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A GENERAL METHOD OF DENSITY ESTIMATION FOR ASSOCIATED RANDOM VARIABLES

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Let $\{X_n, n \geq 1\}$ be a sequence of stationary associated random variables having a common marginal density function $f(x)$. Let $\phi_n(x,y)$, $n = 1,2,\ldots$, be a sequence of Borel-measurable functions defined on R^2. Let $f_n(x) = 1/n\sum_{k=1}^{n} \phi_n(x, X_k)$ be the empirical density function. Here we study a set of sufficient conditions under which the probability $\Pr(\sup_{a+\delta \leq x \leq b-\delta}|f_n(x) - f(x)| > \varepsilon) \to 0$ at an exponential rate as $n \to \infty$ where the rate possibly depends on ε, δ and f and $[a, b]$ is a finite or an infinite interval.

Keywords: Density estimation; associated random variables; uniform consistency; exponential rate

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1. INTRODUCTION

Let $\{X_n, n \geq 1\}$ be a sequence of associated random variables defined on a probability space $(\Omega, \mathcal{F}, \mathcal{P})$. A set of random variables $\{X_1, \ldots, X_n\}$ is said to be associated it for every pair of functions $h(\mathbf{x})$ and $g(\mathbf{x})$ from R^n to R, which are nondecreasing componentwise,

$$\mathrm{Cov}(h(\mathbf{X}), g(\mathbf{X})) \geq \mathbf{0},$$

whenever it is finite, where $\mathbf{X} = (X_1, X_2, \ldots, X_n)$. An infinite sequence $\{X_n\}$ of random variables is said to be associated if every finite subset is associated.

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The concept of association was introduced by Esary, Proschan and Walkup (1967). Associated random variables are of considerable interest in reliability studies, percolation theory and statistical mechanics. For a review of several probabilistic and statistical results for associated sequences, see Prakasa Rao and Dewan (1998).

Suppose $\{X_n, n \geq 1\}$ is a stationary sequence of associated random variables and the marginal density f of X_1 exists. We now consider the problem of estimation of f based on $(X_1, \ldots, X_n)$. Let $\phi_n(x, y)$, $n = 1, 2, \ldots$, be a sequence of Borel-measurable functions defined on R^2. Then the empirical density function is defined as follows:

$$f_n(x) = \frac{1}{n}\sum_{k=1}^{n} \phi_n(x, X_k). \tag{1.1}$$

This function can be considered as an estimator of f. This estimator is a generalization of the histogram type density estimator, the kernel type density estimators and the density estimator obtained by the method of orthogonal series. Properties of the empirical density function or a variation of it were considered by Foldes and Revesz, (1974) and Walter and Blum (1979) in the case of independent and identically distributed random variables, by Foldes (1974) for the case of stationary ϕ-mixing sequences and by Prakasa Rao (1978) for stationary Markov processes, among others (*cf.* Prakasa Rao, 1983).

Here we study conditions leading to the exponential rate of convergence for the uniform consistency in probability of the estimator $f_n(x)$, that is, the conditions under which

$$\Pr\left(\sup_{a+\delta \leq x \leq b-\delta} |f_n(x) - f(x)| > \varepsilon\right) \to 0 \quad \text{as } n \to \infty, \tag{1.2}$$

at an exponential rate. Sufficient conditions for the asymptotic property (1.2) to hold have been studied earlier for the case of a sequence of independent and identically distributed random variables (Foldes and Revesz, 1974), for a ϕ-mixing sequence of random variables (Foldes, 1974) and for absolutely regular sequence of identically distributed random variables (Yoshihara, 1984).

For some recent work on density estimation for associated sequences see Bagai and Prakasa Rao (1991, 1995) and Roussas (1991). Bagai and Prakasa Rao (1995) have considered uniform consistency of kernel-type density estimator $\tilde{f}_n(x)$ of $f(x)$ based on stationary associated sequences. However, no rates were obtained. Roussas (1991) showed that for $\theta > 0$ and for any compact interval $[a, b]$,

$$\sup_{x \in [a,b]} n^{\theta} |\tilde{f}_n(x) - f(x)| \to 0 \text{ a.s.} \quad \text{as } n \to \infty,$$

under some conditions. We have obtained the exponential type bounds for the rate of uniform consistency for a larger class of estimators which include the kernel-type estimator as a special case.

2. THE MAIN THEOREM

THEOREM 2.1 *Let $\{X_n, n \geq 1\}$ be a stationary sequence of associated random variables with the common one-dimensional marginal density function f for which*

$$|f(x_1) - f(x_2)| \leq k|x_1 - x_2| \quad \text{if } x_1, x_2 \in [a, b], \tag{2.1}$$

$$\int_{-\infty}^{\infty} |x|^{\gamma} f(x) dx < \infty \quad \text{for some} \quad \gamma > 0. \tag{2.2}$$

Let $\{\phi_n(x, y)\}$ be a sequence of Borel measurable functions which are of bounded variation in y for a fixed x. Then,

$$\phi_n(x, y) = \phi_{1n}(x, y) - \phi_{2n}(x, y), \tag{2.3}$$

where $\phi_{in}(x, y)$, $i = 1, 2$ is monotone in y for fixed x. Suppose that there exists two positive numbers α and τ and an interval $[c, d]$ containing $[a, b]$ such that for each n the interval $[c, d]$ can be divided into disjoint left closed intervals $I_s^{(n)}$, $s = 1, 2, \ldots,$ for which

$$|I_s^{(n)}| \geq \frac{1}{n^{\alpha}}, \quad \bigcup_{s=1}^{n} I_s^{(n)} = [c, d], \tag{2.4}$$

$$|\phi_n(x_1, y) - \phi_n(x_2, y)| \leq n^{\tau} |x_1 - x_2| \tag{2.5}$$

provided that x_1 and x_2 belong to the same interval $I_s^{(n)}$.

Suppose that

$$\int_a^b \phi_n(x,y)f(y)dy \to f(x) \quad \text{as } n \to \infty \tag{2.6}$$

uniformly in $[a+\delta, b-\delta]$ for some $\delta > 0$.

Suppose that for each n

$$\operatorname{Var}(\phi_{in}(x, X_1)) \leq h_n, \quad i = 1,2 \tag{2.7}$$

$$h_n \leq \frac{n}{w(n)\log n}, \tag{2.8}$$

where $w(n) = O(n^{\beta'})$ for some $\beta' > 0$ and $w(n) \to \infty$ as $n \to \infty$, and for a positive constant C

$$|\phi_{in}(x,y)| \leq Ch_n, \quad i = 1,2. \tag{2.9}$$

Further suppose that there exists a $v > 0$ and a sequence of positive numbers $\varepsilon_n \to 0$ such that

$$|\phi_n(x_n, y_n)| \leq \varepsilon_n \tag{2.10}$$

whenever,

$$|x_n - y_n| > n^v, \tag{2.11}$$

and

$$n > n_0(\varepsilon). \tag{2.12}$$

Suppose that for $i = 1, 2$, $\phi_{in}(x, y)$ is differentiable with respect to y and

$$|\phi'_{in}(x,y)| \leq b_n \tag{2.13}$$

where $\phi'_{in}(x,y)$ denotes the partial derivative of $\phi_{in}(x,y)$ with respect to y and there exists $\beta > 0$ such that

$$\frac{b_n}{h_n} = O(n^{\beta}). \tag{2.14}$$

Finally assume that

$$\frac{1}{n}\sum_{j=1}^{n} \mathrm{Cov}(X_1, X_j) = O(\mathrm{e}^{-n\theta}) \tag{2.15}$$

for some $\theta > 3/2$. Then

$$\Pr\left(\sup_{a+\delta \le x \le b-\delta} |f_n(x) - f(x)| \ge \varepsilon \right) \le \mathrm{e}^{-\frac{k_1 n}{h_n}}. \tag{2.16}$$

as $n \to \infty$, where k_1 is a positive constant depending on ε, δ, and f.

Remarks 2.2 The list of conditions assumed above on $\phi_n(x, y)$ is long and they are similar to those of Foldes and Revesz (1974) in the i.i.d. case to include histogram type density estimator, kernel type density estimator and the density estimator obtained by the method of orthogonal series *etc.* In addition we have assumed here that $\phi_n(x, y)$ is a function of bounded variation in y for a fixed x to deal with the dependence of association type. Covariance structure of an associated sequence plays an important role in the study of limit theorems for associated random variables. Our condition (2.15) on the covariance structure is of this type. The inequality (2.16) gives an exponential bound for the uniform convergence of the density estimator f_n.

3. SOME LEMMAS

The proof of Theorem 2.1 is based on the following lemmas.

LEMMA 3.1 *Let $X_1, X_2, \ldots, X_n$ be associated random variables that are bounded by a constant δ'. Then, for any $\lambda > 0$,*

$$\left| E[e^{\lambda \sum_{i=1}^{n} X_i}] - \prod_{i=1}^{n} E[e^{\lambda X_i}] \right| \le \lambda^2 k^n \sum_{1 \le i < j \le n} Cov(X_i, X_j), \tag{3.1}$$

where

$$k = e^{\lambda \delta'}. \tag{3.2}$$

Proof Using Newman's (1980) inequality, we get that for $n = 2$ and any $\lambda > 0$

$$|\mathrm{Cov}(e^{\lambda X_1}, e^{\lambda X_2})| \leq \lambda^2 k^2 |\mathrm{Cov}(X_1, X_2)| \tag{3.3}$$

The result follows by induction and using the fact that if X, Y and Z are associated then so are X and $Y + Z$ as they are increasing functions of associated random variables. ■

Note that because of (2.3)

$$\begin{aligned} f_n(x) &= \frac{1}{n}\sum_{k=1}^{n} \phi_n(x, X_k) \\ &= \frac{1}{n}\sum_{k=1}^{n}\{\phi_{1n}(x, X_k) - \phi_{2n}(x, X_k)\} \\ &= f_{1n}(x) - f_{2n}(x) \quad \text{(say)}. \end{aligned} \tag{3.4}$$

LEMMA 3.2 *Under the conditions of Theorem* 2.1, *there exists* $\alpha' > 0$ *such that for any* $x \in [a, b]$,

$$E\left[e^{\lambda_n(f_{1n}(x)) - E(f_{1n}(x))}\right] \leq e^{\lambda_n^2 h_n / n} + cn^2 \frac{b_n^2}{h_n^2} e^{-\alpha' n}, \tag{3.5}$$

provided that

$$0 < \lambda_n < \frac{n}{4Ch_n}, \tag{3.6}$$

where C *is the constant in Condition* (2.9) *and* c *denotes a positive constant.*

Proof Note that

$$\begin{aligned} E\left[e^{\lambda_n(f_{1n}(x)) - E(f_{1n}(x))}\right] &= E\left[e^{(\lambda_n/n)\sum_{j=1}^{n}(\phi_{1n}(x, X_j) - E\phi_{1n}(x, X_j))}\right] \\ &= E\left[e^{(\lambda_n/n)\sum_{j=1}^{n} Y_{nj}(x)}\right], \end{aligned} \tag{3.7}$$

where

$$Y_{nj}(x) = \phi_{1n}(x, X_j) - E\phi_{1n}(x, X_j). \tag{3.8}$$

Observe that $Y_{nj}(x)$, $j = 1, 2, \dots, n$, are increasing functions of associated random variables ((2.3)) and hence are associated.

Then

$$\left|E\left[e^{\lambda_n(f_{1n}(x)-E(f_{1n}(x)))}\right]\right| \le \left|E\left[e^{(\lambda_n/n)\sum_{j=1}^n Y_{nj}(x)}\right] - \prod_{j=1}^n E\left[e^{(\lambda_n/n)Y_{nj}(x)}\right]\right| + \prod_{j=1}^n E\left[e^{(\lambda_n/n)Y_{nj}(x)}\right]\Bigg|. \tag{3.9}$$

Thus, by using the inequality $e^u \le 1 + u + u^2$ for $|u| \le 1/2$, we get

$$\begin{aligned}\prod_{j=1}^n E\left[e^{(\lambda_n/n)Y_{nj}(x)}\right] &\le \prod_{j=1}^n E\left[1 + \frac{\lambda_n}{n} Y_{nj}(x) + \frac{\lambda_n^2}{n^2} Y_{nj}^2(x)\right] \\ &\le \prod_{j=1}^n \left(1 + \frac{\lambda_n^2}{n^2} h_n\right) \quad \text{(by (2.7))} \\ &\le e^{(\lambda_n^2/n)h_n}. \end{aligned} \tag{3.10}$$

Further using Lemma 3.1 and (2.9) and the fact that $0 < \lambda_n < n/4Ch_n$, we get that

$$\begin{aligned}&\left|E\left[e^{(\lambda_n/n)\sum_{j=1}^n Y_{nj}(x)}\right] - \prod_{j=1}^n E\left[e^{(\lambda_n/n)Y_{nj}(x)}\right]\right| \\ &\le \frac{\lambda_n^2}{n^2} e^{n/2} \sum_{1\le i<j\le n} \mathrm{Cov}(Y_{ni}(x), Y_{nj}(x)) \quad \text{(by Lemma 3.1)} \\ &\le \frac{\lambda_n^2}{n^2} e^{n/2} b_n^2 \sum_{1\le i<j\le n} \mathrm{Cov}(X_i, X_j) \quad \text{(by Newman's (1980) inequality)} \\ &\le \frac{\lambda_n^2}{n^2} e^{n/2} b_n^2 n \sum_{j=1}^n \mathrm{Cov}(X_1, X_j) \quad \text{(by stationarity of } X_j\text{)} \\ &\le \frac{\lambda_n^2}{n^2} e^{n/2} b_n^2 n^2 e^{-n\theta} \quad \text{(using (2.15))} \\ &\le n^2 \frac{b_n^2}{h_n^2} e^{-\alpha' n}, \quad \alpha' > 0. \end{aligned} \tag{3.11}$$

The result follows by combining (3.10) and (3.11). ■

A similar result holds for $f_{2n}(x)$.

LEMMA 3.3 *Under the conditions of Theorem* 2.1, *for any* $x \in [a, b]$ *and for every* $\varepsilon > 0$,

$$Pr(|f_n(x) - Ef_n(x)| \geq \varepsilon) \leq e^{-(k_1(\varepsilon)n)/h_n} + n^2 \frac{b_n^2}{h_n^2} e^{-(k_1(\varepsilon)n)/h_n}, \quad (3.12)$$

where the constant $k_1(\varepsilon)$ *does not depend on n and x.*

Proof Using (3.4), we get that

$$\Pr(|f_n(x) - Ef_n(x)| \geq \varepsilon) \leq \Pr\left(|f_{1n}(x) - Ef_{1n}(x)| \geq \frac{\varepsilon}{2}\right) + \Pr\left(|f_{2n}(x) - Ef_{2n}(x)| \geq \frac{\varepsilon}{2}\right) \quad (3.13)$$

Note that $0 < \lambda_n < n/4Ch_n$ and suppose that $\lambda_n h_n/n \to 0$ as $n \to \infty$. Then by the Markov inequality and Lemma 3.2, we get that, for $i = 1$, 2, there exists a positive constant $k_2(\varepsilon)$ such that

$$\begin{aligned}\Pr\left(f_{in}(x) - Ef_{in}(x) \geq \frac{\varepsilon}{2}\right) &\leq E\left[e^{\lambda_n(f_{in}(x)) - E(f_{in}(x))}\right]/e^{\lambda_n\varepsilon/2} \\ &= \frac{e^{\lambda_n^2 h_n/n} + cn^2 \frac{b_n^2}{h_n^2} e^{-\alpha' n}}{e^{\lambda_n \varepsilon/2}} \\ &\leq e^{-k_2(\varepsilon)n/h_n} + cn^2 \frac{b_n^2}{h_n^2} e^{-k_2(\varepsilon)n/h_n} \quad (3.14)\end{aligned}$$

One can choose $k_2(\varepsilon) = \varepsilon/16C$. The result now follows from the fact that if W_j', are associated, then so are $-W_j'$s. ■

LEMMA 3.4 (Bagai and Prakasa Rao (1991)) *Let X and Y be associated random variables, with bounded continuous density functions given by* f_X *and* f_Y. *Then there exists a constant C such that*

$$\begin{aligned}&\sup_{x,y} |Pr[X \leq x, Y \leq y] - Pr[X \leq x]\, Pr[Y \leq y]| \\ &\leq C\left\{t^2 Cov\,(X, Y) + \frac{1}{t}\right\}, \quad (3.15)\end{aligned}$$

for every $t > 0$.

4. PROOF OF THEOREM 2.1

The proof of Theorem 2.1 will be along the same lines as that given in Foldes and Revesz (1974) and Foldes (1974).

Let for each positive integer n

$$-n^T = z_0^{(n)} < z_1^{(n)} < \cdots < z_{l(n)}^{(n)} = n^T$$

(the number T will be determined later on) be a partitioning of the interval $[-n^T, n^T]$ having the following properties:

(a) $\frac{c_1}{n^{\alpha+2\tau}} \leq z_i^{(n)} - z_{i-1}^{(n)} \leq \frac{c_2}{n^{\alpha+2\tau}}$,
$0 < c_1 < c_2 < \infty, \quad i = 1, \ldots, l(n)$,

(b) those end points of the interval $I_s^{(n)}$ which belong to $[-n^T, n^T]$ are elements of the sequence $z_0^{(n)}, z_1^{(n)}, \ldots, z_{l(n)}^{(n)}$.

By (a), (1.1) and (2.5), we get that

$$|f_n(x) - f_n(y)| \leq c\frac{1}{n^{\alpha+\tau}} \quad \text{if } x, y \in \left[z_{i-1}^{(n)}, z_i^{(n)}\right], \tag{4.1}$$

and

$$|Ef_n(x) - Ef_n(y)| \leq c\frac{1}{n^{\alpha+\tau}} \quad \text{if } x, y \in \left[z_{i-1}^{(n)}, z_i^{(n)}\right]. \tag{4.2}$$

Note that,

$$|f_n(x) - Ef_n(x)| \leq \left|f_n(x) - f_n\left(z_{i-1}^{(n)}\right)\right| + \left|f_n\left(z_{i-1}^{(n)}\right) - Ef_n\left(z_{i-1}^{(n)}\right)\right| + \left|Ef_n\left(z_{i-1}^{(n)}\right) - Ef_n(x)\right|.$$

Therefore, using (4.1) and (4.2) we have

$$\sup_{z_{i-1}^{(n)} \leq x \leq z_i^{(n)}} |f_n(x) - Ef_n(x)| \leq \frac{c}{n^{\alpha+\tau}} + \sup_{z_{i-1}^{(n)} \leq x \leq z_i^{(n)}} \left|f_n\left(z_{i-1}^{(n)}\right) - Ef_n\left(z_{i-1}^{(n)}\right)\right| + \frac{c}{n^{\alpha+\tau}}.$$

Hence,

$$\Pr\Big(\sup_{z_{i-1}^{(n)}\le x\le z_i^{(n)}} |f_n(x) - Ef_n(x)| \ge \varepsilon\Big)$$
$$\le \Pr\Big(2\frac{c}{n^{\alpha+\tau}} \ge \frac{\varepsilon}{2}\Big) + \Pr\Big(\big|f_n(z_{i-1}^{(n)}) - Ef_n(z_{i-1}^{(n)})\big| \ge \frac{\varepsilon}{2}\Big). \quad (4.3)$$

But, for large n $\Pr(2c/n^{\alpha+\tau} \ge \varepsilon/2)$ is zero. Therefore, using (4.3) we have

$$\Pr\Big(\sup_{-n^T\le x\le n^T} |f_n(x) - Ef_n(x)| \ge \varepsilon\Big)$$
$$\le \sum_{i=1}^{l(n)} \Big\{\Pr\Big(\sup_{z_{i-1}^{(n)}\le x\le z_i^{(n)}} |f_n(x) - Ef_n(x)| \ge \varepsilon\Big)\Big\}$$
$$\le l(n)\max_i \Big\{\Pr\Big(\sup_{z_{i-1}^{(n)}\le x\le z_i^{(n)}} |f_n(x) - Ef_n(x)| \ge \varepsilon\Big)\Big\}$$
$$\le l(n)\max_i \Big\{\Pr\Big(\big|f_n(z_{i-1}^{(n)}) - Ef_n(z_{i-1}^{(n)})\big| \ge \frac{\varepsilon}{2}\Big)\Big\}$$
$$\le l(n)\mathrm{e}^{-k_1(\varepsilon)n/h_n} + l(n)n^2\frac{b_n^2}{h_n^2}\,\mathrm{e}^{-k_1(\varepsilon)n/h_n} \quad \text{(using (3.12))}$$
$$\le \mathrm{e}^{-k_4(\varepsilon)n/h_n} \quad \text{(using (3.14))} \quad (4.4)$$

for large n. Note that $l(n)$ is the number of partitioning intervals of an interval length $2n^T$. Therefore, $l(n) \simeq 2n^{\alpha+2\tau+T}$. Furthermore, $Ef_n(x)\to f(x)$ as $n\to\infty$ by (2.6). Therefore

$$\Pr\Big(\sup_{\substack{|x|\ge n^T\\ x\in[a+\delta,b-\delta]}} |f_n(x) - Ef_n(x)| \ge \varepsilon\Big)$$
$$\le \Pr\Big(\sup_{\substack{|x|\ge n^T\\ x\in[a+\delta,b-\delta]}} |f_n(x)| \ge \frac{\varepsilon}{2}\Big) + \Pr\Big(\sup_{\substack{|x|\ge n^T\\ x\in[a+\delta,b-\delta]}} |Ef_n(x)| \ge \frac{\varepsilon}{2}\Big)$$
$$= \Pr\Big(\sup_{\substack{|x|\ge n^T\\ x\in[a+\delta,b-\delta]}} |f_n(x)| \ge \frac{\varepsilon}{2}\Big) \quad (4.5)$$

for large n, since $\Pr(\sup |f(x)| \geq \varepsilon/2)$ is zero. Now

$$\begin{aligned}
\Pr\Big(\sup_{|x|\geq n^T} |f_n(x)| \geq \frac{\varepsilon}{2}\Big)
&\leq \Pr\Big(\sup_{|x|\geq n^T} \frac{1}{n} \sum_{k:|X_k-x|\leq\delta} |\phi_n(x,X_k)| \geq \frac{\varepsilon}{4}\Big) \\
&\quad + \Pr\Big(\sup_{|x|\geq n^T} \frac{1}{n} \sum_{\substack{k:|X_k-x|>\delta \\ X_k\in S(x,nT/2)}} |\phi_n(x,X_k)| \geq \frac{\varepsilon}{8}\Big) \\
&\quad + \Pr\Big(\sup_{|x|\geq n^T} \frac{1}{n} \sum_{\substack{k:|X_k-x|>\delta \\ X_k\notin S(x,nT/2)}} |\phi_n(x,X_k)| \geq \frac{\varepsilon}{8}\Big) \\
&\leq 2\Pr\Big(\frac{1}{n}Ch_n \sum_{k:|X_k|\geq nT/2} 1 \geq \frac{\varepsilon}{8}\Big) \\
&\quad + \Pr\Big(\sup_{|x|\geq n^T} \frac{1}{n} \sum_{\substack{k:|X_k-x|>\delta \\ X_k\notin S(x,nT/2)}} |\phi_n(x,X_k)| \geq \frac{\varepsilon}{8}\Big) \qquad (4.6)
\end{aligned}$$

where $S(x, n^T/2)$ denotes the interval $[x - n^T/2, x + n^T/2]$. This inequality is a consequence of (2.9) and the fact that, for large n,

$$|x| \geq n^T, \quad |X-x| \leq \delta \Rightarrow |X| \geq \frac{n^T}{2}$$

and

$$|x| \geq n^T, \quad |X-x| > \delta, \quad X \in S\Big(x, \frac{n^T}{2}\Big) \Rightarrow |X| \geq \frac{n^T}{2}.$$

Denote by $J_n(u)$ the following indicator function

$$J_n(u) = \begin{cases} 1, & \text{if } |u| \geq n^T/2 \\ 0, & \text{otherwise.} \end{cases} \qquad (4.7)$$

Note that J_n can be expressed as sum of two monotone functions I_n and I'_n, where

$$I_n(u) = \begin{cases} 1, & \text{if } u \geq n^T/2 \\ 0, & \text{otherwise,} \end{cases}$$

and

$$I'_n(u) = \begin{cases} 1, & \text{if } u \le -n^T/2 \\ 0, & \text{otherwise.} \end{cases}$$

Then we have to estimate the following probability

$$\Pr\left(\sum_{k=1}^{n} J_n(X_k) \ge \frac{cn\varepsilon}{h_n}\right) \le \Pr\left(\sum_{k=1}^{n} I_n(X_k) \ge \frac{cn\varepsilon}{2h_n}\right) + \Pr\left(\sum_{k=1}^{n} I'_n(X_k) \ge \frac{cn\varepsilon}{2h_n}\right) \tag{4.8}$$

for some positive constant c. Since X'_ks are associated, $Y_k = I_n(X_k)$, $k = 1, \dots, n$ are associated and so are $Z_k = I'_n(X_k)$, $k = 1, \dots, n$. Therefore, we will estimate (4.8) using Lemma 3.3 for the associated random variables $Y_1, \dots, Y_n$ and $Z_1, \dots, Z_n$.
Now

$$\Pr\left(\sum_{k=1}^{n} I_n(X_k) \ge \frac{cn\varepsilon}{2h_n}\right) \le \Pr\left(\sum_{k=1}^{n} (I_n(X_k) - EI_n(X_k)) \ge \frac{cn\varepsilon}{4h_n}\right) + \Pr\left(\sum_{k=1}^{n} EI_n(X_k) \ge \frac{cn\varepsilon}{4h_n}\right). \tag{4.9}$$

Note that

$$\begin{aligned} E(I_n(X_k)) &= \Pr\left(X_k > \frac{n^T}{2}\right) \\ &\le \Pr\left(|X_k| > \frac{n^T}{2}\right) \\ &= \Pr\left(|X_k|^\gamma > \frac{n^{T\gamma}}{2^\gamma}\right) \\ &\le \frac{E(|X_k|^\gamma)2^\gamma}{n^{T\gamma}} \\ &\le \frac{c(\gamma)}{n^{T\gamma}} \text{ (using (2.2))} \end{aligned} \tag{4.10}$$

where $c(\gamma)$ is a constant depending on γ and f.

Choose T to be so large that $T\gamma-1 > 0$. The sequence $(cn\varepsilon/4h_n)$ tends to infinity by (2.8). Therefore

$$\Pr\left(\sum_{k=1}^{n} EI_n(X_k) \geq \frac{cn\varepsilon}{4h_n}\right) = 0 \tag{4.11}$$

for large n.

Furthermore,

$$\mathrm{Var}(I_n(X_k)) \leq \frac{c(\gamma)}{n^{T\gamma}}. \tag{4.12}$$

For $0 < \lambda_n^* \leq 1/4$,

$$\Pr\left(\sum_{k=1}^{n}(I_n(X_k) - EI_n(X_k)) \geq \frac{cn\varepsilon}{4h_n}\right) \leq \frac{E\left(e^{\lambda_n^* \sum_{k=1}^{n}(I_n(X_k)-EI_n(X_k))}\right)}{e^{\lambda_n^* cn\varepsilon/4h_n}}. \tag{4.13}$$

Now

$$\begin{aligned} E\left(e^{\lambda_n^* \sum_{k=1}^{n}(I_n(X_k)-EI_n(X_k))}\right) &= E\left(e^{\lambda_n^* \sum_{k=1}^{n}(I_n(X_k)-EI_n(X_k))}\right) \\ &\quad - \prod_{k=1}^{n} E\left(e^{\lambda_n^*(I_n(X_k)-EI_n(X_k))}\right) \\ &\quad + \prod_{k=1}^{n} E\left(e^{\lambda_n^*(I_n(X_k)-EI_n(X_k))}\right). \end{aligned} \tag{4.14}$$

Therefore, for $0 < \lambda_n^* \leq 1/4$ and using the inequality $e^u \leq 1 + u + u^2$ for $|u| \leq 1/2$, we get

$$\begin{aligned} &\prod_{k=1}^{n} E\left(e^{\lambda_n^*(I_n(X_k)-EI_n(X_k))}\right) \\ &\quad \leq \prod_{k=1}^{n} E[1+\lambda_n^*(I_n(X_k)-EI_n(X_k))+(\lambda_n^*)^2(I_n(X_k)-EI_n(X_k))^2] \\ &\quad = \prod_{k=1}^{n}[1+(\lambda_n^*)^2\mathrm{Var}(I_n(X_k))] \\ &\quad \leq \left[1+\frac{(\lambda_n^*)^2 c(\gamma)}{n^{T\gamma}}\right]^n \\ &\quad \leq e^{(\lambda_n^*)^2 c(\gamma)/n^{T\gamma-1}} \end{aligned} \tag{4.15}$$

since $(1+x_n)^n = (1+nx_n/n)^n \approx \mathrm{e}^{nx_n}$. Applying Lemma 3.1 for the associated random variables $Y_1, \ldots, Y_n$, we get

$$\begin{aligned}&\left|E\left(\mathrm{e}^{\lambda_n^* \sum_{k=1}^n (I_n(X_k)-EI_n(X_k))}\right) - \prod_{k=1}^n E\left(\mathrm{e}^{\lambda_n^*(I_n(X_k)-EI_n(X_k))}\right)\right| \\ &\quad \le (\lambda_n^*)^2 \mathrm{e}^{2n\lambda_n^*} \sum_{1\le i<j\le n} \mathrm{Cov}(I_n(X_i), I_n(X_j)).\end{aligned} \tag{4.16}$$

Then, for $0 < \lambda_n^* \le 1/4$ and any $t > 0$, we get that

$$\begin{aligned}&\left|E\left(\mathrm{e}^{\lambda_n^* \sum_{k=1}^n (I_n(X_k)-EI_n(X_k))}\right) - \prod_{k=1}^n E\left(\mathrm{e}^{\lambda_n^*(I_n(X_k)-EI_n(X_k))}\right)\right| \\ &\quad \le c(\lambda_n^*)^2 \mathrm{e}^{2n\lambda_n^*} \sum_{1\le i<j\le n} \left(t^2 \mathrm{Cov}(X_i, X_j) + \frac{1}{t}\right) \\ &\quad \le c(\lambda_n^*)^2 \mathrm{e}^{2n\lambda_n^*} \left(t^2 n \sum_{j=1}^n \mathrm{Cov}(X_1, X_j) + \frac{n^2}{t}\right) \\ &\qquad \text{(by using the stationarity of } \{X_j\}) \\ &\quad \le c(\lambda_n^*)^2 \mathrm{e}^{2n\lambda_n^*} n^2 \left(t^2 \mathrm{e}^{-n\theta} + \frac{1}{t}\right) \quad \text{(by using (2.15))} \\ &\quad \le c(\lambda_n^*)^2 n^2 \mathrm{e}^{2n\lambda_n^*} \mathrm{e}^{-n\theta/3} \quad \text{(by choosing } t = \mathrm{e}^{n\theta/3}).\end{aligned} \tag{4.17}$$

Using (4.15) and (4.17) in (4.13), we get that for $0 < \lambda_n^* \le \frac{1}{4}$,

$$\begin{aligned}&\Pr\left(\sum_{k=1}^n (I_n(X_k) - EI_n(X_k)) \ge \frac{cn\varepsilon}{4h_n}\right) \\ &\quad \le \mathrm{e}^{\frac{(\lambda_n^*)^2 c(\gamma)}{n^{T\gamma-1}} - \frac{cn\lambda_n^*\varepsilon}{2h_n}} + c(\lambda_n^*)^2 n^2 \mathrm{e}^{2n\lambda_n^*} \mathrm{e}^{\frac{-n\theta}{3}} \mathrm{e}^{-\frac{cn\lambda_n^*\varepsilon}{2h_n}} \\ &\quad \le \mathrm{e}^{\frac{-k_3(\varepsilon)n}{h_n}} + \mathrm{e}^{-\frac{k_4(\varepsilon)n}{h_n}} \quad \text{(by using(2.8))} \\ &\quad \le \mathrm{e}^{-\frac{k(\varepsilon)n}{h_n}}.\end{aligned} \tag{4.18}$$

Substituting (4.11) and (4.18) in (4.9) we get an estimate for $\Pr(\sum_{k=1}^n I_n(X_k) \ge \frac{cn\varepsilon}{2h_n})$. Similarly, we can get an estimate for $\Pr(\sum_{k=1}^n I'_n(X_k) \ge \frac{cn\varepsilon}{2h_n})$. Combining the two we can get an estimate for the expression on the left hand side of (4.8).

Finally for $T > v$ and by (2.10) and (2.11), we get

$$\Pr\left(\sup_{|x| \geq n^T} \frac{1}{n} \sum_{\substack{k:|X_k - x| > \delta \\ X_k \notin S\left(x, \frac{n^T}{2}\right)}} |\phi_n(x, X_k)| \geq \frac{\varepsilon}{8} \right) \leq \Pr\left(\varepsilon_n > \frac{\varepsilon}{8} \right) = 0, \tag{4.19}$$

for large n. Using (4.8) and (4.19) in (4.6) we get an estimate for (4.5).

Choose $T > \max(v, 1/\gamma)$. Then, for large n, and from (2.6), (4.4) and the estimate of (4.5) given by (4.19) we have the following inequality proving the theorem:

$$\begin{aligned} &\Pr\left(\sup_{a+\delta \leq x \leq b-\delta} |f_n(x) - f(x)| \geq \varepsilon \right) \\ &\leq \Pr\left(\sup_{a+\delta \leq x \leq b-\delta} |f_n(x) - Ef_n(x)| \geq \frac{\varepsilon}{2} \right) \\ &\quad + \Pr\left(\sup_{a+\delta \leq x \leq b-\delta} |Ef_n(x) - f(x)| \geq \frac{\varepsilon}{2} \right) \\ &= \Pr\left(\sup_{a+\delta \leq x \leq b-\delta} |f_n(x) - Ef_n(x)| \geq \frac{\varepsilon}{2} \right) \\ &\leq e^{-\frac{k_1 n}{h_n}}. \end{aligned} \tag{4.20}$$

Remarks 4.1 Various examples of the estimator $f_n(x)$ have been discussed by Foldes and Revesz (1974) in the i.i.d. case. Similar examples can be given for the associated case. For instance the standard normal density is a kernal which is a function of bounded variation and it can be checked that it satisfies all the conditions of Theorem 2.1 and we obtain exponential rates for uniform convergence of the kernel type density estimator.

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