambda$, $D_1(\Lambda, n, c) + \partial D_2(\Lambda, n, c)$ considered as a function of c is a minimum (Condition (B_∂) of Section 2.7) at c given by

$$L(\Lambda n\,,\, c+1) > (1+\partial)^{-1} \geq L(\Lambda\, n, c)$$

i.e., the optimum c should satisfy

$$\Lambda_{o\partial}\,(n, c+1) > \Lambda \geq \Lambda_{o\partial}\,(n, c) \qquad (3.68)$$

This is a generalization of the condition (3.42) given in Section 3.2c for $\partial = 1$ and can be proved in the same way as (3.42).

3.7 The EQL and the weighted error-areas

3.7a: The EQL:-

The c.d.f. and the p.d.f. of the prior distribution of λ are denoted by $F(\lambda)$ and $f(\lambda)$ respectively. For a given plan (n,c) the area under the weighted OC, $A_F(n,c)$; the EQL, $\Lambda_{eF}(n,c)$; the weighted error-areas at the point $\lambda = \Lambda$ i.e., $J_{1F}(\Lambda, n, c)$, $J_{2F}(\Lambda, n, c)$ and $J_F(\Lambda, n, c)$ are all defined as in Sections 1.3 and 2.1. As usual the suffix F and the symbols (n,c) are sometimes dropped from the notations for the sake of convenience.

We have

$$A_F = F(\Lambda_e) = F(m) + R \tag{3.69}$$

and

$$|R| \leq \begin{cases} \frac{1}{2n} Mm & \text{if Theorem 1.1 is applicable} \\ \frac{1}{2} M^* D(m) & \text{if Theorem 1.2 is applicable} \end{cases}$$

where

$$m = \frac{c+1}{n} \; ; \quad D(m) = 2mb(mn, c+1);$$

$$M = \sup_{0 \leq \lambda < \infty} \left| \frac{d^2 F(\lambda)}{d\lambda^2} \right| \quad \text{and} \quad M^* = -\sup_{0 \leq \lambda < \infty} f(\lambda)$$

Note that the values of $D(m)$ are shown in Table 3.7.

On substituting the expressions for the central moments of ξ as given in Section 3.1b the relation (1.20) yields

$$F(\Lambda_e) = F(m) + \frac{c+1}{2n^2} F_2(m) + \frac{c+1}{3n^3} F_3(m)$$

$$+ \frac{(c+1)(c+3)}{8n^4} F_4(m) + \ldots \qquad (3.70)$$

where $F_j(\lambda)$ is the j-th derivative of $F(\lambda)$.

The relation (1.22) gives an asymptotic expansion of Λ_e in terms of n and c. If m is kept fixed, the expansion can be written as

$$\Lambda_e = m + \frac{m}{2n} \frac{F_2(m)}{F_1(m)} + \frac{m}{3n^2} \frac{F_3(m)}{F_1(m)}$$

$$+ \frac{1}{8} \frac{m^2}{n^2} \left[\frac{F_4(m)}{F_1(m)} - \left\{ \frac{F_2(m)}{F_1(m)} \right\}^3 \right] + 0(n^{-3}) \qquad (3.71)$$

3.7b: The weighted error-areas:-

Substituting the expressions for the incomplete central moments of ξ as given in (3.11) and (3.12) of Section 3.1b, the relations (2.48) and (2.49) of Theorem 2.4b (Section 2.4c) give asymptotic expansions for the weighted error-areas at m in terms of n and c. However when m is kept fixed

$$b(mn; c+1) = D(m)/2m = O(n^{-1/2}) \qquad \text{by (3.45)}$$

Therefore the expansions take the form

$$\begin{aligned} J_2(m,n,c) &= \frac{1}{2} D(m)f(m) + \frac{m}{2n} L(mn,c+1)F_2(m) \\ &+ \frac{m}{6n} D(m) F_3(m) + \frac{m}{3n^2} L(mn,c+1) F_3(m) \\ &+ \frac{\bar{3m}^2}{8n^2} L(mn,c+1) F_4(m) + O(n^{-5/2}) \end{aligned} \qquad (3.72)$$

and

$$\begin{aligned} J(m,n,c) &= D(m)f(m) + \frac{\bar{m}}{2n} [2L(mn,c+1) - 1 \quad F_2(m)] \\ &+ \frac{m}{3n} D(m) F_3(m) + \frac{m}{3n^2} [2L(mn,c+1) - 1] F_3(m) \\ &+ \frac{m^2}{8n^2} [6L(mn,c+1) - 1] F_4(m) + O(n^{-5/2}) \end{aligned} \qquad (3.73)$$

3.7c: Alternative formulae:-

Suppose $F(\lambda)$ can be expanded in a Taylor series around the origin $\lambda = 0$ i.e.,

$$F(\lambda) = \sum_{r=1} \frac{\lambda^r}{r!} F_r(0) \tag{3.74}$$

Expanding $F(\xi)$ in (1.14) of Lemma 1.1 around the origin and noting that $E_G \xi^r = (c+r)!/c!$ - cf. (3.7) - we obtain an alternative expansion for $F(\Lambda_c)$:

$$F(\Lambda_c) = \sum_{r=1} \binom{c+r}{c} \frac{F_r(0)}{n^r} = \sum_{r=1} \frac{\Lambda_c^r}{r!} F_r(0) \tag{3.75}$$

Expanding $F(\xi)$ in (2.35) of Lemma 2.3 around the origin and using (3.7) we also get alternative expansions for the weighted error-areas.

$$J_2(\Lambda,n,c) = -L(\Lambda n,c)F(\Lambda) + \sum_{r=1} \binom{c+r}{c} \frac{F_r(0)}{n^r} L(\Lambda n,c+r) \tag{3.76.1}$$

$$= -L(\Lambda n,c-1)F(\Lambda) + \sum_{r=1} \binom{c+r}{c} \frac{F_r(0)}{n^r} L(\Lambda n,c+r-1) \tag{3.76.2}$$

The second relation (3.76.2) follows from the first because of (3.74). In particular, the error-area at the EQL is obtained as (3.77)

$$J_2(\Lambda_c,n,c) = \sum_{r=1} \binom{c+r}{c} \frac{F_r(0)}{n^r} [L(\Lambda_c n,c+r) - L(\Lambda_c n,c)]$$

$$= \sum_{r=1} \binom{c+r}{c} \frac{F_r(0)}{n^r} [L(\Lambda_c n,c+r-1) - L(\Lambda_c n,c-1)]$$

3.8 The case of a Gamma PC

3.8a: The EQL:-

The p.d.f. of the prior distribution of λ is

$$f(\lambda) = \frac{e^{-\alpha\lambda}\,\lambda^{v}\,\alpha^{v+1}}{\Gamma(v+1)}, \quad \alpha > 0,\ v > 0,\ 0 \leq \lambda < \infty \qquad (3.78)$$

Whether v is an integer or not, we will use the notation in (3.2) and write the c.d.f. as

$$F(\lambda) = 1 - L(\alpha\lambda, v) \qquad (3.79)$$

The mean and the variance of the distribution are $(v+1)\alpha^{-1}$ and $(v+1)\alpha^{-2}$ respectively.

Since $v > 0$, $f(\lambda)$ is a maximum at $\lambda = v\alpha^{-1}$. Therefor the bound M^* in (3.69) can be taken as

$$M^* = \frac{\alpha e^{-v}\,v^{v}}{\Gamma(v+1)}, \quad \alpha > 0,\ v > 0 \qquad (3.80)$$

Values of $M^*\alpha^{-1}$ are shown in Table 3.12 for $v = 0(.1).9$.

If $v \geq 1$, $\left|\frac{d^2F(\lambda)}{d\lambda^2}\right|$ is bounded and Theorem 1.1 is applicable.

$$F_2(\lambda) = \frac{d^2F(\lambda)}{d\lambda^2} = \frac{\alpha^2 e^{-\alpha\lambda} (\alpha\lambda)^{v-1} (v-\alpha\lambda)}{\Gamma(v+1)}$$

$F_2(\lambda)$ is a maximum at $\lambda = (v-\sqrt{v})\,\alpha^{-1}$. Therefore the bound M in (3.69) is

$$M = \left| F_2\left(\frac{v-\sqrt{v}}{\alpha}\right) \right| \qquad (3.81).$$

values of $M\alpha^{-2}$ are given in Table 3.12 below for selected values of v.

TABLE 3.12

Values of $M\alpha^{-1}$ and $M^*\alpha^{-2}$ in Theorems 1.1 and 1.2 for a Gamma PC with parameters α and v+1.

v	$M^*\alpha^{-1}$	v	$M\alpha^{-2}$	v	$M\alpha^{-2}$
0.0	1.000	1.0	1.000	2	0.231
0.1	0.755	1.1	0.707	3	0.131
0.2	0.646	1.2	0.570	4	0.090
0.3	0.575	1.3	0.481	5	0.069
0.4	0.533	1.4	0.416	6	0.055
0.5	0.484	1.5	0.367		
0.6	0.452			8	0.039
0.7	0.426			10	0.031
0.8	0.404				
0.9	0.384				

Note: Values of $M\alpha^{-2}$ are taken from Mitra and Subrahmanya (1968).

3.8b: Exact formulae for the EQL and the weighted error-areas:.

The error-area at $\lambda = \Lambda$ is given by

$$J_2(\Lambda,n,c) = \int_{\Lambda}^{\infty} \sum_{r=o}^{c} b(\lambda n,r)f(\lambda)d\lambda$$

$$= \sum_{s=o}^{c} \frac{\Gamma(r+v+1)}{\Gamma(r+1)\Gamma(v+1)} q^r p^{v+1} L(\Lambda n(1+u),v+r) \tag{3.82}$$

where

$$u = \frac{\bar{c}}{n}\,; \quad p = \frac{u}{1+u} \quad \text{and} \quad q = 1-p \tag{3.83}$$

Hence

$$A_F = J_2(0,n,c) = \sum_{r=o}^{c} \frac{\Gamma(r+v+1)}{\Gamma(r+1)\Gamma(v+1)} q^r p^{v+1}$$

$$= \int_{o}^{p} \frac{x^v\,(1-x)^c}{\beta(v+1,c+1)}\,dx = \gamma \text{ say} \tag{3.84}$$

The EQL Λ_e is therefore given by

$$F(\Lambda_e) = \int_{o}^{\alpha\Lambda_e} \frac{e^{-x}\,x^v}{\Gamma(v+1)} = \gamma \tag{3.85}$$

$2\alpha\Lambda_e$ is the γ-fractile of a χ^2-distribution with 2γ degrees of freedom. γ is the lower tail area of a Beta distribution with parameters $v+1$ and $c+1$. Tables of χ^2 (or Gamma) and

Beta distributions are given in Owen (1962), Pearson (1922) and(1934), Pearson and Hartley (1957) etc.

If v is an integer, we have (cf. page 264 in Owen 1962)

$$\gamma = \sum_{r=0}^{c} \binom{r+v}{v} q^r p^{v+1} = \sum_{v+1}^{c+v+1} \binom{c+v+1}{r} p^r q^{c+v+1-r} \qquad (3.86)$$

γ can now be obtained from the tables of cumulative binomial distribution (Harvard, 1955).

When v is an integer, we can derive alternative formulae for the error-areas.

$$J_2(\Lambda, n, c) = \int_{\Lambda}^{\infty} L(\lambda n, c)\, f(\lambda)\, d\lambda$$

$$= - L(\Lambda n, c) F(\Lambda) + \int_{\Lambda}^{\infty} \frac{e^{-n\lambda}(n\lambda)^c}{c!} F(\lambda)\, d\lambda .$$

Since

$$F(\lambda) = 1 - L(\alpha\lambda, v) = 1 - \sum_{r=0}^{v} b(\alpha\lambda, v)$$

we get

$$J_2(\Lambda, n, c) = L(\Lambda n, c) L(\alpha \Lambda, v)$$

$$- q^{c+1} \sum_{r=0}^{v} \binom{c+r}{c} p^r L(\Lambda(n+\alpha), c+r) \qquad (3.87)$$

In particular, the error-area at the EQL can be given as

$$J(\Lambda_e,n,c) = 2q^{c+1} \sum_{r=0}^{v} \binom{c+r}{c} p^r [\, L(\Lambda_e n,c) - L(\Lambda_e\, n(1+u), c+r) \,] \qquad (3.88)$$

Table 3.13 gives $F(\Lambda_e)$ and $F(m)$, $m = (c+1)n^{-1}$ for selected values of $u = \omega n^{-1}$, v and c.

3.8c: Condition (B_∂):-

The average probability of getting r defects for lots of quality worse than Λ is given by

$$\bar{b}_b(\Lambda n,r) = \int_{\Lambda}^{\infty} b(\lambda n, r) f(\lambda)\, d\lambda$$

$$= \bar{b}(n,r) L(\Lambda n(1+u), v+r) \qquad (3.89)$$

where

$$\bar{b}(n,r) = \frac{\Gamma(r+v+1)}{\Gamma(r+1)\,\Gamma(v+1)}\, q^r p^{v+1}$$

Therefore from (2.66) of Theorem 2.6, the condition (B_∂) leads to the relationship

$$L(\Lambda n(1+u), c+v+1) > (1+\partial)^{-1} \geq L(\Lambda n(1+u), c+v) \qquad (3.90)$$

When $\partial = 1$ - that is, the condition (B) of minimizing $J(\Lambda, n, c)$ for given values of Λ and n -

$$\Lambda_0 (n + \alpha, c+v+1) > \Lambda \geq \Lambda_0 (n + \alpha, c + v)$$

or approximately

$$\frac{c + v + 1 + \frac{2}{3}}{n + \alpha} > \Lambda \geq \frac{c + v + \frac{2}{3}}{n + \alpha} \qquad (3.91)$$

This relationship is asymptotically an IQL relationship.

TABLE 3.13

Values of. F(m) and $F(\Lambda_e)$ for Poisson single sampling plans. (n,c) when λ follows a Gamma distribution with parameters α and v+1

$\alpha n^{-1} = 0.2$

c	v = 1 F(m) %	v = 1 $F(\Lambda_e)$ %	v = 2 F(m) %	v = 2 $F(\Lambda_e)$ %	v = 3 F(m) %	v = 3 $F(\Lambda_e)$ %	v = 4 F(m) %	v = 4 F(Λ
0	98.2	97.2	99.9	99.5	100.0	99.9	100.0	10
1	93.8	92.6	99.2	98.4	99.9	99.7	100.0	9
2	87.8	86.8	97.7	96.5	99.7	99.1	100.0	9
3	80.9	80.4	95.3	93.8	99.1	98.2	99.9	9
4	73.6	73.7	92.0	90.4	98.1	96.9	99.6	9
6	59.2	60.5	83.3	82.2	94.6	93.0	98.6	9
8	46.3	48.5	73.1	72.7	89.1	87.5	96.4	9
10	35.4	38.1	62.3	62.8	81.9	80.6	92.8	9
15	17.1	19.8	38.0	40.3	60.3	60.7	78.1	7
20	7.8	9.8	21.0	23.7	39.5	41.6	59.0	5

v = 10 c	v = 10 F(m) %	v = 10 $F(\Lambda_e)$ %	v = 20 c	v = 20 F(m) %	v = 20 $F(\Lambda_e)$ %	v = 30 c	v = 30 F(m) %	v = 30 $F(\Lambda_e)$ %
0	100.0	100.0	0	100.0	100.0	0	100.0	100.0
19	99.7	99.3	39	100.0	100.0	69	100.0	100.0
29	95.7	94.2	49	99.8	99.6	79	99.9	99.8
39	81.6	79.9	59	98.8	98.0	89	99.7	99.3
49	58.3	58.3	69	95.2	93.6	99	98.7	97.8
			79	86.8	84.8	109	95.9	94.4
			89	73.1	71.7	119	90.4	88.4
			99	55.9	55.9	129	81.3	79.4
						139	69.0	67.9
						149	54.8	54.9

TABLE 3.13 (contd.)

$\alpha n^{-1} = 0.5$

v = 1		v = 2		v = 3		v = 4	
F(m) %	$F(\Lambda_e)$ %	F(m) %	$F(\Lambda_e)$ %	F(m) %	$F(\Lambda_e)$ %	F(m) %	$F(\Lambda_e)$ %
91.0	88.9	98.6	96.3	99.8	98.8	100.0	99.6
73.6	74.1	92.0	88.9	98.1	95.5	99.6	98.2
55.8	59.3	80.9	79.0	93.4	90.0	98.1	95.5
40.6	46.1	67.7	68.0	85.7	82.7	94.7	91.2
		54.4	57.1	75.8	74.1	89.1	85.5
		42.3	46.8	64.7	65.0	81.5	78.7
				53.7	55.9	72.5	71.1
				43.3	47.3	62.9	63.2
						53.2	55.2
						44.0	47.6

v = 5		v = 10		v = 15		v = 20	
F(m) %	$F(\Lambda_e)$ %	F(m) %	$F(\Lambda_e)$ %	F(m) %	$F(\Lambda_e)$ %	F(m) %	$F(\Lambda_e)$ %
100.0	99.9	100.0	100.0	100.0	100.0	100.0	100.0
91.6	87.8	100.0	99.6	100.0	100.0	100.0	100.0
52.9	54.7	97.5	94.4	100.0	99.7	100.0	100.0
19.1	24.9	81.6	78.0	99.2	97.3	100.0	99.9
		58.3	58.5	95.1	91.3	99.8	99.0
		17.6	26.1	66.9	65.3	95.2	93.0
				28.7	29.9	73.1	70.2
						38.7	41.8

3.9 The case of an exponential PC

The p.d.f. and the c.d.f. of an exponential. PC. are given by

$$f(\lambda) = \alpha e^{-\alpha\lambda} \quad \text{and} \quad F(\lambda) = 1 - e^{-\alpha\lambda} ;$$

$$\alpha > 0, \; 0 \leq \lambda < \infty \tag{3.92}$$

The formula for the EQL and the error-areas are obtained by putting $v = 0$ in those corresponding to a Gamma PC given in the previous section.

Writing $u = \alpha n^{-1}$ and putting $v = 0$ in (8.84), we see that

$$F(\Lambda_e) = 1 - (1 + u)^{-(c+1)} \tag{3.93}$$

and therefore

$$_e = mu^{-1} \log (1 + u), \qquad m = (c+1)n^{-1} \tag{3.94}$$

The error-area of the second kind at the point $\lambda = \Lambda$ i

$$J_2(\Lambda, n, c) = L(\Lambda n, c)e^{-\Lambda nu} - L(\Lambda n(1+u), c)e^{-\Lambda_e nu} \tag{3.95.1}$$

$$= L(\Lambda n, c+1)e^{-\Lambda nu} - L(n(1+u), c+1)e^{-\Lambda_e nu} \tag{3.95.2}$$

The first relation (3.95.1) is obtained by putting $v=0$ in (3.87). The second relation (3.95.2) follows from the first because of (3.94).

The second relation (3.95.2) is also valid for the error-area $J_2(\Lambda, n, c, c+1; a)$ of the composite plan obtained by randomizing between (n,c) and $(n,c+1)$ (Procedure (2) dealt with in Section 3.4) provided that Λ_e in (3.95.2) is taken as $\Lambda_e(n,c,c+1;\ a)$, the EQL for the composite plan i.e.,

$$F(\Lambda_e(n,c,c+1;\ a)) = aF(\Lambda_e(n,c)) + (1-a)F(\Lambda_e(n,c+1)$$

i.e.,

$$\exp\left\{-nu\Lambda_e(n,c,c+1;\ a)\right\} = a(1+u)^{-(c+1)} + (1-a)(1+u)^{-(c+2)} \tag{3.96}$$

The error-area at the EQL is given by

$$J(\Lambda_e, n, c) = 2\left\{L(\Lambda_e n, c+1) - L(\Lambda_e n(1+u), c+1)\right\} e^{-\Lambda_e nu} \tag{3.97}$$

It can be seen that the values of $F(m)$ and $D(m)f(m)$ can be taken as approximations to those of $F(\Lambda_e)$ and $J(\Lambda_e)$ and that the approximations are good for small values of αn^{-1}. The exact values of Λ_e and $J(\Lambda_e)$ were computed to five significant figures on an IBM 1620 computer belonging to the University of Ghana for $c = 0(1)25$ and various values of αn^{-1}. Typical values are shown in Table 3.14. Corresponding values of $F(m)$ and $D(m)f(m)$ are also given for the purpose of comparisons.

TABLE 3.14

The EQL and the error-area at the EQL **for** Poisson single sampling plans (n,c) when λ follows an exponential distribution with density $f(\lambda) = \alpha \exp(-\alpha\lambda)$

(m and D(m) refer to the unweighted OC)

αn^{-1}	$mn = c+1$	$\Lambda_e n$	F(m) %	$F(\Lambda_e)$ %	D(m)f(m) %	$J(\Lambda_e)$ %
0.05	1	0.98	4.9	4.8	3.5	3.4
	5	0.49	22.1	21.7	6.8	6.7
	10	9.76	39.3	38.6	7.6	7.5
	15	14.64	52.8	51.9	7.3	7.2
	20	19.52	63.2	62.3	6.5	6.5
	25	24.40	71.3	70.5	5.7	5.7
0.10	1	0.95	9.5	9.1	6.7	6.4
	5	4.77	39.3	37.9	10.6	10.4
	10	9.53	63.2	61.4	9.2	9.2
	15	14.30	77.7	76.1	6.9	7.0
	20	19.06	86.5	85.1	4.8	5.0
	25	23.83	91.8	90.8	3.3	3.5
0.25	1	0.89	22.1	20.0	14.3	13.1
	5	4.46	71.3	67.2	12.6	12.7
	10	8.93	91.8	89.3	5.1	5.9
	15	13.39	97.7	96.5	1.8	2.3
	20	17.85	99.3	98.8	0.6	0.9
0.50	1	0.81	39.3	33.3	22.3	19.8
	2	1.62	63.2	55.6	19.9	19.2
	3	2.43	77.7	70.4	15.0	15.8
	4	3.24	86.5	80.2	10.6	12.2
	6	4.87	95.0	91.2	4.8	6.6
	8	6.49	98.2	96.1	2.0	3.3
	10	8.11	99.3	98.3	0.8	1.6
	12	9.73	99.8	99.2	0.3	0.8

CHAPTER 4

BINOMIAL SINGLE SAMPLING OC CURVES

4.1 The binomial OC and the error-areas

4.1a: The sampling procedure:-

A random sample of size n is drawn from the lot submitted for inspection and the number of defectives in the sample d_n is obtained. d_n is the decision variable. The lot is accepted if $d_n \leq c$ where c is an acceptance number, $0 \leq c \leq n$. The lot is not accepted if $d_n > c$. The plan is determined by the two elements n and c. The plan itself ca be denoted by (n,c).

The lot quality is measured by p, the proportion of defectives in the lot, $0 \leq p \leq 1$. Given p, d_n follows a binomial distribution with parameters n and p, if the sampling is done with replacement. d_n is a hypergeometric variable in the case of sampling without replacement. When the lot size N is large compared to the sample size (say $n/N < 0.1$), the hypergeometric can be approximated by the binomial (Cowden 1960). In this chapter we will consider d_n to be distributed as a binomial variable.

An OC curve under the condition of a binomial distribution for the decision variable is termed the binomial OC curve. The properties of binomial OC curves - the definitions and the behaviour of the error-areas - are essentially the same as those of Poisson and will not be discussed in detail. However the exact formulae are different and will be given mostly without explanations.

Throughout this chapter, the symbols p and π play the same role as λ and Λ in Chapter 3 or w and W in the first two Chapters.

4.1b: <u>The binomial OC</u> :-

$$P\left\{d_n = r, \text{ given } p\right\} = \binom{n}{r} p^r (1-p)^{n-r}, \quad r = 0,1,\ldots,n$$

$$= b(p,n,r), \text{ say} \qquad (4.1)$$

$$L(p,n,c) = \sum_{r=0}^{c} b(p,n,r) = \int_p^1 \frac{x^c (1-x)^{n-c-1}}{\beta(c+1,n-c)}\, dx$$

$$= P\left\{\xi > p\right\} \qquad (4.2)$$

where

$$\beta(a_1, a_2) = \frac{\Gamma(a_1)\,\Gamma(a_2)}{\Gamma(a_1 + a_2)}$$

and ξ is a random variable following a Beta distribution with parameters $c+1$ and $n-c$. The p.d.f. and c.d.f. of ξ are

$$g(x,n,c) = \frac{x^c(1-x)^{n-c-1}}{\beta(c+1,\ n-c)}, \quad 0 \le x \le 1$$

and

$$G(x,\ n,\ c) = 1 - L(x,\ n,\ c).$$

Note that $(c+1)^{-1}(1-\xi)^{-1}(n-c)\xi$ follows a 'F' (variance ratio) distribution with $2(c+1)$ and $2(n-c)$ as the degrees of freedom of numerator and denominator respectively. (4.4)

Unlike Poisson OCs which are functions of only two parameters λn and c, the binomial OCs are functions of three parameters p, n and c. Therefore a full tabulation of the OCs and the error-areas requires tables of triple entry.

The following relations can be used to simplify formulae in certain cases. (Harvard, 1955)

$$L(p,n,c) - L(p,n+1,c) = pb(p,n,c) \qquad (4.5.1)$$

$$L(p,n+1,c+1) - L(p,n,c) = (1-p)b(p,n,c+1) \qquad (4.5.2)$$

$$(c+1)b(p,n+1,c+1) = (n+1)pb(p,n,c) \qquad (4.5.3)$$

and $(c+1)(1-p)b(p,n,c+1) = (n-c)pb(p,n,c)$ (4.5.4)

Note that

$$L(p,n,c+1) > L(p,n+1,c+1) > L(p,n,c) > L(p,n+1,c)$$

(4.6)

For the random variable ξ we have (4.7)

the mean, $m(n,c)$ or $m = \frac{c+1}{n+1}$

the variance, $V = \frac{(c+1)(n-c)}{(n+1)^2(n+2)}$

the third central moment, $\mu_3 = \frac{2(c+1)(n-c)(n-2c-1)}{(n+1)^3(n+2)(n+3)}$

and the fourth central moment,

$$\mu_4 = \frac{3(c+1)(n-c)\,[\,(n-c)(nc-3c+3n-1) + 2(c+1)^2\,]}{(n+1)^4(n+2)(n+3)(n+4)}$$

The incomplete moments about the origin are (4.8)

$$E_G\, \epsilon_\pi\, \xi^r = \frac{(c+r)!}{r!} \frac{n!}{(n+r)!} \,[\,1 - L(\pi,n+r,c+r)\,]$$

and

$$E_G(1-\epsilon_\pi)\xi^r = \frac{(c+r)!}{r!} \frac{n!}{(n+r)!} L(\pi,n+r,c+r)$$

where ϵ_π is 1 or 0 according as $\xi \leq \pi$ or $> \pi$. Hence

$$E_G\ \epsilon_m(m-\xi) = E_G(1-\epsilon_m)(\xi - m)$$

$$= m[L(m,n+1,c+1) - L(m,n,c)]$$

$$= m(1-m)b(m,n,c), \text{ using } (4.5) \qquad (4.9)$$

The mean deviation about the mean of ξ is

$$D = E_G|\xi - m| = 2m(1-m)b(m,n,c)$$

$$= 2m(1-m)b(m,n,c+1)$$

$$= 2m(1-m)b(m,n+1,c+1) \qquad (4.10)$$

Further

$$E_G\ \epsilon_m(\xi - m)^2 = (n+1)^{-2}m(1-m)[1 - L(m,n+2,c+1)] \qquad (4.11)$$

and

$$E_G(1-\epsilon_m)(\xi - m)^2 = (n+1)^{-2}\ m(1-m)L(m,n+2,c+1)\ ,c+1$$

4.1c: <u>The IQL π_o and the point $\pi_{o\partial}$</u>:-

The point $\pi_{o\partial} = \pi_{o\partial}\ (n,c)$ is defined by

$$L(\pi_{o\partial}, n, c) = (1 + \partial)^{-1}, \quad \partial > 0 \qquad (4.12)$$

Note that

$$\pi_{o\partial} = \frac{(c+1)\,{}'F'_\partial}{n-c+(c+1)\,{}'F'_\partial} \text{ and } {}'F'_\partial = \exp\ (2Z_\partial) \qquad (4.13)$$

where $'F'_\partial$ is the upper $(1+\partial)^{-1}$ fractile of the 'F' distribution mentioned in (4.4) and Z_∂ is the corresponding fractile of Fisher's Z-distribution (RMM tables (1966); Fisher and Yates, 1953) .

The IQL, π_0 is of course given by

$$L(\pi_0, n, c) = \frac{1}{2}$$

Using (1.11) and (1.10)

$$L(m,n,c) = \frac{1}{2} - \frac{1}{3\sqrt{2\pi}} \frac{1-2m}{n+3} \sqrt{\frac{n+2}{m(1-m)}} + 0(n^{-3/2}) \tag{4.14}$$

where $1/\sqrt{2\pi} = 0.39894 \ldots$, and

$$\pi_0 = m - \frac{1-2m}{3(n+3)} + 0(n^{-2}) \tag{4.15}$$

Applying Fisher-Cornish expansion to the Z-distribution (RMM tables 1966) and then solving for π_0 from (4.13) we find - after simplifications - the following alternative expansion for π_0

$$\pi_0 = m - \frac{1-2m}{3(n+1)} + \frac{8}{405} \frac{1-2m}{m(1-m)(n+1)^2} - \frac{86}{405} \frac{1-2m}{(n+1)^2} + 0(n^{-3}) \tag{4.16}$$

Since $m = (c+1)/(n+1)$, it follows that in terms of n and c $\pi_0 \doteq (n+1)^{-1}(c+2/3)$. However it is found empirically (Hald, 1967) that a better approximation is provided by

$$\pi_0(n,c) \doteq \frac{c+\frac{2}{3}}{n+\frac{1}{3}} \tag{4.17}$$

From (4.6) and (4.12)

$$\pi_{0\partial}(n,c) > \pi_{0\partial'}(n,c) \quad \text{if} \quad \partial > \partial'$$

and

$$\pi_{0\partial}(n+1,c+1) > \pi_{0\partial}(n,c) > \pi_{0\partial}(n+1,c) \tag{4.18}$$

Using (4.17) we may write

$$\pi_0(n+1,c+1) > m(n,c) > \pi_0(n+1,c) \tag{4.19}$$

4.1d: <u>The error-areas</u>:-

Using (4.8) and (4.5) the formulae for the error-areas at the point $p = \pi$ can be written in various ways.

$$D_2(\pi,n,c) = mL(\pi,n+1,c+1) - \pi L(\pi,n,c) \tag{4.20.1}$$

$$= mL(\pi,n+1,c) - \pi L(\pi,n,c-1) \tag{4.20.2}$$

$$= mL(\pi,n+2,c+1) - \pi L(\pi,n+1,c) \tag{4.80.3}$$

and also

$$D_2(\pi,n,c) = (1-\pi)L(\pi,n,c+1)-(1-m)L(\pi,n+1,c+1) \quad (4.21.1)$$

$$= (1-\pi)L(\pi,n,c) - (1-m)L(\pi,n+1,c) \quad (4.21.2)$$

$$= (1-\pi)L(\pi,n+1,c+1) - (1-m)L(\pi,n+2,c+1) \quad (4.21.3)$$

where m stands for $m(n,c) = (c+1)/(n+1)$.

The corresponding formulae for $D_1(\pi,n,c)$ and $D(\pi,n,c)$ are obtained by changing L in (4.20) and (4.21) to $L-1$ and $2L-1$ respectively. This is due to the relation

$$D_1(\pi,n,c) - D_2(\pi,n,c) = \pi - m = \pi - \frac{c+1}{n+1} \quad (4.22)$$

The alternative forms of the formulae make it possible to calculate easily the error-areas for composite plans obtained by randomizing between (n,c) and $(n,c+1)$ or (n,c) and $(n+1,c)$ or (n,c) and $(n+1, c+1)$ (Section 4.5). Further, the first set of formulae (4.20) gets extended to weighted error-areas with respect to the process density $(u+1)p^u$ whereas the second set (4.21) can be extended to the case of process density $(v+1)(1-p)^v$. (Sections 4.8d and 4.8e).

The error-areas at m, π_0 and also at the IQLs of the neighbouring plans are simple-to-compute as can be seen from the following formulae. (4.23)

$$D_1(m,n,c) + \partial D_2(m,n,c) = (1+\partial)m(1-m)b(m,n+1,c+1)$$

$$D_1(\pi_{o\partial},n,c) + \partial D_2(\pi_{o\partial},n,c) = (1+\partial)m(1-m)b(\pi_{o\partial},n+1,c+1)$$

$$D_1(\pi'_{o\partial},n,c) + \partial D_2(\pi'_{o\partial},n,c) = (1+\partial)m(1-\pi'_{o\partial})b(\pi'_{o\partial},n+1,c+1)$$

$$D_1(\pi''_{o\partial},n,c) + \partial D_2(\pi''_{o\partial},n,c) = (1+\partial)\pi''_{o\partial}(1-m)b(\pi''_{o\partial},n+1,c+1)$$

and

$$D_1(\pi'''_{o\partial},n,c) + \partial D_2(\pi'''_{o\partial},n,c) = (1+\partial)\pi'''_{o\partial}(1-\pi'''_{o\partial})\,b(\pi'''_{o\partial}, n+1,c+1)$$

where

$$\partial > 0; \quad m = m(n,c); \quad \pi_{o\partial} = \pi_{o\partial}(n,c);$$

$$\pi'_{o\partial} = \pi_{o\partial}(n+1,c) \; ; \quad \pi''_{o\partial} = \pi_{o\partial}(n+1,c+1)$$

and $$\pi'''_{o\partial} = \pi_{o\partial}(n+2,c+1) \qquad (4.24)$$

In the context of optimality condition (B_∂) the following relation is of interest.

$$\begin{aligned} D_1(\pi'_{o\partial}, n,c) + \partial D_2(\pi'_{o\partial}, n,c) &= D_1(\pi'_{o\partial}, n,c-1) + \partial D_2(\pi'_{o\partial}, n,c-1) \\ &= D_1(\pi'_{o\partial}, n-1,c-1) + \partial D_2(\pi'_{o\partial},n-1,c-1) \end{aligned} \qquad (4.25)$$

The error-areas to any desired value of π can be calculated from the available tables of binomial distribution (Harvard (1955); Weintraub (1963); etc.)

4.1c: <u>The behaviour of the error-areas with respect to p:-</u>

For a given plan (n,c) as p increases from zero to unity, $L(p)$ and $D_2(p)$ decrease respectively from 1 and m to zero whereas $D_1(p)$ increases from zero to $1-m$. Also $p^{-1} D_2(p)$ decreases from ∞ to 0 while $p^{-1}D_1(p)$ increases from zero to $1-m$. (4.26)

The above statements correspond to (3.33) and can be proved similarly.

It follows that for a given number h,

$$hD_2(p,n,c) < D_2(hp, n,c) \quad \text{if} \quad 0 < h < 1 \tag{4.27}$$

We shall prove a stronger inequality which will be needed in Section 4.6. Since $0 < h < 1$

$$\int_p^1 L(p,n,c)dp < \int_p^1 L(hp,n,c) = \int_{hp}^h h^{-1}L(p,n,c)dp$$

Therefore we arrive at the desired result, namely

$$hD_2(p,n,c) < D_2(hp,n,c) - D_2(h,n,c) \tag{4.28}$$

As in the case of Poisson OCs, the following properties can be verified easily. (Theorems 2.1 and 2.2)

$$D_1(m,n,c) = D_2(m,n,c) = \min_p \max [D_1(m,n,c),\ D_2(m,n,c)] \qquad (4.29)$$

$$D_1(\pi_{0\partial},n,c) + \partial D_2(\pi_{0\partial},n,c)$$

$$\leq D_1(p,n,c) + \partial D_2(p,n,c), \quad \text{for all } p \text{ and } \partial > 0 \qquad (4.30)$$

and

$$(\pi''_{0\partial})^{-1}\left[D_1(\pi''_{0\partial},n,c) + \partial D_2'(\pi''_{0\partial},n,c)\right]$$

$$\leq p^{-1} [D_1(p,n,c) + \partial D_2(p,n,c)] \qquad (4.31)$$

for all p and $\partial > 0$ where $\pi''_{0\partial} = \pi_{0\partial}(n+1,c+1)$.

4.2 Optimality conditions

4.2a: Condition (A):-

The behaviour of the OC and the error-areas with respect to n and c is the same as in the case of Poisson OCs (3.40). As a consequence - given n and $p = \pi$ the condition (A) leads to the relationship

$$m(n,c) = \frac{c+1}{n+1} = \pi \qquad (4.32)$$

4.2b: <u>Condition (B_∂):-</u>

Given n and $p = \pi$, the value of $c - c_i = c_i(\pi, n, \partial)$ say - at which $D_1(\pi,n,c) + \partial\, D_2(\pi,n,c)$, $\partial > 0$, is a minimum with respect to c is given by (4.33)

$$L(\pi,n+1,c_i+1) > (1+\partial)^{-1} \geq L(\pi,n+1,c_i)$$

i.e., $\pi_{0\partial}(n+1,c_i+1) > \pi \geq \pi_{0\partial}(n+1,c_i) = \pi'_{0\partial}$

If there exists a plan such that $\pi'_{0\partial} = \pi$, then it follows from (4.25) that the value of the risk function $D_1(\pi,n,c_i) + \partial D_2(\pi,n,c_i)$ remains unaltered when either n or c_i or both are diminished by unity. (cf. Remark (1) following the proof of (3.42)) (4.34)

Given c and $p = \pi$, the value of n - $n_i = n_i(\pi,c,\partial)$ say - at which $D_1(\pi,n,c) + \partial D_2(\pi,n,c)$ is a minimum with respect to n is given by

$$L(\pi,n_i+1,c+1) > (1+\partial)^{-1} \geq L(\pi,n+2,c+1)$$

i.e., $\pi_{0\partial}(n_i+1,c+1) > \pi \geq \pi_{0\partial}\,(n_i+2,c+1) = \pi'''_{0\partial}$ (4.35)

Suppose that there exists a plan for which $\pi'''_{0\partial} = \pi$, the given value. It follows from (4.25) that once the risk function $D_1(\pi,n,c) + \partial D_2(\pi,n,c)$ has attained its minimum at n_i, its value remains unaltered when either n_i or c or both are increased by unity. (4.36)

It is relevant to note that in the context of determining OCs which pass through a given point (π,α) in the $\{p, L(p)\}$ plane, Hald has discussed in detail the solution of the binomial equation $L(\pi,n,c) = \alpha$. (Hald (1967) and Hald and Kousgaard, 1967).

4.2c: Asymptotic properties of plans satisfying conditions (A) and (B):-

From (4.22) and (4.23)

$$D_1(m,n,c) = D_2(m,n,c) = mb(m,n+1,c+1), \qquad m = \frac{c+1}{n+1}$$

$$= \frac{m}{\sqrt{2\pi\, m(1-m)(n+1)}}\left[1 - \frac{1-m(1-m)}{12m(1-m)(n+1)}\right] + 0(n^{-2})$$

(4.37)

using the asymptotic relation given by Govindarajulu in Patil (1965). Note that $1/\sqrt{2\pi} = 0.39894 \ldots$

Therefore as $n \to \infty$, $c \to \infty$ such that $m \doteq (c+1)/(n+1)$ remains constant at a given value (Condition (A)), the or-areas at m tend to zero with the order of $n^{-1/2}$.

Further

$$\sqrt{n+1}\ D(m) \rightarrow 0.79788 \ \sqrt{m(1-m)} \tag{4.38}$$

The following table shows the behaviour of $\sqrt{n+1}\ D(m)$ when m is kept fixed at 25 °/. or 5 °/. .

TABLE 4.1

Values of $\sqrt{n+1}\ D(m)$ under the optimality condition (A)

$\frac{c+1}{n+1} = m = 25$ °/.			$\frac{c+1}{n+1} = m = 5$ °/.		
n+1	c+1	$\sqrt{n+1}\ D(m)$	n+1	c+1	$\sqrt{n+1}\ D(m)$
4	1	.316	20	1	.160
8	2	.330	40	2	.167
12	3	.335	60	3	.169
16	4	.338	80	4	.170
36	9	.3420	100	5	.1709
100	25	.3443	200	10	.1724
200	50	.3449	300	15	.1729
	∞	.3455		∞	.1739

It is suggested that interpolation, where necessary, be carried out with respect to $\sqrt{n+1}\ D(m)$.

From (4.16), (4.17), (4.22) and (4.23)

$$\pi \doteq \frac{c+2/3}{n+1/3}$$

$$D(\pi_0, n,c) = 2m(1-m)b(\pi_0, n+1,c+1)$$

and $D_1(\pi_0,n,c) - D_2(\pi_0,n,c) = \pi_0 - m$

$$= -\frac{1-2m}{3(n+1)} + 0(n^{-2}) \qquad (4.39)$$

As $n \to \infty$, $c \to \infty$ such that $\pi_0(n,c)$ remains constant at a given value, the condition (B) approaches the condition (A) with the order of n^{-1} whereas each of the error-areas tends to zero with $0(1/\sqrt{n})$. The OC curve also tends to the ideal OC (Fig. 2.1).

4.3 The $\{m_e, D_e\}$ system of sampling plans

Sampling plans can be constructed so as to satisfy the specifications

$$m(n,c) = m_e$$

and

$$D(m_e, n,c) = D_e \qquad (4.40)$$

where m_e and. D_e are the specified numbers. The simple relations

$$m_e = \frac{c+1}{n+1} \qquad (4.41)$$

and $$D_e = 2m_e(1-m_e)\,b(m_e, n+1, c+1) \qquad (4.42)$$

$$= 2m_e(1-m_e)\,b(m_e, n, c+1) \qquad (4.43)$$

give the transformation from the set (n,c) to the set $\{m_e, D_e\}$ and vice-versa. Selection of m_e fixes the ratio $(c+1)/(n+1)$. We then have to select that plan for which D_e equals a given number.

Example:- Suppose $m_e = 4$ %. and $D_e = 2$ %. .

Since $m_e = 4$ %. $= \frac{1}{25} = \frac{2}{50}$ etc., we have only to consider the plans (24,0), (49,1), (74,2) etc. From the binomial tables (Harvard 1955) we get

n	c	D_e %.
24	0	4.3
49	1	2.1
74	2	1.8

etc.

Therefore the required plan is (49,1).

Tightened and reduced inspection in the $\{m_e, D_e\}$ system can be installed as in examples (iii) and (iv) of Section 3.3b.

It may be noted that the absolute value of the slope of the OC curve at m_e is

$$g_e = g(m_e,n,c) = (n+1)b(m_e,n+1,c+1) \qquad (4.44)$$

A set of 31 plans belonging to the $\{m_e, D_e\}$ system was originally constructed in Subrahmanya (1966) so as to cover adequately the space of binomial OC curves. The set is given below in the form of Table 4.2. The IQL, the slope at m_e, the AQL (.95), the LTPD (.10) and also the AOQL of the plans are shown in the table for purposes of comparisons with other systems of sampling plans.

The plans in Table 4.2 can be grouped into three sets according to the steepness of the OC curves; in each set m_e ranges from 1 °/. to 25 °/. whereas the corresponding D_e in the first set are smaller than those in the second set whic in turn are smaller than those in the third set. Thus for m_e = 10 °/. we have D_e = 1 °/. , 2.5 °/. and 5 °/. in the first, second and third sets respectively.

TABLE 4.2

A set of 31 binomial single sampling plans belonging to to the $\{m_e, D_e\}$ system

plan n	c	m_e	D_e °/.	g_e	AQL(.95) °/.	IQL π_0 °/.	LTPD(.10) °/.	AOQL °/.
rst set:-								
1000	9	1 °/.	0.25	125	0.5	1	1	0.5
500	9	2 °/.	0.50	63	1	2	3	1
400	15	4 °/.	0.75	41	3	4	5	3
320	19	1/16	1	29	4	6	8	5
480	39	1/12	1	32	6	8	10	7
550	54	10 °/.	1	31	8	10	12	8
320	39	1/8	1.5	22	9	12	15	9
240	39	1/6	2	17	13	17	20	12
160	31	20 °/.	2.5	13	15	20	24	14
200	49	25 °/.	2.5	13	20	25	29	19
cond set:-								
200	3	2 °/.	0.75	39	0.5	2	3	1
150	5	4 °/.	1.25	25	2	4	6	2
160	9	1/16	1.5	21	3	6	9	4
120	9	1/12	2	16	4	8	12	5
90	8	10 °/.	2.5	13	5	10	14	6
160	19	1/8	2	15	8.5	12	16	8

contd.

TABLE 4.2 (contd.)

no.	plan n	c	m_e	D_e ./.	g_e	AQL(.95) ./.	IQL π_0 ./.	LTPD(.10) ./.	AOQL ./.
Second set:-(contd)									
17	120	17	15./.	2.5	12	10	15	19	10
18	65	10	1/6	3.5	9	10	16	23	10
19	63	11	3/16	4	8	12	18	25	12
20	39	7	20./.	5	6	11	20	28	12
21	43	10	25./.	5	6	15	25	34	16
Third set:-									
22	200	1	1 ./.	0.5	55	0.2	1	2	0.5
23	100	1	2 ./.	1	27	0.5	1.5	4	1
24	49	1	4 ./.	2	14	1	3	8	2
25	31	1	1/16	3.5	9	1	5	12	3
26	35	2	1/12	3.5	8	2	8	15	4
27	19	1	10./.	5	6	2	9	19	4
28	23	2	1/8	5	6	4	11	22	6
29	11	1	1/6	8	4	3	15	31	7
30	15	2	3/16	7.5	4	6	17	32	9
31	11	2	25./.	9.5	3	8	24	42	12

4.4 Poisson approximation to the binomial with respect to the $\{m_e, D_e\}$ system

Table 4.3 given below shows that the $\{m_e, D_e\}$ system of plans can be constructed using Poisson approximations to the binomial when m_e is small and n large.

When $m_e \leq 10$ •/. and $n \geq 50$; or $m_e \leq 15$ •/. and $n \geq 100$; or $m_e \leq 20$ •/. and $n \geq 150$, the difference between the exact and the approximate values of D_e is atmost 0.003.

Even in other cases one can conveniently find the Poisson plan (n_0, c_0) and then search for the corresponding binomial plan in the neighbourhood of n_0 and c_0.

TABLE 4.3

Poisson approximation to the binomial with respect to the $\{m_e, D_e\}$ system

plan n	plan c	m_e	value of D_e: binomial	value of D_e: Poisson	value of D_e: difference
200	19	0.10	0.0169	0.0178	0.0009
	29	0.15	0.0201	0.0218	0.0017
	39	0.20	0.0225	0.0252	0.0027
	49	0.25	0.0244	0.0282	0.0038
150	2	0.02	0.0089	0.0090	0.0001
	14	0.10	0.0194	0.0205	0.0011
	24	1/6	0.0242	0.0265	0.0023
	29	0.20	0.0260	0.0291	0.0031
100	1	0.02	0.0107	0.0108	0.0001
	4	0.05	0.0171	0.0175	0.0004
	9	0.10	0.0237	0.0250	0.0013
	14	0.15	0.0283	0.0307	0.0023
	19	0.20	0.0318	0.0355	0.0037
49*	1	0.04	0.0212	0.0217	0.0005
	4	0.10	0.0334	0.0351	0.0017
	7	0.16	0.0409	0.0447	0.0038

* The corresponding Poisson plan has $n = 50$ so that the point of equal error-areas are exactly the same for the binomial and the Poisson plans.

4.5 Methods of randomization

With regard to the methods of randomization as discussed in Section 3.4, nothing needs to be said about procedure (1) of randomizing between two plans (n_1, c_1) and (n_2, c_2) having the same m because no new theory or formulae are involved in the binomial case.

Three other procedures of randomization will be considered in this section.

Procedure (2):-

This is the same as procedure (2) of Section 3.4, that is, plans (n, c) and $(n, c+1)$ are chosen with probabilities a and $1-a$ respectively. The point of equal error-areas is given by

$$m(n, c, c+1;\ a) = \frac{c+2-a}{n+1} = m_e \qquad (4.45)$$

The error-areas at any desired value π can be obtained from (4.20.1) just by replacing m by $m(n, c, c+1;\ a)$ as given by the above relation. (The proof is similar to that of (3.53) and therefore omitted). In particular, the error-area at m_e is given by (4.43) i.e., by

$$D_e = D(m_e, n, c, c+1;\ a) = 2m_e(1-m_e)b(m_e, n, c+1) \qquad (4.46)$$

This follows by putting $\pi = m_e$ in (4.20.1) and then using (4.5.2).

Procedure (3):-

This method randomizes between n and $n+1$ keeping c fixed. Plans (n,c) and $(n+1,c)$ are chosen with probabilities a and $1-a$ respectively. The operational procedure is as follows:

Draw a sample of size n from the lot and observe the number of defectives in the sample d_n

accept the lot if $d_n < c$;

reject the lot if $d_n > c$;

inspect one more item if $d_n = c$;

if this item turns out to be non-defective,

accept thelot;

if it turns out to be defective,

accept the lot with probability a and

reject the lot with probability $1-a$.

The probability of acceptance under the above procedure is

$$L(p,n,c,n+1;a) = L(p,n,c-1) + (1-p+ap)b(p,n,c)$$
$$= aL(p,n,c) + (1-a)L(p,n+1,c) \qquad (4.47)$$

using (4.5.1).

The point of equal error-areas is given by

$$m(n,c,n+1;\ a) = \frac{a(c+1)}{n+1} + \frac{(1-a)(c+1)}{n+2} = m_e \qquad (4.48)$$

The error-area of the second kind at $p = \pi$ is

$$D_2(\pi,n,c,n+1;\ a) = aD_2(\pi,n,c) + (1-a)D_2(\pi,n+1,c)$$

Applying (4.20.3) to $D_2(\pi,n,c)$ and (4.20.1) to $D_2(\pi,n+1,c)$ we get a formula of the same form as (4.20.3) i.e.,

$$D_2(\pi,n,c,n+1;a) = m(n,c,n+1;a)L(\pi,n+2,c+1) - \pi L(\pi,n+1,c) \qquad (4.49)$$

Setting $\pi = m(n,c,n+1;\ a) = m_e$ and then applying (4.5.2) we obtain

$$D_e = D(m_e,n,c,n+1;\ a) = 2m_e(1-m_e)b(m_e,n+1,c+1) \qquad (4.50)$$

is complete analogy with the corresponding formula (4.42) for the non-randomized plan.

Procedure (4):-

Plans (n,c) and (n+1,c+1) are chosen with probabilities a and 1-a respectively. The operational procedure can be as follows:

Draw a sample of size n and observe d_n;

if $d_n \leq c$, accept the lot;

if $d_n > c+1$, reject the lot;

if $d_n = c+1$, inspect one more item ;

if this item is defective, reject the lot;

if it is non-defective, accept with probability 1-a and reject with probability a.

The probability of acceptance under this scheme is

$$\begin{aligned} L(p,n,c,n+1,c+1;a) &= L(p,n,c)+(1-a)(1-p)b(p,n,c+1) \\ &= aL(p,n,c)+(1-a)L(p,n+1,c+1) \end{aligned} \quad (4.51)$$

using (4.5.2)

The point of equal error-area is given by

$$m(n,c,n+1,c+1;\ a) = \frac{a(c+1)}{n+1} + \frac{(1-a)(c+2)}{n+2} \quad (4.52)$$

The error-areas at any point π can be calculated from (4.20.3) just by replacing m by m(n,c,n+1,c+1; a) (This can be proved as in the case of procedure (3)). The error-area at $\pi = m(n,c,n+1,c+1;\ a) = m_e$ is therefore given by

$$D(m_e,n,c,n+1,c+1;\ a)$$
$$= 2m_e\left[\,L(m_e,n+2,c+1) - L(\pi,n+1,c)\,\right]$$

Applying (4.5.2) we get

$$D_e = D(m_e,n,c,n+1,c+1;\ a) = 2m_e(1-m_e)b(m_e,n+1,c+1) \qquad (4.53)$$

in analogy with the corresponding formula (4.42) for the non-randomized plan.

4.6 Adjusting the plans of the $\{m_e, D_e\}$ system when the inspection technique is not exact

The defectives that are found out by an inspection technique are called 'assumed defectives'. The number of actual defectives in a sample of size n is a binomial variable with parameters n and p whereas the number of assumed defectives is considered to be a binomial with parameters n and hp, h being a positive number. In other words, given p, the probability of finding r defectives in a sample of size n is given by

$$\begin{cases} b(hp,n,r) & \text{if } 0 \leq p \leq \min(1, h^{-1}) \\ 0 & \text{if } h>1,\ r \neq n \text{ and } \min(1,h^{-1}) \leq p \leq 1 \\ 1 & \text{if } h>1,\ r=n \text{ and } \min(1,h^{-1}) \leq p \leq 1 \end{cases} \qquad (4.54)$$

Specifications of good and bad lots are made on the basis of p, the proportion of actual defectives in the lot. hp is the proportion of assumed defectives in the lot, i.e., the proportion of defectives that will be found in the lot when subjected to the particular technique of inspection.

The technique is liberal if $h<1$, exact if $h=1$ and strict if $h>1$.

The following model leads to a liberal technique of inspection.

Probability of detecting a defective item = h

probability of not detecting a defective item = 1-h

and probability of detecting a non-defective item = 1

Under the model the probability of counting r defectives in a sample of size n is

$$\sum_{s=0}^{n-r} b(p,n,r+s)\binom{r+s}{s} h^r (1-h)^s$$

$$= b(hp,n,r), \qquad 0 < h \leq 1$$

The following situation can also lead to inexact techniques of inspection.

Suppose that an item is defective if its quality characteristic x exceeds an upper specification limit u. Suppose further that x is an exponential variable with mean α^{-1}. Then the proportion of actual defectives is

$$p = \int_{u}^{\infty} \alpha\, e^{-\alpha x}\, dx$$

If, during inspection, the gauge is set at $u + \gamma\alpha^{-1}$ the proportion of assumed defectives under such a setting is given by

$$\int_{u+\gamma_\alpha}^{\infty} \alpha\, e^{\alpha x} = hp$$

where $h = \exp(-\gamma)$

For the plan (n,c) the OC under the exact technique is

$$L(p,n,c) = \sum_{r=0}^{c} b(p,n,c)$$

Under the inexact technique-h, the probability of accepting a lot of quality p is the probability of getting not more than c assumed defectives. It is given by

$$L^h(p,n,c) = \begin{cases} L(hp,n,c), & \text{if } 0 \le hp \le 1 \\ 0 & \text{otherwise} \end{cases} \qquad (4.5$$

Here and elsewhere in this section, h is used as a super-script in the notations L^h, m^h, D_1^h etc.

The area under the OC $L^h(p,n,c)$ is given by

$$m^h(n,c) = \int_0^1 L^h(p,n,c)dp = \int_0^{\min(1,h^{-1})} L(hp,n,c)dp$$

$$= \frac{m(n,c) - \epsilon\, D_2(h,n,c)}{h} \qquad (4.5$$

where $m(n,c) = (c+1)/(n+1)$ and $\epsilon = 0$ or 1 according as $h > 1$ or ≤ 1.

If $h\pi \le 1$, the error-areas under L^h at the point p may be calculated from

$$h^{-1} D_1^h(\pi,n,c) = D_1(h\pi,n,c)$$

and

$$h^{-1} D_2^h(\pi,n,c) = D_2(h\pi,n,c) - \epsilon\, D_2(h,n,c) \qquad (4.5$$

It is reasonable to assume that $h < 2$ and that the specified quality level $m_e < 50$ °/. so that $hm_e < 1$ and the formula (4.57) can be used to calculate the error-areas at the target value $m_e = m(n,c)$.

Using the inequalities (4.26) when $h \geq 1$ and also the stronger inequality (4.28) when $h < 1$ we arrive at inequalities similar to (3.60) i.e.,

$$D_1^h(m_e) \lessgtr D_1(m_e) = D_2(m_e) \lessgtr D_2^h(m_e) \qquad (4.58)$$

according as $h \lessgtr 1$.

All the remarks made in the case of Poisson plans in Section 3.5 regarding liberal and strict techniques and also reduced and tightened inspection are valid for binomial plans also.

It may be noted that the exact formulae for m^h and $D^h(\pi)$ are similar to the corresponding Poisson formulae (3.56) and (3.57) but for the term $h^{-1} D_2(h,n,c)$ in the case of $h < 1$. The effect of this term gets larger as h gets smaller. Table 4.4 gives its value for certain selected plans when $h = \frac{1}{2}$.

TABLE 4.4

Values of $2D_2(\frac{1}{2}, n, c)$

Plan n	c	m(n,c)	$2D_2(\frac{1}{2},n,c)$	Plan n	c	m(n,c)	$2D_2(\frac{1}{2},n,c)$
1	0	$\frac{1}{2}$	.1250	7	1	$\frac{1}{4}$	.0098
2	0	$\frac{1}{3}$	.0417	9	1	$\frac{1}{5}$	.0023
3	0	$\frac{1}{4}$	.0313	11	1	$\frac{1}{6}$	.0006
4	0	$\frac{1}{5}$	.0125	11	2	$\frac{1}{4}$	.0038
5	0	$\frac{1}{6}$	.0052	14	2	$\frac{1}{5}$	.0006
7	0	$\frac{1}{8}$	.0010	15	3	$\frac{1}{4}$	.0016
9	0	$\frac{1}{10}$	.0002				

We see that the term $h^{-1}D_2(h)$ may be ignored when $h \geq \frac{1}{2}$, $n \geq 10$ and $m \leq \frac{1}{4}$.

Two examples are given below to illustrate the effect an inexact technique of inspection on the error-areas at the target value $m_e = m(n,c)$.

Example (i):-

$n = 19$ and $c = 1$ so that $m_e = 10$ %. and $D(m_e) = 5.1$ %. .

For selected values of h in the interval $\frac{1}{2} \leq h \leq 2$, the error-areas under $L^h(p,19,1)$ at the target value 10 °/. are shown in the following table. Values of $m^h(19,1)$, the point of equal error-areas for the OC $L^h(p,19,1)$ and also the corresponding error-area $D^h(m^h,19,1)$ are given in the last two columns of the table.

TABLE 4.5

The effect of technique -h on the error-areas of the binomial plan n = 19 and c=1;
$m_e = m(19,1) = 10$ °/. and $D(m_e,19,1) = 5.1$ °/.

h	at the target value 10 °/. $D_1^h(10$ °/.) °/.	$D_2^h(10$ °/.) °/.	$D^h(10$ °/.) °/.	m^h °/.	$D^h(m^h)$ °/.
$\frac{1}{2}$	0.95	10.9	11.9	20.0	10.3
$\frac{2}{3}$	1.5	6.7	7.9	15.0	7.7
$\frac{3}{4}$	1.7	5.1	6.8	13.3	6.8
$\frac{4}{5}$	1.9	4.4	6.3	12.5	6.4
$\frac{9}{10}$	2.2	3.4	5.6	11.1	5.7
$\frac{19}{20}$	2.4	2.9	5.3	10.5	5.4
1	2.6	2.6	5.1	10.0	5.1
$\frac{5}{4}$	3.3	1.3	4.7	8.0	4.1
$\frac{3}{2}$	4.0	0.7	4.8	6.7	3.4
2	5.2	0.2	5.4	5.0	2.6

The values do not differ from those of the correspondi Poisson plan (20,1) given in Table 3.10 by more than .003.

Example (ii):-

$n = 500$ and $c = 9$ so that $m_e = 2$ %. and $D(m_e) = \frac{1}{2}$ %.

It is not necessary to tabulate the error areas under L^h in this case, because they do not differ from those of t corresponding Poisson plan (500, 9) given in Table 3.11 by more than .001.

The above discussions suggest a method of adjusting a binomial plan (n, c) when h is known. The sample size n first changed to n_0 where

$$n_0 + 1 = h^{-1} (n+1) \qquad (4.59$$

so that $m^h(n_0, c) = m(n, c) = m_e$, the target value. In most of the cases - whenever Poisson approximation is good - the plan (n_0, c) is adequate in the sense that $D^h(m^h, n_0, c)$ is close to the target value $D(m_e, n, c) = D_e$. If n is small and m is large, further adjustments will be necessary.

The above method is demonstrated with the help of examples (iii)-(vi) given below. It may be noted that if the method (4.59) is satisfactory for a particular value of $h=h_0$ say, it is also satisfactory for any value of h that lies between h_0 and 1. Since it is unlikely that $h < \frac{1}{2}$ or > 2, we have given examples only for the two extreme cases $h = \frac{1}{2}$ and $h = 2$.

	h	Plan n	Plan c	m^h	$D^h(m^h)$ %
Example (iii):-	1	19	1	$\frac{1}{10} = m_e$	$5.3 = D_e$ %
	$\frac{1}{2}$	$n_0 = 39$	1	$\frac{1}{10}$	5.3
Example (iv):-	1	160	19	$\frac{1}{8} = m_e$	$2.1 = D_e$ %
	2	$n_0 = 79$	19	$\frac{1}{8}$	1.9
	2	75	18	$\frac{1}{8}$	2.0
	2	67	16	$\frac{1}{8}$	2.1
Example (v):-	1	11	2	$\frac{1}{4} = m_e$	$9.7 = D_e$ %
	$\frac{1}{2}$	$n_0 = 23$	2	$\frac{1}{4}$	10.5
	$\frac{1}{2}$	31	3	$\frac{1}{4}$	9.1
Example (vi):-	1	11	1	$\frac{1}{6} = m_e$	$8.2 = D_e$ %
	2	$n_0 = 5$	1	$\frac{1}{6}$	7.3
	2	2	0	$\frac{1}{6}$	9.9
	2	3	0	$\frac{1}{8}$	7.9

The plan (n_0,c) is satisfactory in examples (iii) a (iv). In the later case however, (67, 16) gives exactly th target value of 2.1 %. In example (v) one should be satisfied with 10.5 % or 9.1 % as compared to the ta get value 9.7 %. In (vi) the two neighbouring plans wit the same value of m^h give 7.3 % and 9.9 % respectively as compared to the target value 8.2 %. We shoul be satisfied with one of these plans unless we agree to a change in m_e in which case the plan (3,0) with $m^h = 1/8$ and $D^h(m^h) = 7.9$ % may be adopted.

It may be remarked that exact specifications may be achieved in all these cases by employing the procedures of randomization.

4.7 The EQL and the weighted error-areas

4.7a: The EQL:-

The c.d.f. and the p.d.f. of the prior distribution lot quality p are denoted by F(p) and f(p) respective

For the EQL $\pi_e = \pi_{eF}(n,c)$ we have

$$A_F = F(\pi_e) = F(m) + R \qquad (4.6$$

and

$$|R| \leq \begin{cases} \frac{M}{2} \frac{m(1-m)}{n+2} & \text{if Theorem 1.1 is applicable} \\ \frac{M^*}{2} D(m) & \text{if Theorem 1.2 is applicable} \end{cases}$$

where

$$m = \frac{c+1}{n+1} ;$$

$$D(m) = 2m(1-m)b(m,n,c);$$

$$M = \sup_{0 \leq p \leq 1} \left| \frac{d^2F(p)}{dp^2} \right| \quad \text{and} \quad M^* = \sup_{0 \leq p \leq 1} f(p)$$

The relations (1.20) and (1.22) give asymptotic expansions of $F(\pi_e)$ and π_e in terms of n and c. When m is kept fixed, using the expressions for the moments of ξ as given in Section 4.1b, we obtain

$$F(\pi_e) = F(m) + \frac{m(1-m)}{2(n+2)} F_2(m) + \frac{m(1-m)(1-2m)}{3(n+2)(n+3)} F_3(m)$$
$$+ \frac{m^2(1-m)^2(n-5)}{8(n+2)(n+3)(n+4)} F_4(m) + O(n^{-3}) \tag{4.61}$$

and

$$\pi_e = m + \frac{m(1-m)}{2(n+2)} \frac{F_2(m)}{F_1(m)} + \frac{m(1-m)(1-2m)}{3(n+2)(n+3)} \frac{F_3(m)}{F_1(m)}$$
$$+ \frac{1}{8} \frac{m^2(1-m)^2}{n+2} \left[\frac{n-5}{(n+3)(n+4)} \frac{F_4(m)}{F_1(m)} - \frac{1}{n+2} \left\{ \frac{F_2(m)}{F_1(m)} \right\}^3 \right]$$
$$+ O(n^{-3}) \tag{4.62}$$

where $F_j(p)$ is the j-th derivative of $F(p)$.

4.7b: The weighted error-areas:-

When m is kept fixed, substituting the expressions for the incomplete moments of ξ as given in Section 4.1b the relations (2.48) and (2.49) of Theorem 2.4b yield

$$J_2(m,n,c) = \frac{\bar{1}}{2} D(m)f(m) + \frac{m(1-m)}{2(n+2)} L(m,n+2,c+1)F_2(m) + 0(n^{-3/2}) \qquad (4.$$

and

$$J(m,n,c) = \frac{1}{2} D(m)f(m) + \frac{m(1-m)}{2(n+2)} [\, 2L(m,n+2,c+1)-1 \,] F_2(m + 0(n^{-3/2}) \qquad (4.$$

4.7c: Alternative formulae:-

If $F(p)$ can be expanded in a Taylor series around origin $p = 0$, then the lemmas 1.1 and 2.3 and the relati (4.8) give - as in (3.75) and (3.76) - the following alter tive formulae for $F(\pi_c)$ and the error-areas.

$$F(\pi_e) = \sum_{r=1} \frac{n!}{(n+r)!} \binom{c+r}{c} F_r(0) = \sum_{r=1} \frac{\pi_e^r}{r!} F_r(0) \qquad (4.$$

and

$$J_2(\pi,n,c) = -L(\pi,n,c)F(\pi) + \sum_{r=1} \frac{n!}{(n+r)!} \binom{c+r}{c} F_r(0) \, L(\pi,n+r,c+r) \qquad (4.$$

$J_1(\pi,n,c)$ is of course obtained from

$$J_1(\pi,n,c) - J_2(\pi,n,c) = F(\pi) - F(\pi_e) \tag{4.67}$$

In particular, the error-area at the EQL is given by

$$J(\pi_e,n,c) = 2 \sum_{r=1} \frac{n!}{(n+r)!} \binom{c+r}{c} F_r(0) \left[L(\pi_e,n+r,c+r) - L(\pi_e,n,c)\right] \tag{4.68}$$

It may be noted that in (4.66) and (4.68) n can be replaced by $n+1$ or c by $c+1$ in the L-functions without altering the values of the error-areas. This follows from (4.5) and (4.65).

4.8 The case of a Beta PC

4.8a: The EQL:-

The case of a Beta PC is considered in this section.

$$f(p) = \frac{p^u(1-p)^v}{\beta(u+1,v+1)}, \quad u \geq 0,\ v \geq 0,\ 0 \leq p \leq 1 \tag{4.69}$$

Whether u and v are integers or not, we will use the notation of 4.2 and write the c.d.f. as

$$F(p) = 1 - L(p,\ u+v+1,\ u) \qquad (4$$

The mean and the variance of the distribution are

$$\frac{u+1}{u+v+2} \text{ and } \frac{(u+1)(v+1)}{(u+v+2)^2\ (u+v+3)} \text{ respectively.}$$

When $u = v = 0$, p is uniform in the interval (0,
This is the case of unweighted OCs.

When $u = 0$ or $v = 0$, the process density is of t
form $(u+1)p^u$ or $(v+1)(1-p)^v$ and there is a considerabl
simplification in the formulae for the EQL and the error-areas. (Sections 4.8d and 4.8e).

When $u, v > 0$, $f(p)$ is bounded and continuous an
therefore Theorem 1.2 is applicable. Since $f(p)$ is a
maximum at $p = u/(u+v)$, the bound M^* in (4.60) can be t
as

$$M^* = f\left(\frac{u}{u+v}\right) = \frac{\Gamma(u+v+2)}{\Gamma(u+1)\,\Gamma(v+1)} \left(\frac{u}{u+v}\right)^u \left(\frac{v}{u+v}\right)^v \qquad (4.$$

Note that $1 \leq M^* \leq 2$ when $0 \leq u \leq 1$ and $0 \leq v \leq 1$. T
values of M^* are given in Mitra and Subrahmanya (1968) f
selected values of u and v. They are recorded below in
Table 4.6.

TABLE 4.6

Values of M^* in Theorem 1.2 for a Beta PC with parameters $u+1$ and $v+1$

u	v			
	0.2	0.4	0.6	0.8
0.2	1.12	1.20	1.30	1.42
0.4		1.22	1.29	1.37
0.6			1.32	1.38
0.8				1.41

Entries below the diagonal are obtained by symmetry.

When $u, v \geq 1$, the second order derivative of the c.d.f. $F_2(p)$ is bounded and Theorem 1.1 can be applied. The bound M in (4.60) is

$$M = \sup_{0 \leq p \leq 1} \left| \frac{p^{u-1}(1-p)^{v-1}}{\beta(u+1, v+1)} [u-(u+v)p] \right| \tag{4.72}$$

Values of M are given in Mitra and Subrahmanya (1968) for selected values of u and v. They are reproduced in Table 4.7 below.

TABLE 4.7

Values of M in Theorem 1.1 for a Beta PC with parameters u+1 and v+1, $u \geq 1$ and $v \geq 1$.

u	v = 1	2	3	4	6	8	10	15	20
1	6	12	20	30	56	90	132	272	462
2		5.8	8.1	11.0	18.1	27.1	38.0	73.3	120.0
3			7.5	9.4	14.1	19.8	26.7	48.3	76.6
4				9.4	13.0	17.4	22.5	38.6	59.4
6					13.2	16.3	20.0	31.1	45.2
8						17.0	19.9	28.8	39.9
10							20.8	28.4	37.6
15								30.5	36.5
20									40.2

Entries below the diagonal are obtained by symmetry.

It is clear that in situations covered by Theorem 1.1 the bound $\frac{M}{2}\,\frac{(c+1)(n-c)}{(n+1)^2(n+2)}$ - as suggested by (4.60) - cannot general be improved upon in large samples, since from (1.18

$$\lim_{n}\ \max_{c}\ \frac{|R|\ (n+1)^2(n+2)}{(c+1)(n-c)} = \frac{M}{2}$$

$F(\pi_c)$ can be evaluated exactly from the formula (4.74) or (4.77) - to be given below - without much difficulty. It i then found that in small samples the actual upper bound of $|R|$ is considerably less than the best bound obtained under Theorem 1.1 (Mitra and Subrahmanya, 1968).

4.8b: Exact formulae for the EQL and the error-areas:

The error-area of the second kind at $p=\pi$ is given by

$$J_2(\pi,n,c) = \int_{p=\pi}^{1} \sum_{r=0}^{c} b(p,n,c)f(p)dp$$

$$= \sum_{r=0}^{c} \binom{n}{r} \frac{\beta(u+r+1,\ v+n+1-r)}{\beta(u+1,v+1)} L(\pi,u+v+n+1,u+r) \tag{4.73}$$

Therefore

$$F(\pi_e) = A_F = J_2(0,n,c)$$

$$= \sum_{r=0}^{c} \binom{n}{r} \frac{\beta(u+r+1,v+n+1-n)}{\beta(u+1,v+1)} \tag{4.74}$$

Alternatively,

$$J_2(\pi,n,c) = \int_{p=\pi}^{1} L(p,n,c)f(p)dp$$

$$= -L(\pi,n,c)F(\pi)+ \int_{\pi}^{1} \frac{p^c(1-p)^{n-c-1}}{\beta(c+1,n-c)} F(p)dp \tag{4.75}$$

If u is an integer, from (4.70)

$$F(p) = \sum_{r=u+1}^{u+v+1} b(p, u+v+1, r)$$

and hence (4.75) yields

$$J_2(\pi,n,c) = L(\pi,n,c)L(\pi,u+v+1,u) - \sum_{r=0}^{u} \frac{\Gamma(u+v+2)}{\Gamma(r+1)\Gamma(u+v+2-r)} \frac{\beta(c+r+1,u+v+n-c-r+1)}{\beta(c+1,n-c)} L(\pi,u+v+n+1,\ c+r) \qquad (4.76)$$

Therefore

$$F(\pi_e) = 1 - \sum_{r=0}^{u} \frac{\Gamma(u+v+2)}{\Gamma(r+1)\Gamma(u+v+2-r)} \frac{\beta(c+r+1,u+v+n-c-r+1)}{\beta(c+1,n-r)} \qquad (4.77)$$

The error-area at the EQL is obtained by putting $\pi = \pi_e$ in (4.73) or (4.76). The later gives

$$J(\pi_e,n,c) = 2 \sum_{r=0}^{u} \frac{\Gamma(u+v+2)}{\Gamma(r+1)\Gamma(u+v+2-r)} \frac{\beta(c+r+1,u+v+n-c-r+1)}{\beta(c+1,n-r)} [L(\pi_e,n,c) - L(\pi_e,\ u+v+n+1,\ c+r)] \qquad (4.78)$$

4.8c: Condition (B_∂):-

The average probability of getting r defectives for lots of quality worse than π is given by

$$\bar{b}_b(\pi,n,r) = \int_\pi^1 b(p,n,r)f(p)\,dp$$
$$= \bar{b}(n,r)L(\pi,u+v+n+1,\ u+r) \qquad (4.79)$$

Applying (2.66) of Theorem 2.6(a), it is seen that the condition (B_∂) is equal to the following relationship:

$$L(\pi,u+v+n+1,\ c+u+1) > (1+\partial)^{-1} \bar{\geq} L(\pi,u+v+n+1,\ c+u) \qquad (4.80)$$

This relation is the same as for the unweighted error-areas (4.33) but with n replaced by $n+u+v$ and c by $c+u$.

When $\partial = 1$ - i.e., the condition (B) of minimizing $J_1(\pi,n,c) + J_2(\pi,n,c)$, given n and $p = \pi$ - we have approximately

$$\frac{c+u+1+2/3}{u+v+n+1+1/3} > \pi \geq \frac{c+u+2/3}{n+u+v+1+1/3} \qquad (4.81)$$

4.8d: The process density $(v+1)(1-p)^v$:-

Consider the process density given by

$$f(p) = (v+1)(1-p)^v,\quad 0 \leq p \leq 1,\quad v \geq 0 \qquad (4.82)$$

The formulae for the EQL and the error-areas are obtained by putting $u = 0$ in those corresponding to the Beta PC given in Section 4.8b.

Suppose v is an integer. We have from (4.77)

$$(1-\pi_e)^{v+1} = \frac{\beta(c+1, n-c+v+1)}{\beta(c+1, n-c)} = \prod_{r=0}^{v} \left(1 - \frac{c+1}{n+1+r}\right) \qquad (4.83)$$

where π_e should be understood as $\pi_e(n,c)$.
Note that

$$\frac{c+1}{n+1} \geq \pi_e \geq \frac{c+1}{n+1+v} \qquad (4.84)$$

and that the error-area of the second kind at $p = \pi$ is

$$J_2(\pi,n,c) = (1-\pi)^{v+1} L(\pi,n,c) - (1-\pi_e)^{v+1} L(\pi,n+v+1,c) \qquad (4.85)$$

where either n or c may be increased by unity without altering the value in virtue of (4.83) and (4.5.1). It may be noted that the above formula is similar to that in the case of unweighted error-areas (cf. (4.21)).

Further, the error-area at the EQL is given by

$$J_{\sim}(\pi_e,n,c) = 2(1-\pi_e)^{v+1}\,[L(\pi_e,n,c+1) - L(\pi_e,n+v+1,c+1)] \tag{4.86.1}$$

$$= 2(1-\pi_e)^{v+1}\,[L(\pi_e,n+1,c+1) - L(\pi_e,n+v+2,c+1)] \tag{4.86.2}$$

The first relation is also valid for the error-area $J(\pi_e,n,c,c+1;\ a)$ of the composite plan obtained by randomizing between (n,c) and $(n,c+1)$ - procedure (2) of Section 4.5 - provided that π_e is taken to be the EQL of the composite plan i.e.,

$$(1-\pi_e)^{v+1} = a\,\frac{\beta(c+1,n-c+v+1)}{\beta(c+1,n-c)} + (1-a)\,\frac{\beta(c+2,n-c+v)}{\beta(c+2,n-c-1)}$$

$$= a\,[1-\pi_e(n,c)]^{v+1} + (1-a)\,[1-\pi_e(n,c+1)]^{v+1}$$

The second relation is also valid for the error-areas of the composite plans obtained by procedures (3) and (4) of randomization provided that π_e is taken as the respective EQLs of the composite plans (Section 4.5) i.e.,

$$(1-\pi_e)^{v+1} = a\,[1-\pi_e(n,c)]^{v+1} + (1-a)\,[1-\pi_e(n+1,c)]^{v+1}$$

in the case of procedure (3) of randomizing between (n,c) and $(n+1,c)$ and

$$(1-\pi_e)^{v+1} = a \; 1-\pi_e(n,c) \quad ^{v+1} + (1-a) \; 1-\pi_e(n+1,c+1) \quad ^{v+1}$$

in the case of procedure (4) of randomizing between (n,c) and (n+1,c+1).

The relationship under condition (B_a) is obtained by putting $u = 0$ in (4.80).

4.8e: <u>The process-density $(u+1)p^u$:-</u>

Consider the process density

$$f(p) = (u+1)p^u, \quad 0 \leq p \leq 1 \quad \text{and} \quad u \geq 0 \tag{4.87}$$

Either by putting $v = 0$ in the general formulae for the Beta PC given in earlier pages and then simplifying or by working out directly from the definitions, we obtain

$$\pi_e^{u+1} = \frac{\beta(c+u+2,\ n-c)}{\beta(c+1,n-c)}$$

where π_e should be understood to mean $\pi_e(n,c)$.

Suppose u is an integer. Then

$$\pi_e^{u+1} = \prod_{r=1}^{u+1} \frac{c+r}{n+r} \tag{4.88}$$

Note that

$$\frac{c+1}{n+1} \leq \pi_c \leq \frac{c+u+1}{n+u+1} \tag{4.89}$$

and that the error-area of the second kind at $p = \pi$ is

$$J_2(\pi,n,c) = \pi_c^{u+1} L(\pi,n+u+1,c+u+1) - \pi^{u+1} L(\pi,n,c) \tag{4.90}$$

where either c may be changed to $c-1$ or n to $n+1$ without altering the value because of (4.88) and (4.5.1). It may be observed that the above formula is similar to that in the case of unweighted error-areas (cf.(4.20))

The error-area at the EQL is given by

$$J(\pi_c,n,c) = 2\,\pi_c^{u+1} [L(\pi_c,n+u+1,c+u+1) - L(\pi,n,c)] \tag{4.91.1}$$

$$= 2\,\pi_c^{u+1} [L(\pi_c,n+u+2,c+u+1) - L(\pi,n+1,c)] \tag{4.91.2}$$

The first relation is also valid for the error-area of the composite plan obtained by procedure (2) whereas the second relation is also valid for the composite plans obtained by procedures (3) and (4) of randomization. (Section 4.5).

CHAPTER 5

NORMAL SINGLE SAMPLING OC CURVES

5.1 The normal OC and the error-areas

5.1a: The sampling procedure:-

Our study in this chapter is restricted to single sampling plans for inspection by variables where an item is defective if its quality characteristic x' exceeds an upper specification limit U. It is supposed that the variable x' - in a given lot - follows a normal distribution with mean μ and standard deviation σ. In symbols $x' \frown N(\mu, \sigma)$. μ is the mean of all items belonging to the lot. It is further supposed that σ is known. (The case of unknown σ is briefly dealt with towards the end of this chapter).

A sample of size n is drawn from the lot submitted for inspection and the quality characteristic x' is measured for each item in the sample. The sample average $\bar{x}'_n$ is computed. $\bar{x}'_n$ is the decision variable. The lot is accepted or rejected according as $\bar{x}'_n$ does not or does exceed an acceptance number c. It is important to note that c need not be an integer. The choice of c in any particular situation depends on n, u, σ and the quality protection sought from the plan. It is customary to write $c = U - k\sigma$ to show its dependence on U and σ. The plan is determined by the two elements n and c or alternatively by n and k when U and σ are given. The plan itself may be denoted by (n,c) or (n,k).

The case of one-sided lower specification limit where an item is defective if its quality characteristic x'' falls below a lower specification limit U' need not be considered separately, because it can be brought to the case of one-sided upper specification limit by putting $x' = -x''$ and $U = -U'$.

In this chapter, X will always denote a standard normal variate i.e., $X \frown N(0,1)$. Thus we may write

$\bar{x}'_n \frown N(\mu, \sigma/\sqrt{n})$ or $\bar{x}'_n = \mu + \sigma X/\sqrt{n}$.

The p.d.f. and the c.d.f. of X are denoted by $\phi(\cdot)$ and $\Phi(\cdot)$ respectively. (5.1)

$$\phi(x) = (2\pi)^{-1/2} \exp(-x^2/2) \quad \text{and} \quad \Phi(x) = \int_{-\infty}^{x} \phi(x)\,dx$$

The α-fractile of X is denoted by k_α:

$$\Phi(k_\alpha) = \alpha, \qquad 0 \leq \alpha \leq 1 \qquad (5.2)$$

5.1b: μ as a measure of lot quality:-

Since σ is known, μ - the mean of all items in the lot - can be taken as a measure of lot quality. The OC considered as a function of μ is denoted by $L^*(\mu,n,k)$ and the unweighted error-areas by $D_1^*(\mu,n,k)$ etc. Given μ, $\bar{x}_n'$ $N(\mu, \sigma/\sqrt{n})$. Therefore

$$L^*(\mu,n,k) = \Phi\left(\frac{U-k\sigma-\mu}{\sigma}\sqrt{n}\right) = P\left\{\xi^* > \mu\right\} \qquad (5.3)$$

where

$$\xi^* = U - k\sigma - \sigma X/\sqrt{n} .$$

The moments of ξ^* are finite and are of decreasing order of magnitude in $\sqrt{n}$. But the area under the unweighted OC does not exist, the integral $\int_{-\infty}^{\infty} L^*(\mu,n,k)d\mu$ being infinite. However the unweighted error-areas are finite for finite values of μ as can be seen from

$$D_1^*(\mu,n,k) = \int_{-\infty}^{\mu} [1-L^*(\mu,n,k)]d\mu$$

$$= \frac{\sigma}{\sqrt{n}} \int_{-\infty}^{\mu-U+k\sigma} \Phi(x)dx$$

$$= (\mu-U+k\sigma)\,\Phi\left(\frac{\mu-U+k\sigma}{\sigma}\sqrt{n}\right)$$

$$+ \frac{\sigma}{\sqrt{n}}\,\phi\left(\frac{\mu-U+k\sigma}{\sigma}\sqrt{n}\right) \qquad (5.4)$$

and similarly,

$$D_2^*(\mu,n,k) = (U-k\,\sigma-\mu)\,\Phi\left(\frac{U-k\sigma-\mu}{\sigma}\sqrt{n}\right)$$

$$+\frac{\sigma}{\sqrt{n}}\,\phi\left(\frac{U-k\,\sigma-\mu}{\sigma}\sqrt{n}\right) \qquad (5.5)$$

Since $\Phi(-x) = 1 - \Phi(x)$ and $\phi(-x) = \phi(x)$, we get

$$D_1^*(\mu,n,k) - D_2^*(\mu,n,k) = \mu - U+k\,\sigma \qquad (5.6)$$

It can be seen that the point $\mu = U-k\,\sigma$ is the point of equal error-areas as well as the mean of the random variable ζ^*. Further, each of the error-areas at $\mu = U-k\,\sigma$ is equal to $\sigma/\sqrt{2\pi n}$.

The total area under the weighted OC A^* is finite, because

$$A^* = \int_{-\infty}^{\infty} L^*(\mu,n,k)\,dF^*(\mu) < 1.$$

where $F^*(\cdot)$ is the c.d.f. of the prior distribution of μ. The weighted error-areas are defined as in earlier chapters:

$$J_1^*(\mu,n,k) = \int_{\mu}^{\infty}[1-L^*(\mu,n,k)]\,dF^*(\mu);\ \text{etc.} \qquad (5.7)$$

The EQL μ_e^* is given by

$$J_1^* (\mu_e^*, n,k) = J_2^*(\mu_e^*,n,k) \tag{5.8}$$

5.1c: <u>The proportion of defectives as a measure of lot quality:-</u>

It is sometimes natural and more convenient to measure the lot quality and also to state the specifications in terms of p, the proportion of defectives in the lot. p is a monotonic increasing function of μ and vice-versa. We may write if necessary $p(\mu)$ and $\mu(p)$ instead of p and μ respectively. The relations

$$p(\mu) = \bar{\Phi}\left(\frac{\mu - \bar{U}}{\sigma}\right) \text{ and } \mu(p) = U + k_p\sigma , \; 0 \leq p \leq 1, \quad -\infty \leq \mu \leq \infty \tag{5.9}$$

establish a one-to-one correspondence between p and μ. In particular,

$$p(-\infty) = 0; \; p(U) = \frac{1}{2} \text{ and } p(\infty) = 1.$$

The following derivatives may be noted.

$$\frac{dp}{d\mu} = \frac{\bar{1}}{\sigma} \phi \left(\frac{\mu - U}{\sigma}\right) = \frac{\bar{1}}{\sigma} \phi(k_p)$$

and
$$\frac{d}{dp} k_p = \frac{1}{\phi(k_p)} \tag{5.10}$$

The process characteristic can be given in terms of either p or μ. Given a distribution for μ, the induced distribution of p can be worked out from (5.9). It is easy to see that

$$F(p) = F^*(U + k_p\sigma) \text{ and } F^*(\mu) = F(\Phi(\frac{\mu - U}{\sigma})) \qquad (5.11)$$

where $F(\cdot)$ and $F^*(\cdot)$ are the c.d.f.s of the lot quality in terms of p and μ respectively.

5.1d: <u>The OC as a function of p:-</u>

The OC considered as a function of p is denoted by L(p,n,k). We have

$$\begin{aligned} L(p,n,k) &= P\left\{\bar{x}'_n \le U - k\sigma, \text{ given } p\right\} \\ &= \Phi(-k\sqrt{n} - k_p\sqrt{n}) \\ &= P\left\{X > (k + k_p)\sqrt{n}, \text{ given } p\right\} \\ &= P\left\{\xi > p\right\} \end{aligned} \qquad (5.12)$$

where ξ is a random variable given by

$$\xi = \Phi(X/\sqrt{n} - k) = \Phi(-X/\sqrt{n} - k) = 1 - \Phi(X/\sqrt{n} + k) \qquad (5.13)$$

since both X and -X are N(0,1).

The unweighted error-areas at the point $p = \pi$ are denoted by $D_1(\pi,n,k)$ etc.; the area under the unweighted OC by m or $m(n,k)$; the EQL by π_e and the weighted error-areas by $J_1(\pi,n,k)$ etc.

It is clear from (5.11) that for a given PC, the value of the weighted error-area under $L(p)$ at the point $p = \pi$ is equal to that under $L^*(\mu)$ at the point $\mu = \mu(\pi)$ i.e.,

$$J_1(\pi,n,k) = J_1^*(\mu(\pi),n,k) \text{ etc.,}$$

The EQL π_e corresponds to μ_e^* i.e., $\pi_e = p(\mu_e^*)$ where $J_1(\pi_e) = J_2(\pi_e)$ and $J_1^*(\mu_e^*) = J_2^*(\mu_e^*)$.

It may be noted that the case of unweighted error-areas under $L(p)$ refers to the case of a uniform prior distribution of p over the interval $(0,1)$ which in turn implies $\mu \frown N(U,\sigma)$. (The case of unweighted error-areas under $L^*(\mu)$ does not refer to any distribution at all).

Whether we work with p or μ, it is obvious that the calculation of the unweighted error-areas under $L(p)$ and also the approximations to the EQL and the weighted error-areas depend on the complete and incomplete moments of ξ as defined in (5.13).

5.2 The moments of $\Phi(ax + b)$

This section deals with the moments of $\Phi(aX + b)$, $X \sim N(0,1)$. First three lemmas are proved which will then be utilized to give formulae for the mean, variance and the mean deviation about the mean of $\Phi(aX + b)$ in Theorem 5.1.

The c.d.f. of a standardized j-variate normal vector is denoted by $\Phi_j(x_1, \ldots x_j\ ;\ (r_{ii'}))$ where $r_{ii'}$ is the correlation coefficient between the i-th and i'-th variables.

We will use, as and when necessary,

$$\Phi(-x) = 1 - \Phi(x)\ ;\ \phi(-x) = \phi(x)\ ;\ \text{and}$$

$$\Phi_2(x_1, x_2; r_{12}) = \Phi(x_1) - \Phi_2(x_1, -x_2; -r_{12}) \tag{5.14}$$

The Hermite polynomial $H_s(x)$ is defined by

$$H_0(x) = 1$$

and $$H_s(x)\,\phi(x) = (-1)^s \frac{d^s}{dx^s}\phi(x), \quad s = 1,2,\ldots \tag{5.15}$$

We have

$$\int_{-\infty}^{x} H_s(x)\, d\Phi(x) = \begin{cases} \Phi(x) & \text{if } s = 0 \\ -H_{s-1}(x)\,\phi(x) & \text{if } s = 1,2,\ldots \end{cases} \tag{5.16}$$

Also

$$H_s(-x) = (-1)^s H_s(x) \; ; \tag{5.17}$$

$$E_X H_s(X) = \int_{-\infty}^{\infty} H_s(x)\, d\,\Phi(x) = 0 \qquad \text{if } s \geq 1 \; ; \tag{5.18}$$

and the orthogonal properties

$$E_X H_r(X) H_s(X) = \int_{-\infty}^{\infty} H_r(x) H_s(x)\, d\,\Phi(x)$$

$$= \begin{cases} 0 & \text{if } r \neq s \\ s! & \text{if } r = s \end{cases} \tag{5.19}$$

The following identity may be noted (page 133 in Cramer, 1946)

$$\frac{1}{\sqrt{1-t^2}} \exp\left[-\frac{t^2x^2 + t^2y^2 - 2txy}{2(1-t^2)}\right] = \sum_{0}^{\infty} \frac{H_s(x)\; H_s(y)}{s!} t^s \tag{5.20}$$

Further, we have from page 787 in Abramowitz and Stegun (1964),

$$|H_s(x)\; \phi(x)| < C_o \sqrt{\frac{s!}{2\pi}}\; e^{-\frac{x^2}{4}} \quad \text{where} \quad C_o \doteq 1.086435 \tag{5.21}$$

Lemma 5.1:-

(a) $$\int_{-\infty}^{c} \prod_{i=1}^{j} \Phi(a_i x + b_i)\, \phi(x) dx$$

$$= \Phi_{j+1}\left(\frac{b_1}{\sqrt{1+a_1^2}}, \ldots \frac{b_j}{\sqrt{1+a_j^2}}, c; (r_{ii'})\right)$$

where

$$r_{ii'} = r_{i'i} = \begin{cases} 1, & i = i' = 1,2,\ldots j, j+1 \\ \dfrac{a_i\, a_{i'}}{\sqrt{(1+a_i^2)(1+a_{i'}^2)}}, & i \neq i', i, i' = 1,2,\ldots, j \\ \dfrac{-a_i}{\sqrt{1+a_i^2}} & i' = j+1,\ i = 1,2,\ldots j \end{cases}$$

(b) $$E_X \prod_{i=1}^{j} \Phi(a_i X + b_i) = \Phi_j\left(\frac{b_1}{\sqrt{1+a_1^2}}, \ldots \frac{b_j}{\sqrt{1+a_j^2}}; (r_{ii'})\right) \tag{5.23}$$

<u>Proof:</u>

(a) The left-side of (5.22) is the joint probability that $X_i \leq a_i X + b_i$, $i = 1,2,\ldots, j$ and $X \leq C$ where X_i and X are the $j+1$ independently distributed standard normal variables. The inequalities may be written as

$$Z_i = \frac{X_i - a_i Z_{j+1}}{\sqrt{1+a_i^2}} \leq \frac{b_i}{\sqrt{1+a_i^2}}, \quad i = 1,2,\ldots, j \quad \text{and}$$

$$X = Z_{j+1} \leq C$$

The result is now proved by noting that Z_i, $i = 1,2,\ldots, j, j+1$ are jointly distributed as a standardized $j+1$ - variate normal distribution with $r_{ii'}$ as the correlation coefficient between Z_i and $Z_{i'}$.

(b) follows by putting $C = \infty$ in (a).

It may be noted that when $a_i = a$ and $b_i = b$, the left sides of (5.22) and (5.23) reduce respectively to the incomplete and complete moments of order j about the origin for the random variable $\Phi(aX + b)$.

By taking $j = 1$, $a_i = a$ and $b_i = b$ in (5.22) we obtain the first incomplete moment as

$$\int_{-\infty}^{C} \Phi(ax+b)\phi(x)dx = \Phi_2\left(\frac{b}{\sqrt{1+a^2}}, C; \frac{-a}{\sqrt{1+a^2}}\right) \qquad (5.24)$$

In the context of using the available tables of bivariate normal distributions, an alternative formula contained in the following lemma may prove useful

<u>Lemma 5.2:-</u>

$$\int_{-\infty}^{C} \Phi(ax+b)\phi(x)dx - \Phi(C)\Phi(aC+b)$$

$$= \begin{cases} -\Phi_2\left(\frac{-b}{\sqrt{1+a^2}}, aC+b; \frac{-1}{\sqrt{1+a^2}}\right), & \text{if } a > 0 \\ 0 & \text{if } a = 0 \\ \Phi_2\left(\frac{b}{\sqrt{1+a^2}}, -(aC+b); \frac{-1}{\sqrt{1+a^2}}\right), & \text{if } a < 0 \end{cases}$$

(5.25)

Proof:-

Integrating by parts, the left-side of (5.25) is seen to be

$$-\int_{-\infty}^{C} a\,\phi\,(ax+b)\,\bar{\Phi}\,(x)dx$$

On changing $ax+b$ to x when $a > 0$ and to $-x$ when $a < 0$, the above expression takes the form

$$-\int_{-\infty}^{aC+b} \bar{\bar{\Phi}}\left(\frac{x-b}{a}\right)\phi(x)dx, \quad \text{if} \quad a > 0 \quad \text{and}$$

$$\int_{-\infty}^{-aC-b} \bar{\bar{\Phi}}\left(\frac{-x-b}{a}\right)\phi(x)dx, \quad \text{if} \quad a < 0$$

In (5.22) take $j = 1$, $b_1 = -b/a$ and $a_1 = 1/a$ or $-1/a$ according as $a > 0$ or < 0. The lemma is now proved if we note that

$$\frac{b}{a\sqrt{1+\frac{1}{a^2}}} = \pm\frac{b}{\sqrt{1+a^2}} \quad \text{according as} \quad a > 0 \quad \text{or} \quad < 0$$

and

$$\frac{1}{a\sqrt{1+\frac{1}{a^2}}} = \pm\frac{1}{\sqrt{1+a^2}} \quad \text{according as} \quad a > 0 \quad \text{or} \quad < 0$$

The next lemma gives a series for calculating the central moments of $\bar{\Phi}\,(aX+b)$.

Lemma 5.3:-

$$\Phi(ax+b) - \Phi(b/\sqrt{1+a^2})$$

$$= \phi(b/\sqrt{1+a^2}) \sum_{s=0}^{\infty} \frac{H_{s+1}(x)\, H_s(-b/\sqrt{1+a^2})}{(s+1)!} \left(\frac{a}{\sqrt{1+a^2}}\right)^{s+1} \tag{5.26}$$

Proof:- Write

$$ax + b = \frac{a'x + b'}{\sqrt{1 - a'^2}}$$

where

$$a' = \frac{a}{\sqrt{1+a^2}} \quad \text{and} \quad b' = \frac{b}{\sqrt{1+a^2}}$$

(5.26) is simply a Taylor expansion of $\Phi(ax+b)$ in powers of a'. It can also be proved by integrating the identity (5.20) with respect to $\Phi(y)$ from $y = -\infty$ to $y = b'$ and then changing t to $-a'$.

The following theorem can now be proved with the help of lemmas 5.1, 5.2 and 5.3.

Theorem 5.1:-

The mean, variance and the mean deviation about the mean of the random variable $\Phi(aX+b)$, $X \frown N(0,1)$ are respectively given by

(a) $E_X \bar{\Phi}(aX+b) = \bar{\Phi}(b')$ (5.27)

(b) $V_X \bar{\Phi}(aX+b) = \bar{\Phi}_2(b', b'; a'^2) - [\bar{\Phi}(b')]^2$ (5.28.1)

$$= [\phi(b')]^2 \sum_{s=0}^{\infty} \frac{[H_s(b')]^2}{(s+1)!} (a')^{2s+2} \quad (5.28.2)$$

$$< \frac{1}{5} \exp\left(-\frac{b'^2}{2}\right) \log(1+a^2) \quad (5.28.3)$$

(c) $E_X |\bar{\Phi}(aX+b) - \bar{\Phi}(b')|$

$$= 2\, \bar{\Phi}_2\left(b', -b'; \frac{-1}{\sqrt{1+a^2}}\right) \quad (5.29.1)$$

$$= 2\partial_a [\bar{\Phi}_2(b', b''; a') - \bar{\Phi}(b')\bar{\Phi}(b'')] \quad (5.29.2)$$

$$= 2\partial_a\, \phi(b')\phi(b'') \sum_{0}^{\infty} \frac{H_s(b')H_s(b'')}{(s+1)!} (a')^{s+1} \quad (5.29.3)$$

$$< \frac{2}{5} \exp\left(-\frac{b'^2 + b''^2}{4}\right) \log \frac{1}{1-|a'|} \quad (5.29.4)$$

where

$$a' = \frac{a}{\sqrt{1+a^2}}; \quad b' = \frac{b}{\sqrt{1+a^2}};$$

$$b'' = \frac{b-b'}{a} = \frac{b}{a}\left(1 - \frac{1}{\sqrt{1+a^2}}\right); \quad \text{and}$$

$$\partial_a = \begin{cases} +1 & \text{if } a > 0 \\ 0 & \text{if } a = 0 \\ -1 & \text{if } a < 0 \end{cases}$$

<u>Proof:-</u>

(a) is a direct consequence of lemma 5.3 and the relation (5.18)

(b) The first part of the relation - namely (5.28.1) - follows from (5.23) if we put $j = 2$, $a_1 = a_2 = a$ and $b_1 = b_2 = b$. The second part - namely (5.30.2) - follows by squaring (5.26) and then integrating with respect to $\Phi(x)$ from $x = -\infty$ to $x = \infty$ because of the orthogonal properties of Hermite polynomials (5.19)

From (5.21) and (5.28.2) we get

$$V_X \, \Phi(aX + b) < \frac{C_0^2}{2\pi} e^{-b'^2/2} \sum_{s=0}^{\infty} \frac{(a')^{2s+2}}{s+1} \tag{5.30}$$

$$= \frac{C_0^2}{2\pi} e^{-b'^2/2} \log \frac{1}{1 - (a')^2}$$

The result (5.28.3) follows because $C_0^2/2\pi < 1/5$ and $1/(1 - a'^2) = 1 + a^2$.

(c) When $X \leq (b' - b)/a = -b''$,

$\Phi(aX+b) - \Phi(b') \geq 0$ or ≤ 0 according as $a \leq 0$ or ≥ 0. Therefore the mean deviation about the mean of $\Phi(aX+b)$ is equal to

$$-2\partial_a \int_{-\infty}^{-b''} [\Phi(ax+b) - \Phi(-b')]\, \phi(x)\,dx \qquad (5.31)$$

In (5.25) of lemma 5.2, take $C = -b''$ so that $aC + b = -ab'' + b = b'$. Now (5.31) together with (5.25) yield (5.29.1). On the other hand, taking $C = -b''$ in (5.24) and applying (5.31) and (5.14) we obtain (5.29.2).

Integrating the identity (5.20) with respect to $\Phi(x)$ from $x = -\infty$ to $x = b'$ and then with respect to $\bar{\Phi}(y)$ from $y = -\infty$ to $y = b''$ we arrive at (5.29.3).

To prove (5.29.4) we find from (5.29.3) and the inequality (5.21)

$$E_X |\Phi(aX+b) - \Phi(b')|$$

$$< \frac{2C_0^2}{2\pi} e^{-\frac{\bar{b}'^2 + b''^2}{4}} \sum_{s=0}^{\infty} \frac{|a'|^{s+1}}{s+1} \qquad (5.32)$$

$$< \frac{2}{5} e^{-\frac{\bar{b}'^2 + b''^2}{4}} \log \frac{1}{1 - |a'|}$$

This completes the proof of (c) and hence the theorem.

Remark: -

It follows from the formulae given in the above theorem that the mean of $\Phi(aX+b)$ is a function of b and $|a|$ - i.e., it is independent of the sign of a - whereas the variance and the mean deviation about the mean are independent of the signs of both a and b. This is only to be expected because both X and -X are N(0,1) so that

$$\Phi(-aX+b) = \Phi(aX+b)$$

and hence

$$\Phi(aX-b) = \Phi(-aX-b) = 1-\Phi(aX+b) \tag{5.33}$$

5.3 Formulae for the unweighted error-areas

The formulae for the unweighted error-areas under the OC, $L(p,n,k) = \Phi(-k\sqrt{n} - k_p\sqrt{n})$ are derived with the help of the results given in the previous section.

In relation to a given plan (n,k) the following notation are used throughout the remainder of the present chapter.

$$h = \frac{k\sqrt{n}}{\sqrt{n+1}}\ ;\quad \rho = \frac{1}{\sqrt{n+1}} \text{ and } h_1 = h(\sqrt{n+1} - \sqrt{n}) \tag{5.34}$$

The error-area of the first kind at the point $p = \pi$ is given by

$$E_G \epsilon_\pi (\pi - \xi) = D_1(\pi,n,k) = \int_0^\pi [1 - L(p,n,k)]\,dp$$

$$= \int_0^\pi \Phi(k\sqrt{n} + k_p\sqrt{n})\,dp$$

$$= \int_{-\infty}^{k_\pi} \Phi(x\sqrt{n} + k\sqrt{n})\phi(x)\,dx \tag{5.35}$$

where $G(\cdot)$ is the c.d.f. of the random variable ξ defined in (5.13) and ϵ_π is 1 or 0 according as $\xi \leq \pi$ or $> \pi$. The error-area of the second kind is given by

$$E_G(1-\epsilon_\pi)(\xi - \pi) = D_2(\pi,n,k) = \int_\pi^1 L(p,n,k)\,dp$$

$$= \int_{-\infty}^{-k_\pi} \Phi(x\sqrt{n} - k\sqrt{n})\phi(x)\,dx \ . \tag{5.36}$$

The above relations together with (5.24) and the lemma 5.2 give the formulae: (5.37)

$$D_1(\pi,n,k) = \Phi_2(h,k_\pi; -\sqrt{1-\rho^2})$$

$$= \pi\,\Phi(k\sqrt{n} + k_\pi\sqrt{n}) - \Phi_2(h,k\sqrt{n} + k_\pi\sqrt{n}; -\rho)$$

and

$$D_2(\pi,n,k) = \Phi_2(-h, -k_\pi ; -\sqrt{1-\rho^2})$$

$$= (1-\pi)\Phi(-k\sqrt{n} - k_\pi\sqrt{n})$$

$$-\Phi_2(-h, -k\sqrt{n} - k_\pi\sqrt{n}; -\rho)$$

The area under the unweighted OC $L(p,n,k)$ is equal to the mean of the random variable ξ. It is therefore denoted by m or $m(n,k)$

$$E_G\xi = m = \int_0^1 L(p,n,k)dp = D_2(o,n,k) = \Phi(-h) \qquad (5.38)$$

Also

$$k_m = -h \text{ so that } \mu(m) = U - h\sigma \qquad (5.39)$$

It is easy to verify that

$$D_1(\pi,n,k) - D_2(\pi,n,k) = \pi - \Phi(-h) = \pi - m \qquad (5.40)$$

Thus m is the point at which the two kinds of error-areas are equal.

The IQL $\pi_0 = \pi_0(n,k)$ is close to m. By definition

$$L(\pi_0,n,k) = \frac{1}{2} \text{ i.e., } \pi_0 = \Phi(-k) \qquad (5.41)$$

Thefefore

$$k_{\pi_0} = -k \text{ so that } \mu(\pi_0) = U - k\sigma \qquad (5.42)$$

Note that

$$L(m,n,k) = \bar{\Phi}(-k\sqrt{n} - k_m\sqrt{n}) = \bar{\Phi}(-h_1)$$

$$= \frac{1}{2} - \frac{k}{2\sqrt{2\pi n}} + O(n^{-3/2}) \qquad (5.43)$$

(Here $1/\sqrt{2\pi} = \phi(0) = .39894$)

The difference between the two kinds of error-areas at $p = \pi_0$ is given by

$$D_1(\pi_0,n,k) - D_2(\pi_0,n,k) = \pi_0 - m$$

$$= \bar{\Phi}(-k) - \bar{\Phi}(-k\sqrt{n/(n+1)})$$

$$= -\frac{h}{2n}\phi(h) + O(n^{-2}), \text{ or}$$

$$-\frac{k}{2n}\phi(k) + O(n^{-2}) \qquad (5.44)$$

Exact values of m, π_0 and $L(p,n,k)$ are determined without any difficulty because extensive tables of $\bar{\Phi}(x)$ and k_p are available in literature. (National Bureau of Standards (1951); BMM tables, 1966) etc. The error-areas can be calculated with the help of tables of bivariate normal distributions (National Bureau of Standards (1959); Owen (1957) and 1962).

5.4 The behaviour of the error-areas and the optimality conditions

The OC and the (unweighted) error-areas in the case of normal sampling plans (n,k) have the same monotonic properties with respect to p,n and $c = U - k\sigma$ as those of the binomial sampling plans (See (2.51)). Besides they are continuous functions of c (or k).

Condition (A):-

Given n amd $p = \pi$, the value of $k = k^{(e)} = k^{(e)}(\pi,n)$ say - for which the bigger of the two error-areas is a minimum should satisfy

$$\pi = m(n,k^{(e)}) = \Phi(-k^{(e)}\sqrt{n/(n+1)})$$

i.e.,
$$k^{(e)} = -k_\pi \sqrt{(n+1)/n} \tag{5.45}$$

This follows from Theorem 2.5. Thus plans satisfying condition (A) should have the stipulated quality level π as their point of equal error-areas.

Condition (B):-

Given n and $p = \pi$, the problem is to determine k such that $D_1(\pi,n,k) + \partial D_2(\pi,n,k)$, $\partial > 0$ is a minimum.

Using (5.35) and (5.36)

$$D_1(\pi,n,k) + \partial D_2(\pi,n,k) = \int_{-\infty}^{k_\pi} \Phi(x\sqrt{n} + k\sqrt{n})\phi(x)dx$$

$$+ \partial \int_{-\infty}^{-k_\pi} \Phi(x\sqrt{n} - k\sqrt{n})\phi(x)dx$$

Differentiating with respect to k and then equating the derivative to zero we obtain after simplifications

$$\Phi\,(-k_\pi \sqrt{n+1} - kn / \sqrt{n+1}\,) = (1+ \partial)^{-1} \qquad (5.46)$$

from which it is easy to determine k.

Let $k^{(i)} = k^{(i)}(\pi,n)$ be the value of k satisfying the condition (B). Taking $\partial = 1$ in (5.46) we get

$$\pi = \Phi(-k^{(i)}n/(n+1) \quad \text{or} \quad k^{(i)} = - k_\pi(n+1)/n \qquad (5.47)$$

Given n and $p = \pi$, if $k^{(o)} = k^{(o)}(\pi,n)$ is the value of k such that π is the IQL of the plan $(n,k^{(o)})$, then from (5.41)

$$\pi = \Phi(-k^{(o)}) \text{ i.e., } k^{(o)} = - k_\pi \qquad (5.48)$$

Comparing (5.45), (5.47) and (5.48) we observe that $k^{(c)}$ lies between $k^{(o)}$ and $k^{(i)}$. It implies that plans satisfying condition (A) are closer to optimum plans under condition (B) as compared to those for which π is the IQL.

Asymptotic properties of plans satisfying conditions (A) and (B):

The error-areas at m are

$$D_1(m,n,k) = D_2(m,n,k) = \frac{1}{2} E_G \mid \xi - m \mid$$

$$= \frac{1}{2} E_X \mid \Phi(X/\sqrt{n} + k) - \Phi(h) \mid, \qquad (5.49)$$

using (5.13) and (5.38). From Theorem 5.1(c) we obtain by putting $a = 1/\sqrt{n}$ and $b = k$ in (5.29.3)

$$D_1(m,n,k) = \phi(h)\phi(h_1)\left(\frac{1}{\sqrt{n+1}} + \frac{hh_1}{2}\frac{1}{n+1} + \dots\right) \quad (5.50)$$

As $n \to \infty$, $k \to 0$ such that $m(n,k) = \Phi(-h)$ is kept fixed at a given value (Condition (A)), each of the error-areas tends to zero with the order of $n^{-1/2}$.

For plans satisfying condition (B), using (5.40) and (5.47)

$$D_1(\pi,n,k) - D_2(\pi,n,k) = \pi - \Phi(-k\sqrt{n/(n+1)})$$

$$= \pi - \Phi(k_\pi \sqrt{(n+1)/n})$$

$$= \frac{k_\pi}{2n}\phi(k_\pi) + 0(n^{-2}) \qquad (5.51)$$

As $n \to \infty$, $k \to 0$ such that $kn/(n+1)$ remains constant at a given value k_π, the condition (B) approaches the condition (A) with the order of n^{-1}, whereas each of the error-areas approach zero with the order of $n^{-1/2}$.

It is clear from (5.45)-(5.48) and the above discussion that the optimality conditions (A) and (B) are asymptotically equal to an IQL relationship.

5.5 The mean, variance and the mean deviation about the mean of ξ

For a given plan (n,k), the hypothetical random variable ξ is defined by (5.13) i.e., by

$$\xi = \Phi(X/\sqrt{n} - k) = \Phi(-X/\sqrt{n} - k) = 1 - \Phi(X/\sqrt{n} + k) \tag{5.52}$$

Using the results given in Section 5.2, the moments of ξ can be expressed in terms of c.d.f.s of suitable multivariate normal distributions. The complete moment of order j and also the incomplete moments of order $j-1$ depend on c.d.f.s of multivariate normal vectors whose dimensions are atmost j.

The moments of ξ depend only on the two elements n and k.

The mean of ξ is the area under the unweighted OC $L(p,n,k)$. It is also the point of equal error-areas. It is denoted by $m(n,k)$ or simply by m.

The mean deviation about the mean of ξ is of course the total error-area under the unweighted OC at the point m. It is therefore denoted by $D(m,n,k)$. Since it is a function of n and k only, we may write in this section $D(n,k)$. It is shown in Section 5.8 that in the case of a normal PC with $\mu \frown N(u,v)$ the value of $J(\pi_e,n,k)$ - the weighted error-area at the EQL π_e - is equal to $D(n',k')$ where n' and k' are suitably defined quantities.

The variance of ξ is denoted by $V(n,k)$.

The mean, variance and the mean deviation about the mean of ξ are the most important moments of ξ from the point of view of error-areas. They are studied in some detail in this section.

Substituting $a = 1 / \sqrt{n}$ and $b = k$ in Theorem 5.1 we obtain the formulae:

$$m = \Phi(-h) \tag{5.53}$$

$$V(n,k) = \Phi_2 \left(-h,-h; \frac{1}{n+1}\right) - [\Phi(-h)]^2 \tag{5.54}$$

$$V(n,k) = \phi(h)^2 \left[\frac{1}{n+1} + \frac{h^2}{2(n+1)^2} + \frac{(h^2-1)^2}{6(n+1)^3} + \frac{(h^3-h)^3}{24(n+1)^4} + \dots \right] \quad (5.54.2)$$

and

$$D(n,k) = 2\,\Phi_2\left(h,-h;\ -\sqrt{\frac{n}{n+1}}\right) \quad (5.55.1)$$

$$= 2\left[\Phi_2\left(-h,-h_1;\ \frac{1}{\sqrt{n+1}}\right) - \Phi(-h)\Phi(-h_1)\right] \quad (5.55.2)$$

$$= 2\phi(h)\phi(h_1)\left[\frac{1}{\sqrt{n+1}} + \frac{hh_1}{2(n+1)} + \frac{(h^2-1)(h_1^2-1)}{6(n+1)\sqrt{n+1}} + \frac{(h^3-h)(h_1^3-h_1)}{24(n+1)^2} + \dots\right] \quad (5.55.3)$$

where

$$h = k\sqrt{\frac{n}{n+1}} \quad \text{and} \quad h_1 = h\left(\sqrt{n+1} - \sqrt{n}\right) \quad (5.56)$$

Table 5.1 gives $\sqrt{n/(n+1)}$ and $\sqrt{n+1} - \sqrt{n}$ for $n = 1(1)20$.

TABLE 5.1

n	$\sqrt{\frac{n}{n+1}}$	$\sqrt{n+1}-\sqrt{n}$
1	0.7071	0.4142
2	0.8165	0.3178
3	0.8660	0.2679
4	0.8944	0.2361
5	0.9129	0.2134
6	0.9258	0.1963
7	0.9354	0.1827
8	0.9428	0.1716
9	0.9487	0.1623
10	0.9535	0.1543

n	$\sqrt{\frac{n}{n+1}}$	$\sqrt{n+1}-\sqrt{n}$
11	0.9574	0.1475
12	0.9608	0.1414
13	0.9636	0.1361
14	0.9661	0.1313
15	0.9682	0.1270
16	0.9702	0.1231
17	0.9718	0.1195
18	0.9733	0.1163
19	0.9747	0.1132
20	0.9759	0.1104

Note:- This table is taken from Subrahmanya (1966).

The variance and the mean deviation about the mean of ξ may be considered as functions of n and h. In that case they are denoted by $V'(n,h)$ and $D'(n,h)$ respectively. This representation is useful for the purpose of tabulations.

Theorem 5.2:-

Given $h \geq 0$ or $k \geq 0$, each of the functions $V(n,k)$, $D(n,k)$, $V'(n,h)$ and $D'(n,h)$ is a decreasing function of each of its arguments.

Proof:-

The theorem is easily proved by showing that the appropriate differential coefficients are negative. To do this, it is sometimes convenient to differentiate the series representations of V and D and then to sum them up using the identity (5.20) and the lemma 5.3. For example we will show that $\frac{\partial}{\partial n} D(n,k)$ is negative. We have from (5.29.3)

$$D(n,k) = 2\phi(h)\phi(h_1) \sum_{s=0}^{\infty} \frac{H_s(h)H_{s-1}(h_1)}{(s+1)!} \rho^{s+1}$$

where h and h_1 are as given in (5.56) and $\rho = 1/\sqrt{n+1}$. Differentiating $D(n,k)$ with respect to ρ.

$$\frac{\partial D}{2\partial \rho} = \phi(h)\phi(h_1) \sum_0^\infty \frac{H_s(h)H_s(h_1)}{s!} \rho^s$$

$$- \phi(h)\phi(h_1) \sum_0^\infty \frac{H_{s+1}(h)H_s(h_1)}{(s+1)!} \rho^{s+1} \frac{\partial h}{\partial \rho}$$

$$- \phi(h)\phi(h_1) \sum_0^\infty \frac{H_s(h)H_{s+1}(H_1)}{(s+1)!} \rho^{s+1} \frac{\partial \bar{h}_1}{\partial \rho} \qquad (5.57)$$

By (5.20), the first term is equal to

$$\frac{1}{2\pi \sqrt{1-\rho^2}} \exp\left(-\frac{h^2+h_1^2-2hh_1\rho}{2(1-\rho^2)}\right) \geq 0$$

Using lemma 5.3, the coefficient of $\frac{\partial h}{\partial \rho}$ in (5.57) is seen to be equal to

$$\phi(h)[\bar{\Phi}(h/\sqrt{n} + h_1\sqrt{(n+1)/n}) - \bar{\Phi}(h_1)]$$

$$= \phi(h)[\bar{\Phi}(-h_1) - \bar{\Phi}(h_1)] < 0$$

because h_1 is positive. Now $h = k\sqrt{1-\rho^2}$ so that $\frac{\partial h}{\partial \rho}$ is negative. It follows that the second term in (5.57) is positive. Similarly it can be shown that the third term is also positive. Thus $\frac{\partial D}{\partial \rho}$ is positive i.e., $\frac{\partial D}{\partial n}$ is negative. This proves that $D(n,k)$ is a decreasing function of n.

The other parts of the theorem can be proved similarly or by using the integral representations of V and D.

Some of the special values of $V'(n,h)$ and $D'(n,h)$ are given below.

TABLE 5.2

n	h	V'(n,h)	D'(n,h)
0	0	0.25	0.50
0	h	$\Phi(h)\,\Phi(-h)$	$2\,\Phi(h_1)\,\Phi(-h_1)$
n	0	$\frac{1}{2\pi}\sin^{-1}\frac{1}{n+1}$	$\frac{1}{\pi}\sin^{-1}\frac{1}{\sqrt{n+1}}$
n	∞	0	0
∞	h	0	0

When $n \geq 1$, we have (5.58)

$$0 \leq V'(n,h) \leq V'(1,\ 0) = \frac{1}{12}\ , \quad \text{and}$$

$$0 \leq D'(n,h) \leq D'(1,\ 0) = \frac{1}{4}$$

Note also that always $D \leq \sqrt{V}$.

Calculation of V and D involves the c.d.f s of bivariate normal distributions with correlation coefficients $(n+1)^{-1}$, $(n+1)^{-1/2}$ or $\sqrt{n/(n+1)}$, $n = 1,2,\ldots$ These are not found in the available tables of bivariate normal distributions - National Bureau of Standards (1959) and Pearson (1931).

Note that $D = 4T(h,a)$, $a = \sqrt{n+1} - \sqrt{n}$ where $T(h,a)$ is the function defined in Owen (1957) and (1962), i.e.

$$T(h,a) = \frac{1}{2\pi} \int_0^a \frac{\exp[-h^2(1+x^2)/2]}{1+x^2}\,dx \,.$$

However $T(h,a)$ is not tabulated in Owen (1957) and (1962) for the values of a as given by $\sqrt{n+1} - \sqrt{n}$, $n = 1,2,\ldots$.

It is not difficult to compute V and D from the series (5.54.2) and (5.55.3) which converge fairly rapidly even for small values of n: if V is computed by taking the first i terms of the series (5.54.2), the error committed will be less than (cf. (5.21) and (5.30))

$$\frac{C_o^2}{2\pi} e^{-h^2/2} \sum_{s=i}^{\infty} \frac{1}{(s+1)(n+1)^{s+1}}, \qquad C_o \doteq 1.086435$$

$$< \frac{C_o^2}{2\pi} e^{-h^2/2} \frac{1}{i+1} \frac{1}{(n+1)^{i+1}} \cdot \frac{1}{1-(n+1)^{-2}}$$

$$< \frac{.19}{i+1} \frac{1}{n(n+1)^i}, \quad \text{for any } h \qquad (5.59)$$

and similarly in the case of D, the absolute error committed in taking the first i terms of the series (5.55.3) is less than (cf. (5.21) and (5.32))

$$\frac{c_o^2}{2\pi} e^{-\frac{h^2+h_1^2}{4}} \frac{1}{i+1} \frac{1}{(\sqrt{n+1})^{i+1}} \cdot \frac{1}{1-(\sqrt{n+1})^{-1}}$$

$$< \frac{.38}{i+1} \frac{1}{(\sqrt{n+1}-1)} \cdot \frac{1}{(\sqrt{n+1})^{i}}, \quad \text{for any } h \qquad (5.60)$$

Using Fortran programming techniques, values of V and D were calculated on an IBM 1401 computer $n = 1(1)\,19$ and $h = .1(.1)4$. Typical values of V and D are presented in Table 5.3 given below. The values of $m = \Phi(-h)$ are also shown in the table.

TABLE 5.3

The mean (m), the variance (V) and the mean deviation about the mean (D) of the random variable $\xi = \Phi(X/\sqrt{n}-k)$, $X \sim N(0,1)$ in connection with normal single sampling plans (n, k).

Note:- $h = k\sqrt{n/(n+1)}$

m	h	n = 1		n = 2		n = 3	
		V	D	V	D	V	D
42	0.2	0.081	0.24	0.052	0.19	0.039	0.16
34	0.4	073	23	047	18	035	15
27	0.6	062	21	040	16	029	14
21	0.8	050	18	031	14	023	12
16	1.0	037	15	023	12	017	10
14	1.1	032	13	019	10	014	090
12	1.2	026	12	016	093	011	080
097	1.3	022	10	013	082	0089	070
081	1.4	017	089	010	071	0070	061
067	1.5	014	077	0079	061	0054	053
055	1.6	011	065	0061	052	0041	045
045	1.7	0084	055	0046	044	0031	038
036	1.8	0064	046	0034	037	0023	032
029	1.9	0048	038	0025	030	0016	026
023	2.0	0035	031	0018	025	0012	022
018	2.1	0026	025	0013	020	00081	017
014	2.2	0018	020	0088	016	00056	014
011	2.3	0013	016	0061	013	00038	011
0082	2.4	00091	012	0041	010	00025	0088
0062	2.5	00063	0094	00027	0078	00016	0068

TABLE 5.3 (contd.)

m	h	n = 5		v = 10		n = 15	
		V	D	V	D	V	D
0.42	0.2	0.026	0.13	0.014	0.096	0.0096	0.079
34	0.4	023	12	012	090	0085	074
27	0.6	019	11	010	081	0070	067
21	0.8	015	097	0079	071	0054	058
16	1.0	011	081	0056	059	0038	049
14	1.1	0087	072	0046	053	0031	044
12	1.2	0070	064	0037	047	0025	039
097	1.3	0056	057	0029	042	0019	034
081	1.4	0044	050	0022	036	0015	030
067	1.5	0033	043	0017	031	0011	026
055	1.6	0025	037	0013	027	00083	022
045	1.7	0019	031	00091	023	00060	019
036	1.8	0013	026	00065	019	00043	016
029	1.9	00096	021	00046	016	00030	013
023	2.0	00067	018	00032	013	00021	011
018	2.1	00046	014	00021	011	00014	0088
014	2.2	00031	011	00014	0085	000091	0071
011	2.3	00020	0091	000092	0068	000059	0056
0082	2.4	00013	0072	000059	0054	000037	0044
0062	2.5	000085	0056	000037	0042	000023	0035

5.6 The $\{m_e, D_e\}$ system of sampling plans

Plans can be constructed so as to satisfy the specifications (5.61)

$$m(n,k) = m_e$$

and $$D(m_e, n, k) = D_e$$

where m_e and D_e are given numbers. The relations (5.53) and (5.55) give the transformation from the set (n,k) to the set $\{m_e, D_e\}$ and vice-versa. Choice of m_e fixes $h = k\sqrt{n/(n+1)}$; h and D_e together determine the elements of the plan n and k.

Some of the plans belonging to MIL-STD 414 (US Defence Department, 1957) are studied from the point of view of error-areas. The results are given in Table 5.4.

TABLE 5.4

Some plans belonging to MIL-STD 414 system (single sampling, one-sided specifications limit, σ known case)

No.	AQL %.	plan n	k	IQL π_0 %.	m %.	D(m) %.
1	0.04	3	2.58	0.5	1.3	1
2	0.04	10	2.83	0.2	0.4	1/4
3	0.25	2	1.94	2.6	5.7	5
4	0.25	19	2.41	0.8	0.9	1/2
5	1.00	2	1.42	7.8	12.5	10
6	1.00	7	1.80	3.6	4.6	3
7	4.00	3	1.01	15.6	22.4	12
8	4.00	5	1.20	11.5	13.6	7
9	4.00	93	1.51	6.6	6.7	1
10	10.00	3	0.573	28.4	30.9	15
11	10.00	5	0.728	23.3	25.5	11
12	10.00	21	0.942	17.4	17.9	4
13	10.00	49	1.03	15.2	15.4	3
14	10.00	127	1.07	14.2	14.2	1

Note: This table is taken from Subrahmanya (1966).

5.7 The EQL and the weighted error-areas

As usual let $F(p)$ and $f(p)$ denote respectively the c.d.f. and the p.d.f. of the prior distribution of p and $F_j(p)$ the j-th derivative of $F(p)$ with respect to p. Assuming the required conditions for their existence and continuity, we have from (1.20), (2.48) and (2.49) the following expansions for $F(\pi_e)$ - π_e being the EQL - and the weighted error-areas $J_i(m,n,k)$, $i = 1,2$.

$$F(\pi_e) = F(m) + \frac{\mu_2}{2!} F_2(m) + \frac{\mu_3}{3!} F_3(m) + \dots \qquad (5.62)$$

$$J_1(m,n,k) = \frac{1}{2} D(m,n,k) - E_G\, \epsilon_m \left[\frac{(\xi - m)^2}{2!} F_2(m) + \frac{(\xi - m)^3}{3!} F_3(m) + \dots \right] \qquad (5.63)$$

and

$$J_2(m,n,k) = \frac{1}{2} D(m,n,k) + E_G(1 - \epsilon_m) \left[\frac{(\xi - m)^2}{2!} F_2(m) + \frac{(\xi - m)^3}{3!} F_3(m) + \dots \right] \qquad (5.64)$$

where $m = \bar{\Phi}\left(-k\sqrt{n/(n+1)}\right)$; $\mu_r = E_G(\xi - m)^r$; ϵ_m is 1 or 0 according as $\xi \leq m$ or $> m$ and $G(\cdot)$ is the c.d.f. of the random variable ξ defined in (5.13).

μ_2 and D(m,n,k), being the variance and the mean deviation about the mean of ξ have been considered already in the previous sections. The other higher complete and incomplete moments about the mean of ξ - which occur in the above expansions - may be calculated with the help of the series given in lemma 5.3.

Since m, the mean of ξ is equal to $\Phi(-h)$, it follows from (5.52) that the inequalities $\xi \leq m$ and $\xi > m$ are respectively equal to $X \geq -h_1$ and $X < -h_1$ where $h = k\sqrt{n/(n+1)}$ and $h_1 = h(\sqrt{n+1} - \sqrt{n})$. Therefore

$$E_G \epsilon_m (m-\xi)^r = E\,\epsilon_m [\,\Phi(-h) - \Phi(-X/\sqrt{n} - k)\,]^r$$

$$= E\,\epsilon_m [\,\Phi(X/\sqrt{n} + k) - \Phi(h)\,]^r$$

$$= \int_{-h_1}^{\infty} \Big[\sum_{s=0}^{\infty} \frac{H_{s+1}(x)\, H_s(-h)\, \phi(h)}{(s+1)!\,(\sqrt{n+1})^{s+1}} \Big]^r \phi(x)\,dx$$

using lemma 5.3

$$\Big[\frac{\phi(h)}{\sqrt{n+1}}\Big]^r \int_{-h_1}^{\infty} \Big[\, x - \frac{h(x^2-1)}{2\sqrt{n+1}}$$

$$+ \frac{(h^2-1)(x^3-x)}{6(n+1)} - \frac{(h^3-h)(x^4-6x^2+3)}{24(n+1)\sqrt{n+1}} + \cdots \Big]^r \phi(x) \qquad (5.65)$$

Similarly,

$$E_G(1-\epsilon_m)(\xi-m)^r = [\frac{\phi(h)}{\sqrt{n+1}}]^r \int_{-\infty}^{-h_1} [-x + \frac{h(x^2-1)}{2\sqrt{n+1}}$$

$$- \frac{(h^2-1)(x^3-x)}{6(n+1)} + \frac{(h^3-h)(x^3-6x^2+3)}{24(n+1)\sqrt{n+1}} - \dots]^r \phi(x)\,dx \qquad (5.66)$$

μ_r is obtained by replacing $-h_1$ in (5.66) by $+\infty$.

It follows from the above relations that

$$E_G\, \epsilon_m(m-\xi)^r = 0(n^{-r/2}); \quad r = 1,2,\dots$$

$$E_G\, (1-\epsilon_m)(\xi-m)^r = 0(n^{-r/2}), \quad r = 1,2,\dots \qquad (5.67)$$

and

$$\mu_r = \begin{cases} 0(n^{-r/2}), & r = 2,4,6,\dots \\ 0(n^{-(r+1)/2}), & r = 1,3,5,\dots \end{cases}$$

In particular, it can be shown that

$$E_G\epsilon_m(m-\xi)^2 = (n+1)^{-1}[\phi(h)]^2 [\Phi(h_1) - h_1\phi(h_1)$$

$$-(n+1)^{-1/2}\, hh_1^2\, \phi(h_1)] + 0(n^{-2})$$

and

(5.68)

$$E_G(1-\epsilon_m)(\xi - m)^2 = (n+1)^{-1} [\,\phi(h)\,]^2 [\,\bar{\Phi}(-h_1) + h_1\phi(h_1)$$
$$+ (n+1)^{-1/2}\, hh_1^2\, \phi(h_1)] + 0(n^{-2})$$

5.8 The case of a normal PC

It is supposed in this section that $\mu \frown N(u,v)$ i.e., in the notations of Section 5.1.

$$F^*(\mu) = \bar{\Phi}\left(\frac{\mu - u}{v}\right) \tag{5.69}$$

From (5. 11), it follows that

$$F(p) = F^*(U + k_p\sigma) = \bar{\Phi}\left(\frac{U - u + k_p\sigma}{v}\right) \tag{5.70}$$

is the c.d.f. of the (induced) distribution of p, the proportion of defectives in the lot.

The process average, that is, the average proportion of defectives among all lots is

$$\int_0 p dF(p) = \bar{\Phi}\left(\frac{u - U}{\sqrt{\sigma^2 + v^2}}\right), \text{ by theorem 5.1(a).} \tag{5.71}$$

Conditional on μ, the quality characteristic $x' \frown N(\mu, \sigma)$. The unconditional distribution of x' is $N(u, \sqrt{\sigma^2 + v^2})$. This fact directly leads to (5.71).

In the theory of sampling plans developed so far, the acceptability or otherwise of a lot depends entirely on the sample(s) drawn from that lot. The OC $L(p,n,k)$ is the conditional probability of accepting lots given that they are of quality p. The problem of sampling from the process - i.e., from the normal distribution $N(u, \sqrt{\sigma^2 + v^2})$, is not considered. Therefore the unconditional distribution of the quality characteristic has no direct role to play in our theory.

The weighted error-area of the second kind at $p = \pi$ is

$$J_2(\pi,n,k) = \int_{\pi}^{1} L(p,n,k)\,dF(p)$$

$$= \int_{\pi}^{1} \Phi(-k\sqrt{n}) - k_p\sqrt{n}\,)\,d\,\Phi\left(\frac{U-u+k_p\sigma}{v}\right)$$

$$= \Phi_2\left(-k'\sqrt{\frac{n'}{n'+1}},\ \frac{u-U-k_\pi\sigma}{v}\ ;\ -\sqrt{\frac{n'}{n'+1}}\right) \qquad (5.72)$$

where

$$k' = \frac{u-U+k\sigma}{v} \quad \text{and} \quad n' = \frac{v^2 n}{\sigma^2}$$

The average probability of acceptance is

$$F(\pi_e) = A_F = \int_0^1 L(p,n,k)\,dF(p) = \bar{\Phi}(-h') \qquad (5.73)$$

where

$$\bar{h}' = \bar{k}' \sqrt{\bar{n}'/(\bar{n}'+1)}$$

The weighted error-area of the first kind at the point $p = \pi$ can be obtained from

$$J_1(\pi,n,k) - J_2(\pi,n,k) = F(\pi) - F(\pi_e) \qquad (5.74)$$

The EQL π_e is given by

$$k_{\pi_e} = \frac{u - U}{\sigma} - \frac{v\bar{h}'}{\sigma} \qquad (5.75)$$

The total error-area at the EQL is

$$2J_1(\pi_e,n,k) = 2J_2(\pi_e,n,k) = J(\pi_e,n,k)$$

$$= 2\,\bar{\Phi}(-\bar{h}', \bar{h}';\ -\sqrt{\frac{\bar{n}'}{\bar{n}'+1}}) \qquad (5.76)$$

$$= D(\bar{m}', \bar{n}', \bar{k}') \qquad (5.77)$$

where $\bar{m}' = \bar{\Phi}(-\bar{h}')$ and $D(\bar{m}',\bar{n}',\bar{k}')$ is the total error-area under the unweighted OC $L(p,\bar{n}',\bar{k}')$ at the point of equal error-areas. The function $D(\bar{m}',\bar{n}',\bar{k}') = D(\bar{n}',\bar{k}')$ has been considered already in Section 5.5.

5.9 The case of unknown σ

The case of unknown σ is briefly considered in this section.

As in earlier sections X denotes a random variable following $N(0,1)$. Let Y_j denote a random variable which is independent of X and is such that $j\,Y_j^2$ follows a χ^2-distribution with j degrees of freedom.

The c.d.f. of a students' t-distribution with j degrees of freedom is denoted by $T_j(x)$ i.e.,

$$T_j(x) = P\left\{ \frac{X}{Y_j} \leq x \right\} \tag{5.78}$$

Further, let $T_{2,j}\,(x_1,x_2;\, r_{12})$ stand for the c.d.f. of a bivariate t-distribution, that is,

$$T_{2,j}(x_1,x_2;\, r_{12}) = P\left\{ \frac{X}{Y_j} \leq x_1,\ \frac{X'}{Y_j} \leq x_2 \right\} \tag{5.79}$$

where (X,X') is a standardized bivariate normal vector with

with correlation coefficient r_{12} and is independent of Y_j

$T_j(\cdot)$ is tabulated in Pearson and Hartley (1957). Details regarding $T_{2,j}$ may be found in Dunnet and Sobel (1954), John (1964) etc.

Let $x_1', x_2', \ldots x_n'$ be the sample observations. Define

$$\bar{x}' = \frac{1}{n} \sum_{i=1}^{n} x_i' \quad \text{and} \quad s^2 = \frac{1}{n-1} \sum_{i=1}^{n} (x_i' - \bar{x}')^2 \tag{5.80}$$

The lot is accepted if $\bar{x}' \leq U - ks$ and rejected if otherwise. Note that $\bar{x}' = \sigma X/\sqrt{n} + \mu$ and $s = \sigma Y_{n-1}$. The statistic s may be replaced by any other statistic $y = \sigma Y_j$.

The OC is

$$L(p,n,k) = P\left\{ \bar{x}' \leq U - ky, \text{ given } p \right\}$$

$$= P\left\{ \frac{X}{\sqrt{n}} \leq -k_p - k\, Y_j \right\}$$

$$= P\left\{ \xi > p \right\} \tag{5.81}$$

where ξ is the random variable given by

$$\xi = \Phi\,(X/\sqrt{n} - k\,Y_j) \tag{5.82}$$

The mean of ξ is given by

$$m = E\,\xi = E_{Y_j}\,E_X\,\Phi\,(X/\sqrt{n} - k\,Y_j)$$

$$= E_{Y_j}\,\Phi\,(-h\,Y_j),$$

using (5.23), where $h = k\sqrt{n/(n+1)}$. Therefore

$$m = P\left\{X \le -hY_j\right\} = P\left\{\frac{X}{Y_j} \le -h\right\}$$

Thus

$$m = T_j(-h) \tag{5.83}$$

Further

$$E\xi^2 = E_{Y_j}\,E_X\,\Phi\,(X/\sqrt{n} - kY_j)^2$$

$$= E_{Y_j}\,\Phi_2\,(-hY_j, -hY_j\,;\,(n+1)^{-1}),$$

using (5.23). Therefore

$$E\xi^2 = P\left\{\frac{X}{Y_j} \le -h,\ \frac{X'}{Y_j} \le -h\right\}$$

where (X,X') is a standardized bivariate normal vector with $\mathrm{cov}\,(X,X') = (n+1)^{-1}$ and is independent of Y_j. Hence

$$E\,\xi^2 = T_{2,j}\,(-h,\ -h;\ (n+1)^{-1}).$$

Thus the variance of ξ is

$$V = T_{2,j}\,(-h,\ -h;\ \frac{1}{n+1}) - [\ T_j(-h)\ ]^2 \qquad (5.84)$$

It may be noted that m and V in the case of unknown σ are obtained by replacing $\bar{\Phi}$ by T and $\bar{\Phi}_2$ by $T_{2,j}$ in the corresponding formulae (5.53) and (5.54.1) for the σ-known case.

CHAPTER 6

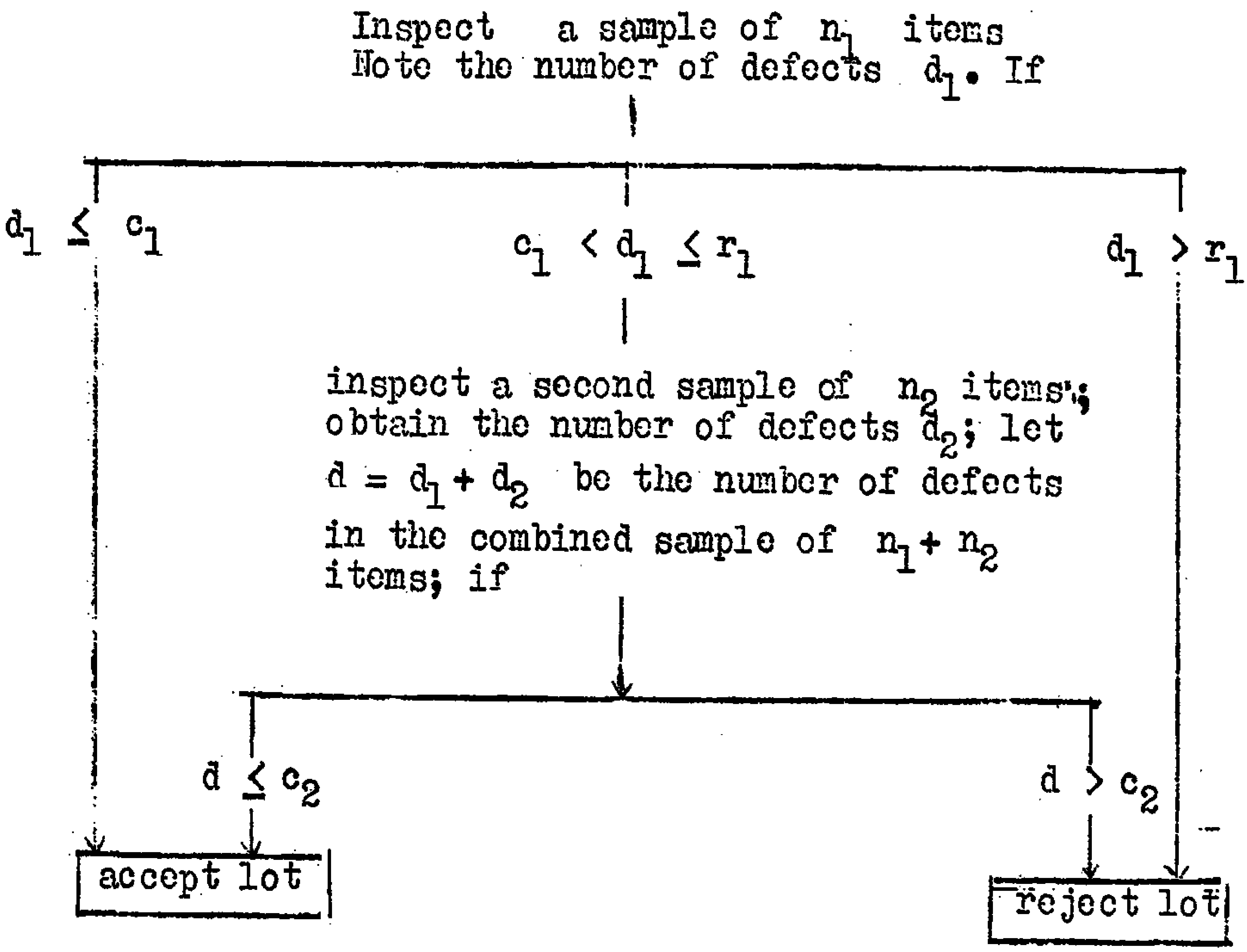
POISSON DOUBLE SAMPLING OC CURVES

6.1: The sampling procedure:-

As in the case of Poisson single sampling plans, the lot quality is measured by λ, the average number of defects per item in the lot. The probability of getting exactly r defects in a sample of size n is given by the Poisson probability,

$$b(\lambda n, r) = e^{-\lambda n} \frac{(\lambda n)^r}{r!}, \; r = 0,1,\ldots; \; 0 \leq \lambda < \infty \qquad (6.1)$$

The procedure of a double sampling plan is schematically shown as follows:



It may be noted that $c_1 < r_1 \leq c_2$. c_2 and r_1 are different for plans given in RMM tables (1966) and MIL-STD, U. S. Defence (1957) whereas $r_1 = c_2$ for plans given in Dodge and Romig (1959) and Philips SSS (Willemze and Fuijt, 1955). We shall take $r_1 = c_2$ so that the four elements n_1, n_2, c_1 and c_2 determine the plan uniquely.

Define

$$p = n_1(n_1 + n_2)^{-1}; \quad q = 1-p \tag{6.2}$$

and the upper cumulative binomial by

$$B(p,u,t) = \sum_{r=t}^{u} \binom{u}{r} p^r q^{u-r} \tag{6.3}$$

The following identities are needed (Harvard, 1955)

$$B(p,u+1,t) = pB(p,u,t-1) + qB(p,u,t)$$

$$B(p,u,t) = 1 - B(q,u,u-t+1) \tag{6.4}$$

and (page 264 in Owen, 1962)

$$p^{-1}B(p,x+1,\ t+1) = \sum_{u=t}^{x} \binom{u}{t} p^t q^{u-t} \tag{6.5}$$

6.2: <u>The OC:-</u>

It is supposed that the decision variables d_1 and d_2 are independent. Therefore the OC is given by

$$L(\lambda,n_1,n_2,c_1,c_2) = L'(\lambda n_1,c_1) + L''(\lambda,n_1,n_2,c_1,c_2) \tag{6.6}$$

where

$$L'(\lambda n_1,\ c_1) = \sum_{r=o}^{c_1} b(\lambda n,r)$$

and

$$L''(\lambda,n_1,n_2,c_1,c_2) = \sum_{r=c_1+1}^{c_2} \sum_{s=o}^{c_2-r} b(\lambda n_1,r)b(\lambda n_2,s)$$

$$\sum_{u=c_1+1}^{c_2} B(p,u,c_1+1)b(\lambda n_1 p^{-1},u)$$

6.3: The random variable ξ:-

The random variable ξ defined by $L(\lambda) = P\{\xi > \lambda\}$ has the p.d.f. given by $g(\lambda) = - dL(\lambda)/d\lambda$ which is actually the absolute value of the slope of the OC curve at λ. We have

$$g(\lambda,n_1,n_2,c_1,c_2) = g'(\lambda n_1,c_1) + g''(\lambda,n_1,n_2,c_1,c_2) \tag{6.7}$$

where

$$g'(\lambda n_1,c_1) = b(\lambda n_1,c_1)$$

and

$$g''(\lambda,n_1,n_2,c_1,c_2) = n_1 p^{-1} \sum_{u=c_1+1}^{c_2} B(p,u,c_1+1) \cdot b(\lambda n_1 p^{-1},u-1)\, [\lambda n_1 p^{-1} u^{-1} - 1]$$

6.4: The point of equal error-areas:-

The mean of ξ is the area under the unweighted OC. It is also the point of equal error-areas. It is given by

$$m = m(n_1,n_2,c_1,c_2) = m'(n_1,c_1) + m''(n_1,n_2,c_1,c_2) \tag{6.8}$$

where

$$m'(n_1,c_1) = \int_0^\infty L'(\lambda n_1,c_1)d\lambda = n_1^{-1}(c_1+1)$$

and

$$m''(n_1,n_2,c_1,c_2) = \int_0^\infty L''(\lambda,n_1,n_2,c_1,c_2)d\lambda$$

$$= n_1^{-1}\, p \sum_{u=c_1+1}^{c_2} B(p,u,c_1+1)$$

m can be calculated from the recurrence formula

$$n_1 m(n_1,n_2,c_1,c_2) = n_1 m(n_1,n_2,c_1,c_2-1) + pB(p,c_2,c_1+1)$$

together with (6.9)

$$n_1 m(n_1,n_2,c_1,c_1+1) = c_1+1 + p^{c_1+2}$$

Using (6.4) and (6.5) it can be shown that

$$n_1 m = c_1 + 1 + [p(c_2+1) - c_1 - 1]\, B(p,c_2,c_1+2) + (c_2-c_1)\binom{c_2}{c_1+1} p^{c_1+2} q^{c_2-c_1-2} \qquad (6.10)$$

It follows that if c_2 is large compared to c_1, m can be approximated by $(n_1+n_2)^{-1}(c_2+1)$. The approximation is better for larger values of p.

6.5: The error-areas:-

The error-area of the second kind at the point $\lambda = \Lambda$ is

$$D_2(\Lambda\,,\, n_1, n_2, c_1, c_2) = D_2'(\Lambda\,, n_1, c_1) + D_2''(\Lambda\,, n_1, n_2, c_1, c_2) \qquad (6.11)$$

where

$$D_2'(\Lambda, n_1, c_1) = \sum_{u=0}^{c_1} \sum_{r=0}^{u} b(\Lambda\, n_1, r)$$

$$\sum_{r=0}^{c_1} (c_1 + 1 - r)\, b(\Lambda\, n_1, r)$$

and

$$D_2''\,(\Lambda, n_1, n_2, c_1, c_2) = n_1^{-1} p \sum_{u=c_1+1}^{c_2} B(p, u, c_1+1) \sum_{r=0}^{u} b(\Lambda n_1 p^{-1}, r)$$

Knowing $D_2(\Lambda)$ the error-area of the first kind is easily calculated from $D_1(\Lambda) - D_2(\Lambda) = \Lambda - m$ and the total error-area from $D(\Lambda) = D_1(\Lambda) + D_2\,(\Lambda)$.

6.6: The $\{m_e,\ D_e\}$ system of sampling plans:-

The error-areas, the slope and the OC at the point of equal error-areas are obtained by putting $\Lambda = m$ in the relevant formulae. Values of $n_1 m$, $n_1 D(m)$, $n_1^{-1} g(m)$ and $L(m)$ were computed on an IBM 1620 computer for $c_1 = 0(1)30$; $c_1 = c_1+1(1)30$ and $p = 1/4, 1/3, 1/2, 2/3$ and $3/4$. They are given in Subrahmanya (1968). Based on these calculations the following observations are made with respect to the behaviour of these functions.

For fixed values of	as the following argument increases	$n_1 m$	$n_1 D(m)$	$n_1^{-1} g(m)$	$L(m)$
c_1 and p	c_2 ↑	↑	↘↗	↗↘	↘↗
c_2 and p	c_1 ↑	↑	↑	↓	↘↗
c_1 and c_2	p ↑	↑	↘↗	↗↘	↘↗

<u>Note:</u> ↑ increases; ↑ decreases; ↘↗ first decreases and then increases; ↗↘ first increases and then decreases.

It may be noted that $n_1 m$, $n_1 D(m)$, $n_1^{-1} g(m)$ and $L(m)$ depend only on the three parameters c_1, c_2 and p and that the case of $p=1$ refers to a single sampling plan (n_1, c_2).

Typical values of $n_1 m$, $n_1 D_2(m) = \frac{1}{2} n_1 D(m)$, $n_1^{-1} g(m)$ and $L(m)$ are presented in Table 6.1. They are taken from the more detailed and extensive tables given in Subrahmanya (1968). The contributions from the second stage sample - $n_1 D_2''(m)$,

$n_1^{-1}\, g''(m)$ and $L''(m)$ - are also shown separately. ($n_1 m''$ is not shown separately because it can be easily obtained from $n_1 m'' = n_1 m - (c_1+1)$).

Since a double sampling plan has four elements n_1, n_2, c_1 and c_2, there are in general more than one plan satisfying the specifications

$$m(n_1, n_2, c_1, c_2) = m_e$$

and

$$D(m_e, n_1, n_2, c_1, c_2) = D_e \qquad (6.12)$$

where m_e and D_e are the specified quantities. Additional conditions must be imposed in order to determine the plan uniquely. A possible condition is to minimize the average amount of inspection at a specified quality level. (Dodge and Romig (1959) and also Section 2.8).

6.7: <u>The average sample size:-</u>

Let N denote the lot size. When all rejected lots are submitted to hundred per cent inspection, the average sample size $NI(\lambda)$ - given that lots are of quality λ - is given by

$$NI(\lambda) = N - (N-n)\, L(\lambda n, c) \qquad (6.13)$$

for a single sampling plan (n,c) and by

$$NI(\lambda) = N-(N-n_1)L(\lambda,n_1,n_2,c_1,c_2)+n_2L''(\lambda,n_1,n_2,c_1,c_2) \quad (6.14)$$

for a double sampling plan (n_1,n_2,c_1,c_2).

6.8: Examples:-

Example (i):-

Single sampling plan; m = 5 °/. and D(m) = 1 °/. .
From Table 3.7 given in Section 3.3 we find that the required plan is n = 320 and c = 15.

Example (ii):-

Single sampling plan; m = 5 °/.; N = 1000 and I is a minimum at m. We find from Table 3.7.

n	c	m °/.	NI(m)
20	0	5	639
40	1	5	610
60	2	5	602
80*	3*	5*	601*
100	4	5	604
120	5	5	608

* The required plan is n = 80 and c = 3.

Example (iii):-

Double sampling plan; $n_2 = 3n_1$; $m = 5$ %; $D(m) = 1$ % $N = 1000$ and I. is a minimum at m.

We find from Subrahmanya (1968) that there are 8 plans satisfying the first three conditions.

n_1	n_2	$n_1 + n_2$	c_1	c_2	m %	D %	NI(m)
80	240	320	0	15	5	1.0	679
86	258	344	1	16	5	1.0	679
93*	279*	372*	2*	17*	5*	1.0*	675*
110	330	440	3	20	5	1.0	686
124	372	496	4	22	5	1.0	687
139	417	556	5	24	5	1.0	685
158	474	632	6	27	5	1.0	690
174	522	696	7	29	5	1.0	685
	single sampling plan:-						
320	-	-	15	-	5	1.0	683

* The required plan is

$$n_1 = 93;\ n_2 = 279;\ c_1 = 2 \text{ and } c_2 = 17.$$

It may be noted that a double sampling plan can have a higher value for NI(m) than that of the corresponding single sampling plan. In the above example, all the double sampling plans except the first three have NI(m) greater than 683.

Example (iv):-

Same specifications as above except that $n_2 = n_1$. From Subrahmanya (1968), we obtain the required plan:

$n_1 = n_2 = 163$; $c_1 = 5$; $c_2 = 15$ and hence $m = 5$ %;
D(m) = 1.0 % and NI(m) = 660.

6.9: Philips Standard Sampling System:-

The double sampling plans belonging to Philips Standard ampling System (Willemze and Fuijt, 1955) have $n_2 = 2n_1$ i.e., $p = 1/3$. Values of m, D(m), g(m) and L(m) are shown in Table 6.2 for all the 48 double sampling plans of the system. They may be compared with those of the 25 single sampling plans of the system given in Table 3.8.

TABLE 6.1

Values of $n_1 m$, $n_1 D_2(m)$ etc. for Poisson double sampling plans (n_1, n_2, c_1, c_2)

$p = \frac{1}{4}$ i.e., $3n_1 = n_2$

c_1	c_2	$n_1 m$	$n_1 D_2(m)$	$\frac{g(m)}{n_1}$	L(m)	$n_1 D_2''(m)$	$\frac{g''(m)}{n_1}$	L''(m)
0	1	1.0625	0.3503	0.3919	0.3607	0.0047	0.0464	0.0152
	2	1.1719	3299	4453	3648	0.0201	1355	0551
	3	1.3164	3147	4929	3789	0.0466	2248	1108
	4	1.4873	3065	5194	3960	0.0806	2934	1700
	5	1.6780	3045	5268	4118	0.1178	3401	2251
	10	2.7922	3415	4635	4536	0.2803	4022	3923
	15	4.0100	3997	3948	4656	0.3816	3766	4475
	20	5.2524	4559	3465	4708	0.4506	3412	4655
	25	6.5006	5071	3119	4739	0.5056	3104	4724
	30	7.7501	5538	2858	4761	0.5534	2854	4757
	5	2.2119	4776	3126	3881	0.0165	0705	0364
	10	2.9893	4062	4005	4278	0.1552	2501	2270
	15	4.0735	4189	3812	4580	0.3155	3119	3717
	20	5.2714	4611	3435	4689	0.4238	3165	4367
	25	6.5060	5085	3112	4735	0.4958	3015	4622
	30	7.7516	5542	2856	4760	0.5500	2823	4722
	10	3.4445	5483	2869	4028	0.0433	0975	0716
	15	4.2706	4806	3339	4371	0.1920	2125	2360
	20	5.3459	4830	3319	4613	0.3496	2638	3629
	25	6.5319	5155	3081	4712	0.4611	2770	4292
	30	7.7601	5564	2849	4754	0.5356	2720	4588
	15	4.6756	6041	2668	4138	0.0736	1080	1003
	20	5.9050	6514	2505	4225	0.1057	1124	1245
	25	7.1335	6931	2370	4295	0.1388	1142	1455
	30	8.3617	7307	2255	4354	0.1725	1146	1641

TABLE 6.1(contd.)

Values of n_1m, $n_1D_2(m)$ etc., for Poisson double sampling plans (n_1, n_2, c_1, c_2)

$p = \frac{1}{3}$ i.e., $2n_1 = n_2$

c_2	$n_1 m$	$n_1 D_2(m)$	$\frac{g(m)}{n_1}$	L(m)	$n_1 D_2''(m)$	$\frac{g''(m)}{n_1}$	L''(m)
1	1.1111	0.3464	0.4124	0.3688	0.0172	0.0832	0.0396
2	1.2963	3319	4580	3861	0.0583	1844	1125
3	1.5309	3296	4729	4062	0.1132	2565	1899
4	1.7984	3366	4659	4226	0.1171	3003	2570
5	2.0878	3496	4488	4345	0.2256	3249	3105
10	3.6782	4409	3560	4589	0.4156	3308	4337
15	5.3349	5295	2975	4667	0.5247	2927	4618
20	7.0002	6070	2601	4710	0.6061	2592	4701
25	8.6667	6759	2340	4739	0.6757	2338	4737
30	10.333	7384	2144	4761	0.7384	2143	4761
5	2.4390	4609	3404	4082	0.0735	1276	1081
10	3.7534	4626	3423	4522	0.3278	2544	3408
15	5.3486	5330	2959	4658	0.4981	2705	4356
20	7.0025	6076	2599	4709	0.5994	2536	4636
25	8.6671	6760	2339	4739	0.6742	2324	4722
30	10.333	7384	2144	4761	0.7380	2140	4757
10	3.9875	5318	3019	4348	0.1808	1545	1949
15	5.4079	5491	2889	4615	0.4217	2234	3673
20	7.0153	6107	2589	4702	0.5733	2368	4409
25	8.6696	6765	2338	4738	0.6666	2273	4657
30	10.334	7385	2144	4761	0.7360	2126	4740
15	5.5739	5957	2697	4504	0.2895	1602	2569
20	7.1827	6552	2445	4601	0.3927	1603	3031
25	8.8065	7113	2246	4666	0.4883	1585	3386
30	10.441	7647	2085	4711	0.5762	1560	3664

TABLE 6.1(contd.)

Values of $n_1 m$, $n_1 D_2(m)$ etc.. for Poison double sampling plans (n_1, n_2, c_1, c_2)

$p = \frac{1}{2}$ i.e., $n_1 = n_2$

c_1	c_2	$n_1 m$	$n_1 D_2(m)$	$\frac{g(m)}{n_1}$	L(m)	$n_1 D_2''(m)$	$\frac{g''(m)}{n_1}$	L''(m)
0	1	1.2500	0.3583	0.4096	0.3891	0.0718	0.1231	0.1026
	2	1.6250	3767	4023	4135	0.1798	2054	2166
	3	2.0625	4096	3742	4291	0.2825	2470	3019
	4	2.5313	4475	3446	4386	0.3679	2651	3590
	5	3.0156	4860	3188	4449	0.4370	2697	3958
	10	5.5005	6567	2387	4599	0.6526	2346	4558
	15	8.0000	7937	1984	4667	0.7934	1981	4664
	20	10.500	9105	1734	4710	0.9104	1734	4709
	25	13.000	1.014	1560	4739	1.014	1560	4739
	30	15.500	1.108	1429	4761	1.108	1429	4761
1	5	3.1250	0.5163	3016	4390	0.2991	1643	2577
	10	5.5063	6580	2383	4597	0.6275	2159	4333
	15	8.0003	7938	1984	4667	0.7904	1957	4637
	20	10.500	9105	1734	4710	0.9101	1731	4707
	25	13.000	1.014	1560	4739	1.014	1559	4739
	30	15.500	1.108	1429	4761	1.108	1429	4761
2	10	5.5391	0.6657	2358	4585	0.5501	1755	3725
	15	8.0024	7942	1983	4667	0.7771	1876	4530
	20	10.500	9105	1734	4710	0.9083	1719	4691
	25	13.000	1.014	1560	4739	1.014	1558	4737
	30	15.500	1.108	1429	4761	1.108	1429	4761
3	15	8.0130	0.7964	1978	4664	0.7375	1694	4244
4	20	10.504	0.9113	1733	4709	0.8810	1594	4499
5	25	13.002	1.014	1559	4739	0.9984	1490	4632
6	30	15.501	1.108	1429	4761	1.099	1393	4706

TABLE 6.2

Philips Standard Sampling System - double sampling plans.
$(2n_1 = n_2)$

Indexed value of A_o ./.	plan n_1	c_1	c_2	m ./.	D(m) ./.	g(m)	L(m) ./.
0.25	330	0	1	0.34	0.21	136	36.9
0.50	150	0	1	0.74	0.46	62	36.9
0.25	425	0	2	0.31	0.16	195	38.6
0.50	200	0	2	0.65	0.33	92	38.6
1.00	110	0	2	1.18	0.60	50	38.6
2.00	55	0	2	2.36	1.21	25	38.6
0.25	525	0	3	0.29	0.13	248	40.6
0.50	260	0	3	0.59	0.25	123	40.6
1.00	135	0	3	1.13	0.49	64	40.6
2.00	70	0	3	2.19	0.94	33	40.6
3.00	45	0	3	3.40	1.46	21	40.6
5.00	25	0	3	6.12	2.64	12	40.6
0.25	875	1	5	0.28	0.11	298	40.8
0.50	440	1	5	0.55	0.21	150	40.8
1.00	220	1	5	1.11	0.42	75	40.8
2.00	110	1	5	2.22	0.84	37	40.8
3.00	70	1	5	3.48	1.32	24	40.8
5.00	45	1	5	5.42	2.05	15	40.8
7.00	30	1	5	8.13	3.47	10	40.8
10.00	22	1	5	11.09	4.19	7	40.8
0.25	1500	2	10	0.27	0.07	453	43.5
0.50	750	2	10	0.53	0.14	226	43.5
1.00	380	2	10	1.05	0.28	115	43.5
2.00	190	2	10	2.10	0.56	57	43.5

TABLE 6.2 (contd.)

no.	indexed value of o %	plan n_1	c_1	c_2	m %	D(m) %	g(m)	L(m) %
25	3.00	125	2	10	3.19	0.85	38	43.5
26	5.00	75	2	10	5.32	1.42	23	43.5
27	7.00	55	2	10	7.25	1.95	17	43.5
28	10.00	40	2	10	9.97	2.66	12	43.5
29	0.25	2200	3	15	0.25	0.05	593	45.0
30	0.50	1100	3	15	0.51	0.11	297	45.0
31	1.00	540	3	15	1.03	0.22	146	45.0
32	2.00	270	3	15	2.06	0.44	73	45.0
33	3.00	180	3	15	3.10	0.66	49	45.0
34	5.00	110	3	15	5.07	1.08	30	45.0
35	7.00	75	3	15	7.43	1.59	20	45.0
36	10.00	55	3	15	10.13	2.17	15	45.0
37	1.00	700	4	20	1.03	0.19	171	46.0
38	2.00	350	4	20	2.05	0.37	86	46.0
39	3.00	240	4	20	2.99	0.55	59	46.0
40	5.00	140	4	20	5.13	0.94	34	46.0
41	7.00	100	4	20	7.18	1.31	24	46.0
42	10.00	70	4	20	10.26	1.87	17	46.0
43	3.00	290	5	25	3.04	0.49	65	46.7
44	5.00	175	5	25	5.03	0.81	39	46.7
45	7.00	120	5	25	7.34	1.19	27	46.7
46	10.00	85	5	25	10.36	1.67	19	46.7
47	7.00	145	6	30	7.20	1.05	30	47.1
48	10.00	105	6	30	9.94	1.46	22	47.1

CHAPTER 7

MULTINOMIAL SINGLE SAMPLING OC CURVES

7.1: The sampling procedure for a trinomial plan:-

It is supposed that there are two categories of defective items and that they are mutually exclusive. Let p_1 and p_2 be the proportions of defectives belonging respectively to the first and second categories in the lot submitted for inspection. An increase in the value of either p_1 or p_2 or both means a deterioration in lot quality.

A sample of size n is drawn from the lot. Let d_i, $i = 1,2$ be the number of defectives belonging to the i-th category in the sample. The lot is accepted if and only if both the conditions $d_1 \leq c_1$ and $d_2 \leq c_2$ are satisfied, where c_1 and c_2 are the acceptance numbers. n, c_1 and c_2 are the elements of the trinomial single sampling plan.

The probability of d_1 and d_2 taking respectively the values r_1 and r_2 is given by the trinomial probability $b(p_1, p_2, n, r_1, r_2)$ which equals

$$\begin{cases} \frac{n!}{r_1!\, r_2!\, (n-r_1-r_2)!}\, p_1^{r_1}\, p_2^{r_2}\, (1-p_1-p_2)^{n-r_1-r_2} & \text{if} \\ \quad r_1, r_2 = 0,1,\ldots n;\ 0 \le r_1 + r_2 \le n; & \\ \quad p_1, p_2 \ge 0 \text{ and } p_1 + p_2 \le 1 & \\ 0 \quad \text{otherwise} & \end{cases} \tag{7.1}$$

It may be noted that the procedure of acceptance sampling mentioned above is equal to the procedure of testing the null hypothesis

$$H_0 : p_1 = \pi_1 \quad \text{and} \quad p_2 = \pi_2$$

against the alternative

H_1: at least one of the following conditions is true:

$$p_1 > \pi_1 \ ; \quad p_2 > \pi_2$$

and that the test is uniformly most powerful.

7.2: The bivariate Dirichlet distribution:-

$(n+2)(n+1)b(x_1, x_2, n, s_1, s_2)$ is the joint p.d.f. of a bivariate Dirichlet distribution with parameters s_1+1, s_2+1 and $n-s_1-s_2+1$. (Wilks, 1962) We also have - if s_i is zero or a positive integer -

$$(n+2)(n+1) \int_{T_{2,x}(p_1,p_2)} b(x_1,x_2,n,s_1,s_2)\,dx_1\,dx_2$$

$$= \sum_{t_2=0}^{s_2} \sum_{t_1=0}^{s_1} b(p_1,p_2,n+2,t_1,t_2) \qquad (7.2)$$

where, the region of integration is the triangle

$$T_{2,x}(p_1,p_2) = \left\{ x_1,x_2 : x_1,x_2 \geq 0;\ x_1 > p_1, x_2 > p_2 \text{ and } x_1 + x_2 \leq 1 \right\} \qquad (7.3)$$

7.3: The trinomial OC and the hypothetical random vector $(\xi^{(1)}, \xi^{(2)})$:-

The trinomial OC is given by

$$L(p_1,p_2,n,c_1,c_2) = \sum_{r_2=0}^{c_2} \sum_{r_1=0}^{c_1} b(p_1,p_2,n,r_1,r_2)$$

$$= P\left\{ \xi^{(1)}, \xi^{(2)} \geq 0;\ \xi^{(1)} > p_1,\ \xi^{(2)} > p_2 \text{ and } \xi^{(1)} + \xi^{(2)} \leq 1 \right\} \qquad (7.4)$$

where the random vector $(\xi^{(1)}, \xi^{(2)})$, follows a bivariate Dirichle distribution with parameters c_1+1, c_2+1 and $n-c_1-c_2-1$. The joint p.d.f. of $(\xi^{(1)}, \xi^{(2)})$ is $n(n-1)b(x_1, x_2, n-2, c_1, c_2)$.

The OC given by the formula (7.4) is an example of a bivariate OC referred to in Sections 1.1 and 1.9.

The value of the OC can be taken as zero or left undefined outside the triangular region T_2 bounded by $p_1 = 0$ $p_2 = 0$ and $p_1 + p_2 = 1$. At $p_1 = p_2 = 0$ the OC takes the value 1; at $p_1 = 1$ or $p_2 = 1$ its value is zero and on the boundary line $p_1 = 0$ or $p_2 = 0$ it reduces to a binomial single sampling OC $L(p_2, n, c_2)$ or $L(p_1, n, c_1)$. Further, the value of the trinomial OC is zero on the boundary line $p_1 + p_2 = 1$ except in the trivial case $c_1 + c_2 = n$ when it is identically equal to 1 over the entire triangular region T.

It may be recalled that the approximations to the EQL depend on the moments of $\xi^{(1)}$ and $\xi^{(2)}$. (Theorems 1.3 and 1.4 given in Section 1.9).

The marginal distribution of $\xi^{(i)}$, $i = 1,2$ is a Beta with parameters $c_i + 1$ and $n - c_i$

We have

$$m^{(1)} = E\xi^{(1)} = \frac{c_1+1}{n+1} \; ; \quad m^{(2)} = E\,\xi^{(2)} \; \frac{c_2+1}{n+1} \; ;$$

$$V_{ii} = E(\xi^{(i)} - m^{(i)})^2 = \frac{(c_i+1)(n-c_i)}{(n+1)^2(n+2)} \, , \quad i = 1,2;$$

$$E\,\xi^{(1)}\,\xi^{(2)} = \frac{(c_1+1)(c_2+1)}{(n+1)(n+2)} \quad \text{and}$$

$$V_{12} = \text{cov}\,(\xi^{(1)}, \xi^{(2)}) = - \frac{(c_1+1)(c_2+1)}{(n+1)^2(n+2)} \, .$$

Further from (4.10), the mean deviation about the mean of $\xi^{(i)}$ is

$$D^{(i)} = E|\xi^{(i)} - m^{(i)}| = 2m^{(i)}(1-m^{(i)})\,b(m^{(i)},n,c_i),$$

$$i = 1,\ 2$$

where $b(p,n,r)$ is the binomial probability defined in (4.1)

7.4: The case of unweighted OCs:-

Consider the situation where the joint prior distribution of p_1 and p_2 is 'uniform' in the triangular region T_2 bounded by the lines $p_1 = 0$, $p_2 = 0$ and $p_1 + p_2 = 1$ The joint p.d.f. is given by

$$f_u(p_1,p_2) = \begin{cases} 2 & \text{if } (p_1,p_2) \text{ lies in } T_2 \\ 0 & \text{otherwise} \end{cases} \tag{7.6}$$

The c.d.f. may be denoted by $F_u(p_1,p_2)$. This case will be referred to as the case of unweighted OCs. (See (1.39) of Section 1.9c).

The volume under the unweighted OC is

$$v_2 = v_2(n,c_1,c_2) = \int_{T_2} L(p_1,p_2,n,c_1,c_2) f_u(p_1, p_2)\, dp_1 dp_2$$

$$= 2 \sum_{r_2=0}^{c_2} \sum_{r_1=0}^{c_1} \int_{T_{2,p}(0,0)} b(p_1,p_2,n,r_1,r_2)\, dp_1\, dp_2$$

where $T_{2,p}(0,0)$ is as defined in (7.3). Using (7.2) and (7.4)

$$v_2 = \frac{2}{(n+1)(n+2)} \sum_{r_2=0}^{c_2} \sum_{r_1=0}^{c_1} L(0,0,n+2,r_1,r_2)$$

$$= \frac{2(c_1+1)(c_2+1)}{(n+1)(n+2)} = 2E\xi^{(1)}\xi^{(2)} \tag{7.7}$$

7.5: <u>The unweighted error-volumes:-</u>

Let a_1 denote the region in the (p_1,p_2) plane bounded by the lines $p_1 = 0$, $p_2 = 0$, $p_1 = \pi_1$ and $p_2 = \pi_2$ subject to the condition that $p_1 + p_2 \leq 1$. (See Fig. given below)

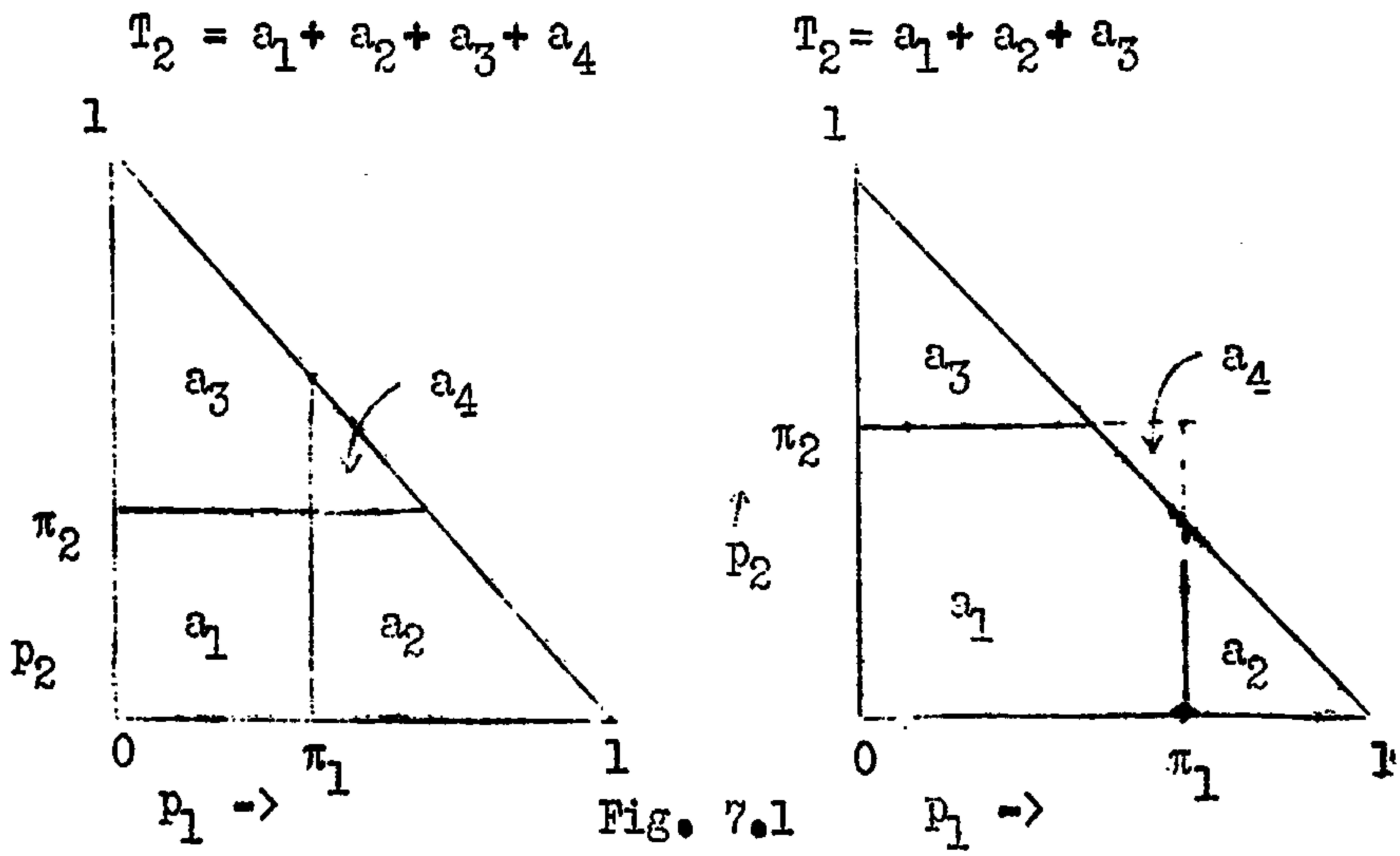


Fig. 7.1

The unweighted error-volume of the first kind at the point $p_1 = \pi_1$ and $p_2 = \pi_2$ is defined to be the volume of the pillar with base a_1 and height $[1 - L(p_1, p_2, n, c_1, c_2)]$ $f_u(p_1, p_2)$ at the point (p_1, p_2) i.e.,

$$D_1(\pi_1, \pi_2, n, c_1, c_2) = \int_{a_1} 1 - L(p_1, p_2, n, c_1, c_2) f_u(p_1, p_2) dp_1 \, dp_2 \qquad (7.8)$$

The unweighted error-volume of the second kind at the point (π_1, π_2) is

$$D_2(\pi_1, \pi_2, n, c_1, c_2) = \int_{a'} L(p_1, p_2, n, c_1, c_2) f_u(p_1, p_2) dp_1 \, dp_2 \qquad (7.9)$$

where the regions a_1 and $\bar{a'}$ together add up to the triangle T_2.

The unweighted error-volumes can be made to depend on suitable sums of trinomial OCs. For example, let us take $\pi_1 + \pi_2 \leq 1$, a case of practical interest. The integral in (7.9) can be split up into three parts:

$$\int_{a'} = \int_{a_2+a_4} + \int_{a_3+a_4} - \int_{a_4}$$

$$(\quad = \int_{T_{2,p}(\pi_1,0)} + \int_{T_{2,p}(0,\pi_2)} - \int_{T_{2,p}(\pi_1,\pi_2)}$$

Using (7.2) and (7.4), we see that

$$\int_{a_2+a_4} L(p_1,p_2,n,c_1,c_2) f_u(p_1,p_2) dp_1 \, dp_2$$

$$= \frac{2}{(n+1)(n+2)} \sum_{r_2=0}^{c_2} \sum_{r_1=0}^{c_1} L(\pi_1,0,n+2,r_1,r_2)$$

$$= \frac{2(c_2+1)}{(n+1)(n+2)} \sum_{r_1=0}^{c_1} L(\pi_1,n+2,r_1)$$

where $L(\pi_1,n+2,r_1)$ is a binomial single sampling OC.

Similarly,

$$\int_{a_3+a_4} = \frac{2(c_1+1)}{(n+1)(n+2)} \sum_{r_2=0}^{c_2} L(\pi_2,\ n+2, r_2)$$

and

$$\int_{a_4} = \frac{2}{(n+1)(n+2)} \sum_{r_2=0}^{c_2} \sum_{r_1=0}^{c_1} L(\pi_1,\pi_2,n+2,r_1,r_2)$$

Therefore we have the formula

$$D_2(\pi_1,\pi_2,n,c_1,c_2) = \frac{2}{(n+1)(n+2)} \sum_{r_2=0}^{c_2} \sum_{r_1=0}^{c_1} [\,L(\pi_1,n+2,r_1)$$

$$+ L(\pi_2,n+2,r_2) - L(\pi_1,\pi_2,n+2,r_1,r_2)\,] \qquad (7.10)$$

It is easy to see that

$$D_1(\pi_1,\pi_2,n,c_1,c_2) - D_2(\pi_1,\pi_2,n,c_1,c_2)$$

$$= F_u(\pi_1,\pi_2) - v_2(n,c_1,c_2)$$

$$= 2\pi_1\pi_2 - v_2(n,c_1,c_2) \qquad (7.11)$$

where v_2 is given by (7.7).

7.6: The 'point of equal error-volumes':-

In general the solution of the equation

$$F_u(p_1,p_2) = v_2(n,c_1,c_2) \qquad (7.12)$$

in terms of p_1 and p_2 is not unique. Let $p_1 = m_1$ and $p_2 = m_2$ be one such solution i.e.,

$$m_1 m_2 = \frac{(c_1+1)(c_2+1)}{(n+1)\ (n+2)} = \frac{\bar{v}_2}{2} \qquad (7.13$$

It follows from (7.11) that the point (m_1, m_2) is a 'point of equal error-volumes'.

The above property is carried deeper into the case of weighted error-volumes.

7.7: <u>The EQL and the weighted error-volumes:-</u>

Suppose $f(p_1, p_2)$ and $F(p_1, p_2)$ are respectively the joint p.d.f. and the c.d.f. of a bivariate prior distribution of p_1 and p_2.

The volume under the weighted OC is defined by

$$A_F(n, c_1, c_2) = \int L(p_1, p_2, n, c_1, c_2) f(p_1, p_2)\, dp_1\, dp_2 \qquad (7.14$$

and the weighted error-volumes at the point (π_1, π_2) by

$$J_{1F}(\pi_1, \pi_2, n, c_1, c_2) = \int [\,1 - L(p_1, p_2, n, c_1, c_2)\,]\; f(p_1, p_2)\, dp_1\, dp_2, \text{ etc.} \qquad (7.15$$

Suppose that there exist values of p_1 and p_2 - $p_1 = \pi_{e1}$ and $p_2 = \pi_{e2}$ say - such that

$$F(\pi_{e1}, \pi_{e2}) = A_F \tag{7.16}$$

The point (π_{e1}, π_{e2}) may be called a two-dimensional EQL-vector for the trinomial plan (n,c_1,c_2) with respect to the PC with the c.d.f. $F(p_1,p_2)$.

Since

$$J_{1F}(\pi_1,\pi_2) - J_{2F}(\pi_1,\pi_2) = F(\pi_1,\pi_2) - A_F \tag{7.17}$$

it follows that the EQL is a point of equal error-volumes. Conversely, any point of equal error-volumes is also an EQL-vector.

7.8: <u>The case of a Dirichlet PC:-</u>

If $f(p_1,p_2)$ is the p.d.f. of a bivariate Dirichlet with parameters α_1+1, α_2+1 and α_3+1 where α_i, $i = 1,2$ or 3, is zero or a positive integer, then the volume under the weighted OC and also the weighted error-volumes can be expressed in terms of suitable sums of trinomial and binomial OCs. We have

$$A_F(n,c_1,c_2) = \frac{\sum_{r_2=0}^{c_2} \sum_{r_1=0}^{c_1} \binom{r_1+\alpha_1}{\alpha_1}\binom{r_2+\alpha_2}{\alpha_2}\binom{n-r_1-r_2+\alpha_3}{\alpha_3}}{\binom{n+2+\alpha_1+\alpha_2+\alpha_3}{n}} \tag{7.18}$$

and

$$J_{2F}(\pi_1,\pi_2,n,c_1,c_2) = \left[\binom{n+2+\alpha_1+\alpha_2+\alpha_3}{n} \right]^{-1}$$

$$\sum_{r_2=o}^{c_2} \sum_{r_1=o}^{c_1} \binom{r_1+\alpha_1}{\alpha_1}\binom{r_2+\alpha_2}{\alpha_2}\binom{n-r_1-r_2+\alpha_3}{\alpha_3}$$

$$[\, L(\pi_1, n+2+\alpha_1+\alpha_2+\alpha_3, \quad r_1+\alpha_1)$$

$$+ L(\pi_2, n+2+\alpha_1+\alpha_2+\alpha_3, \quad r_2+\alpha_2)$$

$$- L(\pi_1,\pi_2, n+2+\alpha_1+\alpha_2+\alpha_3, \quad r_1+\alpha_1, r_2+\alpha_2)\,]$$

(7.19

7.9: Multinomial OCs:-

It is clear that a trinomial OC is a bivariate analogue of the binomial OC. More generally k-nomial probabilities and OCs may be considered. They are related to k-1 variate Dirichlet distributions with k parameters. All the results given so far in this chapter can be extended to k-nomial OCs without any difficulty.

R E F E R E N C E S

1. Abramowitz, M. and Stegun, I.A., eds. (1964): Handbook of Mathematical functions. Washington: National Bureau of Standards.

2. Barnard, G. A. (1954): Sampling inspection and statistical decisions. J. R. Statist. Soc., B, 16, 151-171.

3. Cramer, H. (1946): Mathematical Methods of Statistics. Princeton, N. J.: Princeton University Press.

4. Cowden, D. J. (1960): Statistical Methods in Quality Control Asia Publishing House, Calcutta.

5. Dodge, H. F. and Romig, H. G. (1959): Sampling Inspection Tables, Single and Double Sampling. John Wiley, New York (Second Edition).

6. Dunnet and Sobel (1954): A bivariate generalization of student's t-distribution. Biometrika, 41, 150-169.

7. Fisher, R. A. and Cornish, E. A. (1960): The percentile points of distributions having known cumulants. Technometrics, 2, 209-225.

8. Fisher, R. A. and Yates, F. (1953): Statistical tables. (Fourth Edition) Oliver and Boyd, London.

9. Fuijt, W. and Willemze, F. G. (1955): Directions on the use of Philips Standard Sampling System. N. V. Philips gloeilampenfabricken, Eindhoven.

0. General Electric Company (1962): Tables of Individual and Cumulative sums of Poisson distribution. Defence Systems Department, London.

1. Hald, A. (1960): The compound hypergeometric distribution and a system of single sampling inspection plans based on prior distributions and costs. Technometrics, 2, 275-340.

2. --- (1964a): Bayesian single sampling attribute plans for discrete prior distributions. Institute of Mathematical Statistics, University of Copenhagen.

3. --- (1964b): Efficiency of sampling inspection plans for attributes. Bull. Int. Statist. Inst., 40, 681-697.

4. --- (1967a): Asymptotic properties of Bayesian single sampling plans. J. R. Statist. Soc., B, 29, 162-173.

5. --- (1967b): The determination of single sampling attribute plans with given producer's and consumer s risk. Technometrics, 9, 401-415.

6. --- (1967c): On the theory of single sampling inspection by attributes based on two quality levels. Rev. Int. Statist. Inst. 35, 1-29.

7. --- (1968): Bayesian single sampling attribute plans for continuous prior distributions. Technometrics, 10, 667-683.

18. Hald, A. and Kousgaard, E. (1967): A Table for solving the binomial equation B(c,n,p) = P. Mat. Fys. Skr. Dan. Vid. Selsk (Danish Acedemy of Sciences), 3, no. 4.

19. Hald, A. and Keiding, N. (1969): Asymptotic properties of Bayesian decision rules for two terminal decisions and multiple sampling I.

20. Hamacker, H. C. (1949): Lot inspection by sampling. Philips Tech. Rev., 11, No.6.

21. --- (1950): Theory of sampling inspection plans. Philips Tech. Rev., 11, No.9.

22. --- (1950): Practical applications of sampling inspec tion plans and tables. Philips Tech. Rev. No.12.

23. --- (1951): Economic principles in industrial samplin problems: a general introduction. Bull. Int. Statist. Inst. 33 part 5, 105-122.

24. --- (1955): Infusing statistics into industry. Bull. Int. Statist. Inst. 35, part 2, 433-443.

25. --- (1958): Some basic principles of sampling inspec- tion by attributes. Applied Statistics, 7

26. --- (1958): Attribute sampling in operation. Bull. Int. Statist. Inst. 37, part 2, 255-281.

27. --- (1959): Adjusting single sampling plans for finit lot size. Applied Statistics, 8, No.3.

- Harvard University, Computation Laboratory (1955): Tables of the cumulative binomial probability distribution. Mass. Harvard Univ. Press.
- Johansen, S. (1969): Asymptotic properties of the restricted Bayesian double sampling plan. Technometrics, 11.
- John, S. (1964): Methods for the evaluation of polygonal and angular regions when the distribution is bivariate t. Sankhyā, A, 26, 47-54.
- Kendall, M. G. and Stuart, A. (1963): Advanced theory of Statistics, Vol. I-Distribution theory (Second Edition). Charles Griffin, London.
- Khatri, C. G. and Mitra, S. K. (1968): Some identities and approximations concerning positive and negative multinomial distributions. Tech. Report No.1 of R.T.S., Indian Statistical Institute, Calcutta. (To be published in Ann. Math. Statist.).
- Mitra, S. K. and Subrahmanya, M. T. (1968): A robust property of the OC of binomial and Poisson sampling inspection plans. Sankhyā, B, 30, 335-342.
- National Bureau of Standards, U.S. Department of Commerce: Applied Mathematics Series, No. 14 (1951): Tables of the exponential function e^x.

  No. 23 (1953): Tables of normal probability functions.

(contd.)
National Bureau of Standards, U.S. Department of Commerce: Applied Mathematics Series:

No. 34 (1954): Tables of the Gamma function for complex arguments.

No. 50 (1959): Tables of the bivariate normal distribution functions and related functions.

No. 55 (1964): Handbook of Mathematical Functions.

Owen, D. B. (1957): Bivariate normal probability distribution. Washington. Govt. Printing.

--- (1962): Handbook of Statistical Tables. Pergamon Press. London.

Patil, G. P. ed. (1965): Classical and contagious discrete distributions. Statistical Publishing Society, Calcutta.

Pearson, E. S. and Hartley, H. O. eds (1954): Biometrika Tables for statisticians, Vol. 1. Cambridge, Biometrika trustee.

Pearson, Karl, ed (1922): Tables of the incomplete Gamma function. H.M.S. Office, London.

--- ed (1930): Tables for statisticians and biometricians, Parts I and II (Third Edition). University College, London.

--- ed (1934): Tables of the incomplete Beta function. Biometrika, London.

44. RMM Tables, edited by Rao, C. R., Mitra, S. K. and Matthai, A. (1966): Formulae and tab for statistical work. Statistical Pu ing Society, Calcutta.

45. Subrahmanya, M. T. (1966): Error areas under the operati characteristic curves. Sankhyā, A, 28, 71 - 86.

46. --- (1968): EQL and related measures of Poisson double sampling plans (n_1, n_2, c_1 Technical publications series No.3, Institute of Statistics, University of Ghana, Legon, Ghana.

47. --- with Mitra, S. K. (1968): A ~~robust~~ prop of the OC of binomial and Poisson sam ing inspection plans. Sankhya, B, 30 335-342.

48. --- (1969): On the equitable quality level normal and other sampling inspection pl Institute of Statistical, Social and Economic Research, University of Ghana, Legon, Ghana.

49. Thyregod, P. (1968): Asymptotic properties of minimax regr single sampling plans. Institute of Mathematical Statistics, University of Copenhagen.

50. U. S. Defence Department (1957): Sampling procedures and tables for inspection by variables for percentage defective, MIL-STD 414. Government Printing, Washington.

fence Department (1959): Sampling procedures and tables for inspection by attributes, MIL-STD 105. Government Printing, Washington.

Waerden (1960): Sampling inspection as a minimum loss problem. Ann. Math. Statist., 31, 369-384.

(1947): Sequential Analysis. John Wiley, New York.

S. (1963): Tables of the cumulative binomial probability distribution for small values of p. Collier-Macmillan, London.

(1962): Mathematical Statistics. John New York.

and Fuijt, W. (1955):
Philips Star
ips' gl