A Generalisation to the Hybrid Fourier Transform and Its Application

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(Received February 2002; revised and accepted July 2002)

Abstract—The hybrid Fourier transform, involving a linear combination of the cosine and sine functions as its kernel, is generalised for discontinuous but integrable functions, in the half-range comprising of the positive real axis. The present generalisation of the hybrid transform is observed to be useful in the area of two-dimensional wave problems involving a two-fluid region as opposed to the well-known hybrid transform, known as Havelock's expansion theorem, whose use is limited to the study of water wave problems involving only a single fluid medium.

Keywords—Hybrid Fourier transform, Generalisation, Havelock's expansion, Application to two-fluid problem.

1. INTRODUCTION

Integral transforms involving only the cosine or the sine functions in their kernels, which are known as the Fourier cosine or the Fourier sine transforms, have been extensively used in the study of a variety of boundary value problems of mathematical physics. The transforms, for which the kernels are linear combinations of the cosine and sine functions in the whole semi-infinite range of their definitions, are called *hybrid* Fourier transforms (cf. [1,2]). Such hybrid Fourier transforms have also been used in the study of a number of mixed boundary value problems occurring in the linearised theory of water waves (cf. [3,4]). The simplest of the hybrid Fourier transforms, along with its inversion formula, is also known as Havelock's expansion theorem. This expansion theorem was originally given by Havelock [5] to solve the classical plane vertical wavemaker problem occurring in the theory of surface water waves involving a single fluid medium. Later on, this has been utilised to analyse a large class of water wave scattering problems involving a thin vertical plane barrier in a single fluid.

In the present paper, we have examined the possibility of obtaining an integral expansion theorem which is useful for functions which are in general discontinuous but integrable and are defined on the positive real axis and having just one point of discontinuity. Such an expansion theorem defines a new generalised hybrid Fourier transform in which the kernel is comprised of two different combinations of the cosine and the sine functions, in the two ranges, separating the point of discontinuity. The corresponding inversion formula is easily derivable from the expansion theorem proved.

As an application to the presently derived expansion theorem, we have generalised the classical wavemaker problem in a single fluid to the case when the fluid medium is comprised of two different immiscible fluids of constant densities. All the results for this simple two-fluid problem are expressed in terms of convergent integrals, and the corresponding classical results for a single fluid are derived as a limiting case.

2. THE GENERALISED EXPANSION THEOREM

THEOREM 2.1. *If $f(y)$ is an integrable function in the range $(0,\infty)$, having a discontinuity at a single point $y = h(>0)$, then $f(y)$ can be expanded as*

$$f(y) = \begin{cases} A_1 e^{-Ky} + A_2 g(y) + \int_0^\infty A(k) L_1(k,y)\,dk, & 0 < y < h, \\ A_1 e^{-Ky} + A_2 e^{v(h-y)} + \int_0^\infty A(k) L_2(k,y)\,dk, & y > h, \end{cases} \tag{2.1}$$

where v is the unique positive root of the transcendental equation

$$(K+v)e^{-vh} + (K\sigma - v)e^{vh} = 0 \tag{2.2}$$

with $K > 0$ and $\sigma = (1+s)/(1-s)$ $(0 < s < 1)$ as two given constants (note that there exists no positive root of equation (2.2) if $s \geq 1$),

$$g(y) = \frac{K\sigma - v}{K(\sigma-1)} e^{-v(y-h)} + \frac{K-v}{K(\sigma-1)} e^{v(y-h)}, \qquad 0 < y < h, \tag{2.3}$$

$$L_1(k,y) = K(k\cos ky - K\sin ky), \tag{2.4}$$

$$L_2(k,y) = L_1(k,y) + (1-s)\left(k^2+K^2\right)\sin kh \cos k(y-h), \tag{2.5}$$

$$A_1 = \frac{2Ke^{2Kh}}{1+s(e^{2Kh}-1)}\left[s\int_0^h f(y)e^{-Ky}\,dy + \int_h^\infty f(y)e^{-Ky}\,dy\right], \tag{2.6}$$

$$A_2 = \frac{s\int_0^h f(y)g(y)\,dy + \int_h^\infty f(y)e^{v(h-y)}\,dy}{s\int_0^h \{g(y)\}^2\,dy + 1/2v}, \tag{2.7}$$

and

$$A(k) = \frac{2}{\pi}\frac{1}{(k^2+K^2)\,D(k)}\left[s\int_0^h f(y)L_1(k,y)\,dy + \int_h^\infty f(y)L_2(k,y)\,dy\right]$$

with

$$D(k) = \{(1-s)k\sin kh + K\cos kh\}^2 + s^2K^2\sin^2 kh. \tag{2.8}$$

PROOF. We prove the theorem by utilizing some standard results involving the delta function and its representations in terms of the trigonometric functions. First, we consider the Hilbert space of complex valued functions $\phi(y), \psi(y)$ of the real variable $y \in (0,\infty)$ and introduce the generalised inner product $\langle\phi,\psi\rangle$ defined by

$$\langle\phi,\psi\rangle = \lim_{\epsilon\to 0}\left[s\int_0^h e^{-\epsilon y}\phi(y)\overline{\psi(y)}\,dy + \int_h^\infty e^{-\epsilon y}\phi(y)\overline{\psi(y)}\,dy\right] \tag{2.9}$$

with the bar denoting the complex conjugates, where ϕ and ψ are two complex-valued functions in the range $(0, \infty)$ with a possible discontinuity at $y = h$. Introduction of such generalised inner products with the aid of an appropriately chosen convergence factor is common in Fourier analysis even though it is not stated explicitly in many published works in this direction, since divergent integrals are always to be understood as limits of convergent integrals involving certain convergence factors.

Next, we consider the following eigenvalue problem, for the discontinuous function χ:

$$L\chi \equiv \frac{d^2\chi}{dy^2} = \lambda\chi, \qquad y \in (0,h) \cup (h,\infty), \tag{2.10}$$

with χ having a discontinuity at $y = h$, such that

$$\begin{aligned} \chi'(0) + K\chi(0) &= 0, \\ \chi'(h+0) &= \chi'(h-0), \\ \chi'(h+0) + K\chi(h+0) &= s\left\{\chi'(h-0) + K\chi(h-0)\right\}, \end{aligned}$$

and

$$\chi(y) \text{ bounded as } y \to \infty. \tag{2.11}$$

It may be noted that λ occurring in the problem described by (2.11) is an eigenvalue. It is easy to show that the operator L with respect to the inner product defined by (2.9), is self-adjoint in the sense that

$$\langle \phi, L\psi \rangle = \overline{\langle \psi, L\phi \rangle}, \tag{2.12}$$

and that the following are the sets of eigenvalues and corresponding independent eigenfunctions of the eigenvalue problem described by (2.11):

(i) $\lambda = K^2, \qquad \chi(y) = e^{-Ky}, \quad 0 < y < \infty,$

(ii) $\lambda = v^2, \qquad \chi(y) = \begin{cases} g(y), & 0 < y < h, \\ e^{v(h-y)}, & h < y < \infty, \end{cases}$

(iii) $\lambda = -k^2 (k > 0), \qquad \chi(y) = \begin{cases} L_1(k,y), & 0 < y < h, \\ L_2(k,y), & h < y < \infty, \end{cases}$

where $g(y)$, $L_1(k,y)$, and $L_2(k,y)$ are given by (2.3), (2.4), and (2.5), respectively. Thus, since the operator L is self-adjoint, the above eigenfunctions corresponding to the different eigenvalues are orthogonal with respect to the inner product defined by (2.9), and hence, the expansion theorem (2.1) is easily proved by using the standard techniques involving eigenfunction expansions.

The expressions for A_1 and A_2 as given in (2.6) and (2.7) are obtained easily, while the main result to determine $A(\alpha)$ as given by relation (2.8) (with k replaced by α) is the following:

$$\lim_{\epsilon \to 0} \left[s\int_0^h e^{-\epsilon y} L_1(k,y) L_1(\alpha,y) + \int_h^\infty e^{-\epsilon y} L_2(k,y) L_2(\alpha,y)\,dy \right] = \frac{\pi}{2}\left[G_1(k,\alpha)\delta(k-\alpha) + G_2(k,\alpha)\delta(k+\alpha)\right]$$

with

$$\begin{aligned} G_{1,2}(k,\alpha) = {} & sK^2\left(k\alpha \pm K^2\right) + (1-s)K(\alpha \sin\alpha h + K\cos\alpha h) \\ & \times \left\{\alpha(k\cos kh - K\sin kh) \pm K(k\sin kh + K\cos kh)\right\} \\ & + \left(k^2 + K^2\right)\sin kh(\alpha\cos\alpha h - K\sin\alpha h) \\ & + (1-s)\left(k^2 + K^2\right)\left(\alpha^2 + K^2\right)\sin kh \sin\alpha h. \end{aligned} \tag{2.13}$$

which is easily obtainable by using

$$\lim_{\epsilon\to 0+}\int_0^\infty e^{-\epsilon x}\cos tx\,dx = \pi\delta(t) \qquad (t \text{ real}),$$
$$\lim_{\epsilon\to 0+}\int_0^\infty e^{-\epsilon x}\sin tx\,dt = \frac{1}{t} \qquad (t\neq 0,\ \text{real}). \tag{2.14}$$

It is observed from the above expansion theorem (2.1) that we can now define the following *generalised hybrid Fourier transform* along with its inversion formula.

The *generalised Fourier transform* of the function $f(y)$ $(y > 0)$ is given by

$$F(k) = s\int_0^h f(y)L_1(k,y)\,dy + \int_h^\infty f(y)L_2(k,y)\,dy, \qquad h > 0, \tag{2.15}$$

where $L_1(k,y)$ and $L_2(k,y)$ are given by relations (2.4) and (2.5), respectively, and s is a known constant such that $0 < s < 1$.

The inversion formula for the transform (2.15) is given by (2.1) with the constants A_1, A_2 being given by (2.6),(2.7) along with the relation

$$A(k) = \{\mu(k)\}^{-1}F(k),$$

where

$$\mu(k) = \frac{\pi}{2}\left(k^2+K^2\right)D(k), \tag{2.16}$$

with $D(k)$ being given in (2.8).

We easily check that in the special circumstances when $s\to 1$ and $h\to\infty$, we get back the well-known Havelock's expansion theorem, giving rise to the hybrid transform

$$F_H(k) = \int_0^\infty f(y)(k\cos ky - K\sin ky)\,dy$$

along with the inversion formula

$$f(y) = Ce^{-Ky} + \frac{2}{\pi}\int_0^\infty \frac{F_H(k)}{k^2+K^2}(k\cos ky - K\sin ky)\,dk$$

with

$$C = 2K\int_0^\infty f(y)e^{-Ky}\,dy. \tag{2.17}$$

We remark here that though our theorem has been proved above for sufficiently smooth functions $f(y)$, it is also possible to prove its validity even if $f(y)$ represents a *generalised function* by using the standard concepts that generalised functions can be associated with equivalent classes of smooth ordinary functions having compact support.

3. AN APPLICATION

As an application of the generalised expansion theorem derived above, we consider the following boundary value problem for the Laplace's equation in two dimensions, occurring in the study of forced waves in a two-fluid region. We have to solve the PDE

$$\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} = 0, \qquad 0 < y < h, \qquad \text{and} \qquad h < y < \infty, \tag{3.1}$$

where $\phi(x,y)$ represents the velocity potential of an irrotational motion, with a discontinuity along the plane $y = h$ which represents the interface (at rest) of two layers of inviscid, incompressible, and immiscible fluids of two different constant densities ρ_1 and ρ_2. The various conditions under which the PDE (3.1) has to be solved are

$$\frac{\partial \phi}{\partial y} + K\phi = 0, \qquad \text{on } y = 0, \tag{3.2}$$

$$\left(\frac{\partial \phi}{\partial y} + K\phi\right)_{y=h+0} = s\left(\frac{\partial \phi}{\partial y} + K\phi\right)_{y=h-0} \tag{3.3}$$

with $s = \rho_1/\rho_2$ $(0 < s < 1)$,

$$\frac{\partial \phi}{\partial x}(0, y) = p(y) = \begin{cases} p_1(y), & \text{for } 0 < y < h, \\ p_2(y), & \text{for } y > h, \end{cases} \tag{3.4}$$

$p_1(y), p_2(y)$ being known functions,

$$\phi(x, y) \sim R_1 e^{-Ky+iK|x|} + R_2\, l(y) e^{iv|x|}, \qquad \text{as } |x| \to \infty \tag{3.5}$$

with

$$l(y) = \begin{cases} g(y), & \text{for } 0 < y < h, \\ e^{v(h-y)}, & \text{for } y > h, \end{cases} \tag{3.6}$$

where $g(y)$ is given by (2.3) and v is the unique positive root of equation (2.2), and R_1 and R_2 are two unknown complex constants representing the amplitudes of waves radiated at infinity with wave numbers K and v, respectively.

$$\phi, \qquad \nabla\phi \to 0 \text{ as } y \to \infty. \tag{3.7}$$

This problem is a generalisation of the classical wavemaker problem of Havelock [5] to a two-layer fluid.

An appropriate solution satisfying (3.1)–(3.3), (3.5), and (3.7) is given by

$$\phi(x, y) = R_1 e^{-Ky+iKx} + R_2\, l(y) e^{ivx} + \int_0^\infty A(k) L(k, y) e^{-kx}\, dk, \qquad \text{for } x > 0, \tag{3.8}$$

where

$$L(k, y) = \begin{cases} L_1(k, y), & \text{for } 0 < y < h, \\ L_2(k, y), & \text{for } y > h, \end{cases} \tag{3.9}$$

and $A(k)$ is an unknown function of appropriate behaviour for the mathematical analysis followed below to hold good.

Use of condition (3.4) produces

$$p_1(y) = iKR_1 e^{-Ky} + ivR_2 g(y) - \int_0^\infty kA(k) L_1(k, y)\, dk, \qquad \text{for } 0 < y < h,$$

and

$$p_2(y) = iKR_1 e^{-Ky} + ivR_2 e^{v(h-y)} - \int_0^\infty kA(k) L_2(k, y)\, dk, \qquad \text{for } y > h. \tag{3.10}$$

Thus, using Theorem 2.1, we find that

$$\begin{aligned} iKR_1 &= \frac{2Ke^{2Kh}}{1 + s\left(e^{2Kh} - 1\right)} \left[s\int_0^h p_1(y) e^{-Ky}\, dy + \int_h^\infty p_2(y) e^{-Ky}\, dy \right], \\ ivR_2 &= \frac{s\int_0^h p_1(y) g(y)\, dy + \int_h^\infty p_2(y) e^{v(h-y)}\, dy}{s\int_0^h \{g(y)\}^2\, dy + 1/2v}, \end{aligned} \tag{3.11}$$

and

$$-kA(k) = \frac{2}{\pi} \frac{1}{(k^2 + K^2) D(k)} \left[s \int_0^h p_1(y) L_1(k, y)\, dy + \int_h^\infty p_2(y) L_2(k, y)\, dy \right]. \tag{3.12}$$

It is interesting to note that in the limiting case when $s \to 1$ and $h \to \infty$, we get back the results of the classical wavemaker problem for a single fluid of infinite depth.

As special cases, if we choose $p_1(y) = p_2(y) = e^{-Ky}$, then we find

$$R_1 = \frac{1}{iK}, \qquad R_2 = 0, \qquad \text{and} \qquad A(k) = 0,$$

so that $\phi(x, y)$ is simply given by

$$\phi(x, y) = \frac{1}{iK} e^{-Ky + iKx},$$

while if we choose $p_1(y) = g(y)$, $p_2(y) = e^{v(h-y)}$, then

$$R_1 = 0, \qquad R_2 = \frac{1}{iv}, \qquad A(k) = 0,$$

so that $\phi(x, y)$ is given by

$$\phi(x, y) = \frac{1}{iv} l(y) e^{ivx}.$$

For the more general case, for prescribed $p(y)$, $\phi(x, y)$, and hence, the forms of the free surface and the interface can be obtained. However, this we have not pursued here.

4. CONCLUSION

It is clear that expansion theorems of the type described here can be further generalised for integrable functions having more than one but finite number of discontinuities in the range $(0, \infty)$ of its definition, by way of defining an appropriate generalised inner product of the type (2.9). Such general expansion theorems will then give rise to hybrid Fourier transforms over multiple intervals, and these will obviously have utility in handling wave problems involving multiple layers of inviscid, incompressible, and immiscible fluids of different constant densities.

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