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A MODEL THEORY FOR q-COMMUTING CONTRACTIVE TUPLES

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ABSTRACT. A contractive tuple is a tuple $(T_1, \ldots, T_d)$ of operators on a common Hilbert space such that

$$T_1T_1^* + \cdots + T_dT_d^* \leqslant \mathbb{1}. \tag{0.1}$$

It is said to be q-commuting if $T_jT_i = q_{ij}T_iT_j$ for all $1 \leqslant i < j \leqslant d$, where q_{ij}, $1 \leqslant i < j \leqslant d$ are complex numbers. These are higher-dimensional and non-commutative generalizations of a contraction. A particular example of this is the q-commuting shift. In this note, we investigate model theory for q-commuting contractive tuples using representations of the q-commuting shift.

KEYWORDS: *Contractions, model theory, dilation, complete positivity.*

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1. INTRODUCTION

Suppose we have a linear contraction T on a separable Hilbert space. (All our Hilbert spaces will be separable.) Consider the usual Toeplitz algebra $\mathcal{T}$ (see [5]), i.e., the unital C^*-algebra generated by the unilateral shift S. Then there is a unique unital completely positive map φ on $\mathcal{T}$ which maps S to T and moreover any "sesqui-polynomial" $\sum a_{k,l}S^k(S^*)^l$ to $\sum a_{k,l}T^k(T^*)^l$. (Keeping powers of S^*, T^* only on the right is important.) Actually this is a way of looking at Sz.-Nagy dilation of contractions. Indeed if we consider the minimal Stinespring representation π of φ, we see that $\pi(S)$ is nothing but the minimal isometric dilation of T. Usual model theory including von Neumann's inequality fail miserably when one has to deal with tuples of operators. However this modified approach has been quite successfully used by Agler ([1]), and Athavale ([6], [7]) to deal with operators as well as tuples of operators satisfying certain conditions coming from the theory of reproducing kernels. The basic steps of this model theory are as follows.

In the given class of operators (or operator tuples) identify a distinguished one, sort of "standard shift", the C^*-algebra generated by that will play the role of Toeplitz algebra. Then obtain a unital completely positive map as above. Apply Stinespring's representation theorem to obtain dilation of any operator (or operator tuple) of our class. Some standard facts of C^*-algebra representation theory come in handy to study all possible representations of new "Toeplitz algebra". Typically every representation breaks up as a direct sum of identity representation with some multiplicity and a "spherical part" (recall Wold decomposition). Recently Arveson has demonstrated as to how beautifully the very same method applies to any commutative contractive (satisfying condition (0.1)) tuple. As a corollary he also obtains a von Neumann's inequality. Our program here is to extend this model theory to q-commuting contractive tuples. Such tuples have received a lot of attention in recent years; q-commuting pairs seem to appear in abundance in quantum theory. We refer to [9], [13] and [16] for many examples with such properties.

A much more general approach applicable to general non-commuting contractive tuples of operators can be found in the papers of Popescu ([14], [15]) and his co-author Arias ([2], [3]). It is possible to obtain most of the results one has for special cases like commuting or q-commuting tuples using their theory of Poisson transforms and dilations on full Fock space through a quotienting procedure. However we closely follow Arveson's methods deviating only at a few places. As it turns out, many essential features for commuting contractive tuples carry over to q-commuting contractive tuples. A standard shift $\underline{S}$ can be defined without difficulty. The existence of a required completely positive map, von Neumann's inequality etc. can be established. The notion of energy sequence remains essentially the same and the operator space generated by the tuple $\underline{S}$ is maximal in the sense that the value of its energy sequence is greater than that of any other d-dimensional operator space generated by a q-commuting contractive tuple.

Any *ordered* d-tuple of non-negative integers $\underline{k} = (k_1, \ldots, k_d)$ will be called a *multi-index*. We shall write $k_1 + \cdots + k_d$ as $|\underline{k}|$. The special multi-index which has 0 in all positions except the ith one, where it has 1, is denoted by $\underline{e}_i$.

Throughout this note, $d > 1$ is a positive integer. Let $z_1, \ldots, z_d$ be d variables satisfying $z_j z_i = q_{ij} z_i z_j$ for $1 \leqslant i < j \leqslant d$, where q_{ij} are complex numbers. We shall call these variables to be *q-commuting*. (We will not need q_{ij} for $i \geqslant j$.)

For any d variables $z_1, \ldots, z_d$ as above and any non-zero multi-index $\underline{k}$, the monomial $z_1^{k_1} \cdots z_d^{k_d}$ will be denoted by $\underline{z}^{\underline{k}}$. Note that since z_i are q-commuting, the order in the monomial is important. So our multi-indices are ordered. For the multi-index $\underline{k} = (0, \ldots, 0)$, we let $\underline{z}^{\underline{k}}$ to be the complex number 1. The linear combinations of the monomials give rise to the vector space of polynomials to be denoted by $\mathcal{P}$. A polynomial f of degree n is determined by some set of constants $\{b_{\underline{k}} : |\underline{k}| \leqslant n\}$, i.e., $f(z_1, \ldots, z_d) = \sum b_{\underline{k}} \underline{z}^{\underline{k}}$.

From now on, unless explicitely stated otherwise, the symbols $z_1, \ldots, z_d$ will always mean these q-commuting variables, which will be called *the co-ordinate functions*. Examples of such variables can be found in quantum theory where in many cases $|q_{ij}| = 1$. Throughout we will denote $|q_{ij}|^2$ by p_{ij}. With the variables $z_1, \ldots, z_d$, we shall associate a new set of variables $w_1, \ldots, w_d$ satisfying the relation $w_j w_i = p_{ij} w_i w_j$ for $1 \leqslant i < j \leqslant d$.

DEFINITION 1.1. Let $\mathcal{P}$ be the vector space of all polynomials in $z_1, \dots, z_d$. Endow it with the following inner product. First declare $\underline{z}^{\underline{k}}$ and $\underline{z}^{\underline{l}}$ orthogonal if $\underline{k}$ is not the same as $\underline{l}$ as ordered multi-indices. Then let $\|\underline{z}^{\underline{k}}\|^2$ to be the reciprocal of the coefficient of $\underline{w}^{\underline{k}}$ in the multinomial expansion of $(w_1 + \cdots + w_d)^n$ where $|\underline{k}| = n$. Now define $\mathcal{H}$ to be the closure of $\mathcal{P}$ with respect to this inner product.

In the commutative case, i.e., if $q_{ij} \equiv 1$, the Hilbert space $\mathcal{H}$ is the space H_2 discussed by Arveson in [5].

For any integer $n \geqslant 1$ let $\mathcal{P}_n$ be the finite-dimensional subspace of $\mathcal{P}$ spanned by the monomials of the form $\underline{z}^{\underline{k}}$ where $|\underline{k}| = n$, while $\mathcal{P}_0$ is defined to be $\mathbb{C}$. Note then that

$$\mathcal{H} = \bigoplus_{n=0}^{\infty} \mathcal{P}_n.$$

Given any multi-index $\underline{k}$ and any permutation π of $\{1, \dots, d\}$, let the multi-index $\underline{l}$ be defined by $l_i = k_{\pi(i)}$. Then $z_{\pi(1)}^{l_1} \cdots z_{\pi(n)}^{l_n}$ is a multiple of $\underline{z}^{\underline{k}}$. So dimension of $\mathcal{P}_n$ is the same as the dimension of the nth symmetric tensor power of $\mathbb{C}^d$. So

$$\dim \mathcal{P}_n = \binom{n+d-1}{n}.$$

We do not really treat $\mathcal{H}$ as any functional Hilbert space. But we call the one-dimensional space spanned by $\underline{z}^{\underline{k}}$ where $\underline{k}$ is the zero multi-index as *the space of constant functions*. This is the space $\mathcal{P}_0$ mentioned above.

When $q_{ij} = q$ for all $i < j$, the norm $\|\underline{z}^{\underline{k}}\|$ is as follows. To begin with, we get rid of a few long expressions by fixing notations for them. For any complex number q and positive integer n, let

$$[n, q] = 1 + q + \cdots + q^{n-1}, \quad [0, q] = 0$$

and

$$[n, q]! = [n, q][n-1, q] \cdots [1, q], \quad [0, q]! = 1.$$

The multinomial expansion of $(w_1 + \cdots + w_d)^n$ in this case is of the form

$$(w_1 + \cdots + w_d)^n = \sum a(\underline{k}, p) w_1^{k_1} \cdots w_d^{k_d},$$

where the sum is over all multi-indices $\underline{k}$ such that $n = k_1 + \cdots + k_d$ and

$$a(\underline{k}, p) = \frac{[|\underline{k}|, p]!}{[k_1, p]! \cdots [k_d, p]!}.$$

The set $\{(a(\underline{k}, p))^{1/2} \underline{z}^{\underline{k}} : |\underline{k}| \geqslant 0\}$ thus forms an orthonormal basis for $\mathcal{H}$.

The organisation of the paper is as follows. In Section 2, we prove a minimality property of the space $\mathcal{H}$. Section 3 is about the special q-commuting contractive tuple $\underline{S}$ which is a generalization of the one-dimensional shift on the Hardy space of the unit disk and also of the d-shift of Arveson. So that will be referred to in this paper as *the q-commuting shift* or simply the shift. Section 3 describes the basic properties of the shift. Section 4 investigates a suitable model for a general q-commuting contractive tuple. It turns out that the shift plays a big role. The model theory is investigated in detail where we show that every q-commuting contractive tuple is, up to unitary equivalence, a compression of a certain special q-commuting contractive tuple to a suitable subspace. Section 5 is about the d-dimensional operator space spanned by the shift and its energy sequence.

Methods of this paper are also applicable to general non-commutative contractive operator tuples. Indeed consider the vector space of polynomials in non-commuting variables $z_1, \ldots, z_d$ (no commutation relations between z_i's assumed). Taking distinct monomials $\{1, z_{i_1} z_{i_2} \cdots z_{i_r} : 1 \leqslant i_j \leqslant d, 1 \leqslant j \leqslant r, r \geqslant 1\}$, as orthonormal and completing the space we have a Hilbert space which is naturally isomorphic (in the obvious way) to the full Fock space over $\mathbb{C}^d$. The operator multiplication by z_i from the left (denote it by S_i) corresponds to the left creation operator of the standard basis vector $\{e_i\}$ for $1 \leqslant i \leqslant d$. Note that S_i's don't commute but they are isometries with orthogonal ranges. Taking $(S_1, \ldots, S_d)$ as "standard shift" we can build a model theory for arbitrary contractive tuples $(T_1, \ldots, T_d)$ very much along the lines of Sections 2 and 4. However we do not elaborate much on this as these ideas (though not necessarily with same terminology) has been explored by many authors. See for example Frazho ([11]), Bunce ([8]) and Popescu ([14]). We also would like to remark that much of the theory can be extended to infinite tuples $(T_1, T_2, \ldots)$, satisfying $\sum T_i T_i^* \leqslant I$, by simply considering polynomials in infinite number of variables.

2. AN INCLUSION PRINCIPLE

Let us begin with a couple of characterizations of the condition (0.1) of a q-commuting contractive tuple $\underline{T} = (T_1, \ldots, T_d)$ acting on a Hilbert space $\mathcal{K}$. First note that this condition is equivalent to demanding that

$$\|T_1 \xi_1 + \cdots + T_d \xi_d\|^2 \leqslant \|\xi_1\|^2 + \cdots + \|\xi_d\|^2 \tag{2.1}$$

for any $\xi_1, \ldots, \xi_d$ in $\mathcal{K}$. Notice too that if we define a completely positive map on $\mathcal{B}(\mathcal{K})$ by $P_{\underline{T}}(X) = T_1 X T_1^* + \cdots + T_d X T_d^*$ then the condition (0.1) on $\underline{T}$ holds if and only if $P_{\underline{T}}$ is a contraction.

Given a q-commuting contractive tuple $\underline{T}$ on a Hilbert space $\mathcal{K}$ and a multi-index $\underline{k}$ we employ the notation $\underline{T}^{\underline{k}}$ in exactly the same way as $\underline{z}^{\underline{k}}$. So $\underline{T}^{\underline{k}}$ will mean the operator $T_1^{k_1} \cdots T_d^{k_d}$. Note that if T_i's q-commute, completely positive maps $P_i(X) := T_i X T_i^*$, $X \in \mathcal{B}(\mathcal{K})$, p-commute, where $p_{ij} = |q_{ij}|^2$, in the sense:

$$P_j(P_i(X)) = p_{ij} P_i(P_j(X)), \quad \text{for } 1 \leqslant i < j \leqslant d.$$

Now as $P_{\underline{T}} = \sum_i P_i$, by multinomial theorem for any $n \geqslant 0$,

$$P_{\underline{T}}^n(X) = \sum_{|\underline{k}|=n} \frac{1}{\|\underline{z}^{\underline{k}}\|^2} \underline{T}^{\underline{k}} X (\underline{T}^{\underline{k}})^*. \tag{2.2}$$

Theorem 2.1. *Let $\underline{T}$ be a q-commuting contractive tuple on a Hilbert space $\mathcal{K}$. Suppose there is a unit vector $v \in \mathcal{K}$ such that for any non-zero multi-index $\underline{k}$, the element $\underline{T}^{\underline{k}} v$ is orthogonal to v. Then there is a contraction $C : \mathcal{H} \to \mathcal{K}$ such that*

$$C \underline{z}^{\underline{k}} = \underline{T}^{\underline{k}} v \tag{2.3}$$

for any multi-index $\underline{k}$.

Proof. We define C on the monomials by (2.3) and then extend, by linearity to the polynomials. If we can prove that C is a contraction from $\mathcal{P}$ to $\mathcal{K}$ then, because $\mathcal{P}$ is dense, C will extend uniquely to $\mathcal{H}$ as a contraction. Let n be any positive integer and $\sum b_{\underline{k}}\underline{z}^{\underline{k}}$ be any polynomial in $\mathcal{H}$ where the sum is over all $\underline{k}$ with $|\underline{k}| \leqslant n$. By definition, $C\big(\sum b_{\underline{k}}\underline{z}^{\underline{k}}\big) = \sum b_{\underline{k}}\underline{T}^{\underline{k}}v$ and hence what we need to show is

$$\Big\| \sum b_{\underline{k}}\underline{T}^{\underline{k}}v\Big\|_{\mathcal{K}}^2 \leqslant \sum_{\underline{k}} |b_{\underline{k}}|^2\|\underline{z}^{\underline{k}}\|^2 \tag{2.4}$$

for any set of constants $\{b_{\underline{k}} : |\underline{k}| \leqslant n\}$. Note that by replacing b_k by $b_k(\|\underline{z}^{\underline{k}}\|)^{-1}$, (2.4) is equivalent to

$$\Big\| \sum \frac{b_{\underline{k}}}{\|\underline{z}^{\underline{k}}\|}\underline{T}^{\underline{k}}v\Big\|_{\mathcal{K}}^2 \leqslant \sum_{\underline{k}} |b_{\underline{k}}|^2 \tag{2.5}$$

for any set of constants $\{b_{\underline{k}} : |\underline{k}| \leqslant n\}$. Let E_0 be the projection onto v. It is now obvious from the discussion at the beginning of this section that showing (2.5) for any set of constants $\{b_{\underline{k}}\}$ is equivalent to showing that the tuple

$$\left\{\frac{1}{\|\underline{z}^{\underline{k}}\|}\underline{T}^{\underline{k}}E_0 : |\underline{k}| \leqslant n\right\}$$

is a contractive tuple. This can be best organised in the following way:

On $\mathcal{B}(\mathcal{K})$, define the completely positive map $P_{\underline{T}}(X) = \sum T_iXT_i^*$. It follows that $\|P_{\underline{T}}\| = \|P_{\underline{T}}(\mathbb{1}_{\mathcal{K}})\| = \big\|\sum T_iT_i^*\big\| \leqslant 1$. Then $P_{\underline{T}}(E_0)$ is a positive contraction. Also $\langle T_iE_0T_i^*v, v\rangle_{\mathcal{K}} = 0$ because $E_0T_i^*v$ is a multiple of v. Thus $\langle P_{\underline{T}}(E_0)v, v\rangle_{\mathcal{K}} = 0$. This implies that $P_{\underline{T}}(E_0) \leqslant \mathbb{1}_{\mathcal{K}} - E_0$. Since $0 \leqslant E_0 + P_{\underline{T}}(E_0) \leqslant \mathbb{1}_{\mathcal{K}}$ we have $0 \leqslant P_{\underline{T}}(E_0) + (P_{\underline{T}})^2(E_0) \leqslant P_{\underline{T}}(\mathbb{1}_{\mathcal{K}}) \leqslant \mathbb{1}_{\mathcal{K}}$. Again $\langle (P_{\underline{T}}(E_0) + (P_{\underline{T}})^2(E_0))v, v\rangle_{\mathcal{K}} = 0$ forcing $E_0 + P_{\underline{T}}(E_0) + (P_{\underline{T}})^2(E_0) \leqslant \mathbb{1}_{\mathcal{K}}$. This way after applying a simple induction,

$$E_0 + P_{\underline{T}}(E_0) + \cdots + P_{\underline{T}}^n(E_0) \leqslant \mathbb{1}_{\mathcal{K}}.$$

Now define

$$Q_{\underline{T}}(X) = E_0XE_0 + P_{\underline{T}}(E_0XE_0) + \cdots + P_{\underline{T}}^n(E_0XE_0).$$

Then $Q_{\underline{T}}$ is a completely positive map and the inequality just proved shows that it is a contraction. Now from (2.2)

$$Q_{\underline{T}}(X) = \sum \frac{1}{\|\underline{z}^{\underline{k}}\|^2}\underline{T}^{\underline{k}}E_0XE_0(\underline{T}^{\underline{k}})^*$$

where now the sum is over all multi-indices $\underline{k}$ such that $|\underline{k}| \leqslant n$. By the discussion before the theorem, we are done. ∎

This theorem compares with the maximality of H^2 norm proved by Arveson ([5]). Also note that the following non-commutative generalization holds. Essentially the same proof with necessary modifications works. Let $\underline{T}$ be a contractive $(\sum T_i T_i^* \leqslant \mathbb{1})$ tuple of operators on a Hilbert space $\mathcal{K}$. No commutation relations between T_i's are assumed. Suppose there is a unit vector v in $\mathcal{K}$ such that for any $1 \leqslant i_1, \ldots, i_r \leqslant d$, the vector $T_{i_1} \cdots T_{i_r} v$ is orthogonal to v. Then there is a contraction C from the full Fock space over $\mathbb{C}^d$ into $\mathcal{K}$ such that $C(1) = v$, and $C(e_{i_1} \otimes \cdots \otimes e_{i_r}) = T_{i_1} \cdots T_{i_r} v$, where 1 is the vacuum vector in the full Fock space and $e_1, \ldots, e_d$ are the standard basis vectors of $\mathbb{C}^d$.

3. BASIC PROPERTIES OF A SPECIAL TUPLE $\underline{S}$

The q-commuting structure of the co-ordinate functions gives rise to the following special tuple of operators which is a q-commuting contractive tuple in the sense of (0.1). The following definition defines the operator tuple $\underline{S}$ only on the polynomials. Our first lemma extends these operators to the whole of $\mathcal{H}$.

Definition 3.1. The *q-commuting shift* is the tuple $\underline{S} = (S_1, \ldots, S_d)$ where each S_i is defined for $f \in \mathcal{P}$ by

$$S_i f(z_1, \ldots, z_d) = z_i f(z_1, \ldots, z_d).$$

Lemma 3.2. *For each $i = 1, \ldots, d$, the operator S_i is bounded on the dense subspace $\mathcal{P}$ of polynomials and hence extends to $\mathcal{H}$ uniquely. Denote the extension also by S_i. Then*

$$S_j S_i = q_{ij} S_i S_j \quad \text{for } 1 \leqslant i < j \leqslant d.$$

Proof. Here and in many other occassions the following simple observation will be useful. Let $\underline{k}$ be any multi-index and let $\underline{l}$ be the multi-index $\underline{k} + \underline{e}_i$. Then note that by writing $(w_1 + \cdots + w_d)^{|\underline{l}|}$ as $(w_1 + \cdots + w_d)(w_1 + \cdots + w_d)^{|\underline{k}|}$ and by computing the coeffiecients, we get

$$\frac{1}{\|\underline{z}^{\underline{l}}\|^2} = \sum p_{1j}^{l_1} \cdots p_{(j-1)j}^{l_{j-1}} \frac{1}{\|\underline{z}^{\underline{k} + \underline{e}_i - \underline{e}_j}\|^2} \tag{3.1}$$

where the sum is over all j for which k_j are non-zero.

Thus

$$\frac{1}{\|\underline{z}^{\underline{l}}\|^2} \geqslant p_{1i}^{k_1} \cdots p_{(i-1)i}^{k_{i-1}} \frac{1}{\|\underline{z}^{\underline{k}}\|^2}.$$

Or,

$$p_{1i}^{k_1} \cdots p_{(i-1)i}^{k_{i-1}} \|\underline{z}^{\underline{k} + \underline{e}_i}\|^2 \leqslant \|\underline{z}^{\underline{k}}\|^2.$$

Note the action of S_i on the monomials:

$$S_i \underline{z}^{\underline{k}} = z_i \underline{z}^{\underline{k}} = q_{1i}^{k_1} \cdots q_{(i-1)i}^{k_{i-1}} \underline{z}^{\underline{k} + \underline{e}_i}.$$

So

$$\|S_i \underline{z}^{\underline{k}}\|^2 = p_{1i}^{k_1} \cdots p_{(i-1)i}^{k_{i-1}} \|\underline{z}^{\underline{k} + \underline{e}_i}\|^2 \leqslant \|\underline{z}^{\underline{k}}\|^2.$$

If two multi-indices $\underline{k}$ and $\underline{l}$ are different, then $\underline{z}^{\underline{k}}$ and $\underline{z}^{\underline{l}}$ are orthogonal and consequently $S_i\underline{z}^{\underline{k}}$ and $S_i\underline{z}^{\underline{l}}$ are orthogonal too. For any family of constants $\{b_{\underline{k}} : |\underline{k}| \leqslant n\}$,

$$\left\| S_i\Big(\sum_{|\underline{k}|\leqslant n} b_{\underline{k}}\underline{z}^{\underline{k}}\Big)\right\|^2 = \sum_{|\underline{k}|\leqslant n} |b_{\underline{k}}|^2\|S_i\underline{z}^{\underline{k}}\|^2 \leqslant \sum_{|\underline{k}|\leqslant n} |b_{\underline{k}}|^2\|\underline{z}^{\underline{k}}\|^2 = \left\|\Big(\sum_{|\underline{k}|\leqslant n} b_{\underline{k}}\underline{z}^{\underline{k}}\Big)\right\|^2.$$

Thus S_i is contractive on $\mathcal{P}$ and hence extends uniquely as a bounded operator to the whole of $\mathcal{H}$. The action of S_i on monomials immediately shows that for $i < j$,

$$S_jS_i\underline{z}^{\underline{k}} = q_{ij}S_iS_j\underline{z}^{\underline{k}}.$$

By linearity, this remains true for polynomials and hence extends to the whole of $\mathcal{H}$ by density. ∎

The lemma above proves that S_i is a contraction for each $1 \leqslant i \leqslant d$ and the equation (3.1) is crucial to that. Actually much more can be said.

LEMMA 3.3. *Let $\underline{k}$ and $\underline{l}$ be any two multi-indices. Then*

$$\|\underline{S}^{\underline{k}}\underline{z}^{\underline{l}}\|^2 = \|\underline{z}^{\underline{k}}\underline{z}^{\underline{l}}\|^2 = \prod_{\substack{i<j\\ l_ik_j\neq 0}} p_{ij}^{l_ik_j}\|\underline{z}^{\underline{k}+\underline{l}}\|^2,$$

and

$$\|\underline{S}^{\underline{k}}\| = \|\underline{z}^{\underline{k}}\|.$$

Proof. The first part is obvious. Now as in the proof of the lemma above, writing $(w_1+\cdots+w_d)^{|\underline{k}|+|\underline{l}|}$ as $(w_1+\cdots+w_d)^{|\underline{k}|}(w_1+\cdots+w_d)^{|\underline{l}|}$, and comparing coefficients of $\underline{w}^{\underline{k}+\underline{l}}$, we get

$$\frac{1}{\|\underline{z}^{\underline{l}+\underline{k}}\|^2} \geqslant \frac{1}{\|\underline{z}^{\underline{k}}\|^2}\frac{1}{\|\underline{z}^{\underline{l}}\|^2}\prod_{\substack{i<j\\ l_ik_j\neq 0}} p_{ij}^{l_ik_j}.$$

Or,

$$\prod_{\substack{i<j\\ l_ik_j\neq 0}} p_{ij}^{l_ik_j}\|\underline{z}^{\underline{k}+\underline{l}}\| \leqslant \|\underline{z}^{\underline{k}}\|^2\|\underline{z}^{\underline{l}}\|^2.$$

Thus,

$$\|\underline{S}^{\underline{k}}\underline{z}^{\underline{l}}\| \leqslant \|\underline{z}^{\underline{k}}\|\,\|\underline{z}^{\underline{l}}\|, \quad \text{for all } \underline{k} \text{ and } \underline{l}.$$

This norm inequality on monomials can easily be extended to $\mathcal{P}$ by noting that orthogonal monomials are taken by $\underline{S}^{\underline{k}}$ to orthogonal monomials. Thus $\|\underline{S}^{\underline{k}}\| \leqslant \|\underline{z}^{\underline{k}}\|$. Since $\underline{S}^{\underline{k}}$ takes the constant function 1 to $\underline{z}^{\underline{k}}$, this norm is actually attained. ∎

We propose to prove that $\underline{S}$ is a q-commuting contractive tuple and the next two lemmas facilitate that.

Lemma 3.4. *The action of the operator S_i^* on the basis elements $\underline{z}^{\underline{k}}$ is as follows:*

$$S_i^* \underline{z}^{\underline{k}} = 0 \text{ if } k_i = 0 \quad \text{ and } \quad S_i^* \underline{z}^{\underline{k}} = \overline{q}_{1i}^{k_1} \cdots \overline{q}_{(i-1)i}^{k_{i-1}} \frac{\|\underline{z}^{\underline{k}}\|^2}{\|\underline{z}^{\underline{k}-\underline{e}_i}\|^2} \underline{z}^{\underline{k}-\underline{e}_i} \quad \text{if } k_i \neq 0.$$

Proof. Let $\underline{l}$ be any multi-index. Then

$$\langle S_i^* \underline{z}^{\underline{k}}, \underline{z}^{\underline{l}} \rangle = \langle \underline{z}^{\underline{k}}, S_i \underline{z}^{\underline{l}} \rangle = \overline{q}_{1i}^{l_1} \cdots \overline{q}_{(i-1)i}^{l_{i-1}} \langle \underline{z}^{\underline{k}}, \underline{z}^{\underline{l}+\underline{e}_i} \rangle.$$

This last quantity is non-zero if and only if $\underline{l} + \underline{e}_i = \underline{k}$. Thus first of all $S_i^* \underline{z}^{\underline{k}} = 0$ if $k_i = 0$. But if $k_i \neq 0$, then $S_i^* \underline{z}^{\underline{k}}$ is a constant multiple of $\underline{z}^{\underline{k}-\underline{e}_i}$. The constant is, $\overline{q}_{1i}^{k_1} \cdots \overline{q}_{(i-1)i}^{k_{i-1}} \|\underline{z}^{\underline{k}}\|^2 / \|\underline{z}^{\underline{k}-\underline{e}_i}\|^2$. ∎

Given a q-commuting contractive tuple $\underline{T}$ acting on a Hilbert space $\mathcal{K}$, one naturally associates the *defect operator* with the tuple which is defined by

$$D_T = \left[\mathbb{1} - \left(T_1 T_1^* + \cdots + T_d T_d^* \right) \right]^{\frac{1}{2}}. \tag{3.2}$$

One of the important preliminary fact of the commutative case which remains true in the new situation is that the projection E_0 onto the one-dimensional space of constant functions is the defect operator for the q-commuting shift.

Lemma 3.5. *Let $\underline{S}$ be the q-commuting shift and $\mathbb{1}$ be the identity on $\mathcal{H}$. Then*

$$\sum_{i=1}^d S_i S_i^* = \mathbb{1} - E_0.$$

Proof. First note that if $\underline{k} = 0$, then from Lemma 3.4, $S_i^* \underline{z}^{\underline{k}} = 0$ for all $i = 1, \ldots, d$. So then $\sum S_i S_i^*$ is identically zero on the range of E_0. If $\underline{k}$ is a non-zero multi-index with $k_i \neq 0$, then applying Lemma 3.4 again, we have

$$S_i^* \underline{z}^{\underline{k}} = \overline{q}_{1i}^{k_1} \cdots \overline{q}_{(i-1)i}^{k_{i-1}} \frac{\|\underline{z}^{\underline{k}}\|^2}{\|\underline{z}^{\underline{k}-\underline{e}_i}\|^2} \underline{z}^{\underline{k}-\underline{e}_i},$$

$$S_i S_i^* \underline{z}^{\underline{k}} = p_{1i}^{k_1} \cdots p_{(i-1)i}^{k_{i-1}} \frac{\|\underline{z}^{\underline{k}}\|^2}{\|\underline{z}^{\underline{k}-\underline{e}_i}\|^2} \underline{z}^{\underline{k}}. \tag{3.3}$$

If $k_i = 0$, then $S_i S_i^* \underline{z}^{\underline{k}} = 0$. So for all non-zero $\underline{k}$,

$$\sum_{\substack{i=1 \\ k_i \neq 0}}^d S_i S_i^* \underline{z}^{\underline{k}} = \|\underline{z}^{\underline{k}}\|^2 \sum_{\substack{i=1 \\ k_i \neq 0}}^d \frac{p_{1i}^{k_1} \cdots p_{(i-1)i}^{k_{i-1}}}{\|\underline{z}^{\underline{k}-\underline{e}_i}\|^2} \underline{z}^{\underline{k}}.$$

Now from (3.1),

$$\sum_{\substack{i=1 \\ k_i \neq 0}}^d \frac{p_{1i}^{k_1} \cdots p_{(i-1)i}^{k_{i-1}}}{\|\underline{z}^{\underline{k}-\underline{e}_i}\|^2} = \frac{1}{\|\underline{z}^{\underline{k}}\|^2}.$$

Hence

$$\sum_{i=1}^{d} S_i S_i^* \underline{z}^{\underline{k}} = \underline{z}^{\underline{k}}.$$

Thus the operator $\sum S_i S_i^*$ acts like identity on the orthogonal complement of the range of E_0. Hence the result. ∎

So, as a result of these lemmas, $\underline{S}$ is a q-commuting contractive tuple. Moreover, whatever be q_{ij}, the operator $\mathbb{1} - \sum S_i S_i^*$ is a one-dimensional projection. In contrast to this, the sum $\sum S_i^* S_i - \mathbb{1}$, which is also a diagonal operator by the next lemma, is not even compact if $p_{ij} \neq 1$ for some $i < j$. If all $p_{ij} = 1$, the diagonal co-efficients tend to zero as $|\underline{k}| \to \infty$. That means it can be approximated by finite rank operators.

LEMMA 3.6. *Let $\underline{S}$ be the q-commuting shift. Then each monomial $\underline{z}^{\underline{k}}$ is an eigenvector for $\sum S_i^* S_i - \mathbb{1}$, so that it is a diagonal operator on the standard basis. In fact,*

$$\sum_{i=1}^{d} S_i^* S_i \underline{z}^{\underline{k}} = \left(\sum_{i=1}^{d} \frac{\left\| \underline{z}^{\underline{k}+\underline{e}_i} \right\|^2}{\left\| \underline{z}^{\underline{k}} \right\|^2} p_{1i}^{k_1} \cdots p_{(i-1)i}^{k_{i-1}} \right) \underline{z}^{\underline{k}}.$$

$\sum S_i^* S_i - \mathbb{1}$ is compact if and only if $p_{ij} = 1$ for all $i < j$.

Proof. For any multi-index $\underline{k}$,

$$S_i^* S_i \underline{z}^{\underline{k}} = q_{1i}^{k_1} \cdots q_{(i-1)i}^{k_{i-1}} S_i^* \underline{z}^{\underline{k}+\underline{e}_i} = p_{1i}^{k_1} \cdots p_{(i-1)i}^{k_{i-1}} \frac{\left\| \underline{z}^{\underline{k}+\underline{e}_i} \right\|^2}{\left\| \underline{z}^{\underline{k}} \right\|^2} \underline{z}^{\underline{k}}. \tag{3.4}$$

Hence the first part of the lemma. Now if $p_{ij} = 1$ for all $i < j$, then $w_1, \ldots, w_d$ are commutative. It then follows from the multinomial expansion of $(w_1 + \cdots + w_d)^{|\underline{k}|}$ that

$$\frac{1}{\left\| \underline{z}^{\underline{k}} \right\|^2} = \frac{|\underline{k}|!}{k_1! \cdots k_d!} \quad \text{and} \quad \frac{1}{\left\| \underline{z}^{\underline{k}+e_i} \right\|^2} = \frac{(|\underline{k}|+1)!}{k_1! \cdots k_{i-1}! (k_i+1)! k_{i+1}! \cdots k_d!}.$$

Now on simplification, it can easily be seen from (3.4) that $\left(\sum S_i^* S_i - \mathbb{1} \right) \underline{z}^{\underline{k}} = (d-1)/(|\underline{k}|+1) \underline{z}^{\underline{k}}$ and hence compactness is clear. It remains to see that for each value of $p_{ij} \neq 1$ for some $i < j$, there is a subsequence of $\underline{k}$ along which the quantity

$$\sum_{i=1}^{d} \frac{\left\| \underline{z}^{\underline{k}+\underline{e}_i} \right\|^2}{\left\| \underline{z}^{\underline{k}} \right\|^2} p_{1i}^{k_1} \cdots p_{(i-1)i}^{k_{i-1}} - 1$$

does not go to 0. If $p_{i_0 j} < 1$ for some $i_0 < j$, take the subsequence $\{(0, \ldots, 0, n, 0, \ldots, 0) : n = 1, 2, \ldots\}$ where n is at the jth. place. Then

$$\underline{z}^{\underline{k}+\underline{e}_i} = \begin{cases} z_i z_j^n & \text{if } i < j, \\ z_j^{n+1} & \text{if } i = j, \\ z_j^n z_i & \text{if } i > j. \end{cases}$$

It is easy to see that $1/\|z_i z_j^n\|^2 = 1 + p_{ij} + \cdots + p_{ij}^n$ for $i < j$, $1/\|z_j^{n+1}\|^2 = 1$ and $1/\|z_j^n z_i\|^2 = 1 + p_{ji} + \cdots + p_{ji}^n$ for $i > j$. Moreover, with this choice of $\underline{k}$, we have $1/\|\underline{z}^{\underline{k}}\|^2 = 1$. So

$$\sum_{i=1}^{d} \frac{\left\|\underline{z}^{\underline{k}+\underline{e}_i}\right\|^2}{\left\|\underline{z}^{\underline{k}}\right\|^2} p_{1i}^{k_1} \cdots p_{(i-1)i}^{k_{i-1}} \geqslant \sum_{i=i_0,j} \frac{\left\|\underline{z}^{\underline{k}+e_i}\right\|^2}{\left\|\underline{z}^{\underline{k}}\right\|^2} p_{1i}^{k_1} \cdots p_{(i-1)i}^{k_{i-1}} = \frac{1}{1 + p_{i_0 j} + \cdots + p_{i_0 j}^n} + 1.$$

Now note that since $p_{i_0 j} < 1$, we have $1 + p_{i_0 j} + \cdots + p_{i_0 j}^n \to (1 - p_{i_0 j})^{-1}$ as $n \to \infty$. So this particular subsequence suffices to show that $\sum S_i^* S_i - 1$ is not compact. For $p_{i_0 j} > 1$ for some $i_0 < j$, take the subsequence $\{(0, 0, \ldots, 0, n, 0, \ldots, 0) : n = 1, 2, \ldots\}$ where now n is in the i_0 place. We observe that

$$\sum_{i=1}^{d} \frac{\left\|\underline{z}^{\underline{k}+\underline{e}_i}\right\|^2}{\left\|\underline{z}^{\underline{k}}\right\|^2} p_{1i}^{k_1} \cdots p_{(i-1)i}^{k_{i-1}} \geqslant \sum_{i=i_0,j} \frac{\left\|\underline{z}^{\underline{k}+\underline{e}_i}\right\|^2}{\left\|\underline{z}^{\underline{k}}\right\|^2} p_{1i}^{k_1} \cdots p_{(i-1)i}^{k_{i-1}} = 1 + \frac{p_{i_0 j}^n}{1 + p_{i_0 j} + \cdots + p_{i_0 j}^n}$$

and the proof can be completed as before. ∎

The next result shows how the commutators $[S_i^*, S_i]$ act. The vector $S_i^* S_i \underline{z}^{\underline{k}}$ is never 0 for any multi-index $\underline{k}$. However, $S_i S_i^* \underline{z}^{\underline{k}}$ is 0 whenever $k_i = 0$.

Lemma 3.7. *The commutator of S_i^* and S_i is as follows:*

$$[S_i^*, S_i]\underline{z}^{\underline{k}} = p_{1i}^{k_1} \cdots p_{(i-1)i}^{k_{i-1}} \left(\frac{\left\|\underline{z}^{\underline{k}+\underline{e}_i}\right\|^2}{\left\|\underline{z}^{\underline{k}}\right\|^2} - \frac{\left\|\underline{z}^{\underline{k}}\right\|^2}{\left\|\underline{z}^{\underline{k}-\underline{e}_i}\right\|^2} \right) \underline{z}^{\underline{k}}, \qquad \text{when } k_i \neq 0.$$

If $k_i = 0$, then $[S_i^*, S_i]\underline{z}^{\underline{k}} = S_i^* S_i \underline{z}^{\underline{k}} = p_{1i}^{k_1} \cdots p_{(i-1)i}^{k_{i-1}} \dfrac{\left\|\underline{z}^{\underline{k}+\underline{e}_i}\right\|^2}{\left\|\underline{z}^{\underline{k}}\right\|^2} \underline{z}^{\underline{k}}$.

Proof. This lemma is straightforward from (3.3) and (3.4) above. ∎

We shall leave at that the computations and reap an interesting corollary. For the first time the inherent asymmetry in the definition of the shift becomes apparent.

Corollary 3.8. *For $1 \leqslant i \leqslant d$, if the commutator $[S_i^*, S_i]$ is compact then $p_{ji} \leqslant 1$ for $1 \leqslant j < i$ and $p_{ij} \geqslant 1$ for $i < j \leqslant d$. All the commutators $[S_i^*, S_i]$ are compact if and only if $p_{ij} = 1$ for all $1 \leqslant i < j \leqslant d$.*

Proof. The first claim follows from computations as in the proof of Lemma 3.6 by considering subsequence $n\underline{e}_j$. This also gives us the "only if" part of the second claim. Now if $p_{ij} = 1$ for all $i < j$, the formula for the commutator tells us that $[S_i^*, S_i]\underline{z}^{\underline{k}} = (|\underline{k}| - k_i)/((|\underline{k}| + 1)|\underline{k}|)\underline{z}^{\underline{k}}$, and these eigenvalues certainly converge to zero as $|\underline{k}| \to \infty$. So then each $[S_i^*, S_i]$ is compact. ∎

It is indeed possible that only some $[S_i^*, S_i]$ are compact, for example if $q_{ij} \equiv q$, with $|q| > 1$, then $[S_1^*, S_1]$ is compact and the rest are not. A similar result holds for $|q| < 1$, when $[S_d^*, S_d]$ is compact and the others are not.

For any complex number z, the z-commutator of two operators A, B is defined as:

$$[A, B]_z = AB - zBA.$$

LEMMA 3.9. *If $p_{ij} \equiv 1$, then $[S_i^*, S_j]_{q_{ij}}$ is compact for all $1 \leqslant i < j \leqslant d$.*

Proof. We observe that when $p_{ij} \equiv 1$, for any multi-index $\underline{k}$, $\left\|\underline{z}^{\underline{k}}\right\|^2$ is equal to

$$\frac{k_1! k_2! \cdots k_d!}{|k|!}$$

(reciprocal of the multinomial coefficient). Direct computation yields

$$[S_i^*, S_j]_{q_{ij}} \underline{z}^{\underline{k}} = q_{ij}\left(\frac{|k|^2}{(|k|+1)|k|} - 1\right) S_j S_i^* \underline{z}^{\underline{k}}$$

and then it is easy to complete the proof. ∎

The main use of this corollary is for $p_{ij} \equiv 1$, as for $i > j$, $[S_i^*, S_j]_{\overline{q}_{ji}}$ is just the adjoint of $[S_j^*, S_i]_{q_{ji}}$, every $S_i^* S_j$ can be written as a linear combination of $S_j S_i^*$ and a compact operator for all i and j. This has some interesting consequences. See Remark 4.6. We shall end this section with a description of basic properties of $\underline{S}$ in the special case when $q_{ij} \equiv 0$.

LEMMA 3.10. *If the variables $z_1, z_2, \ldots, z_d$ are such that*

$$z_j z_i = 0 \quad \text{for } i < j,$$

(i.e., $q_{ij} \equiv 0$), then given a multi-index $\underline{k}$,

$$S_i \underline{z}^{\underline{k}} = \begin{cases} \underline{z}^{\underline{k}+\underline{e}_i} & \text{if } k_1 + \cdots + k_{i-1} = 0, \\ 0 & \text{otherwise;} \end{cases}$$

$$S_i^* \underline{z}^{\underline{k}} = \begin{cases} \underline{z}^{\underline{k}-\underline{e}_i} & \text{if } k_1 + \cdots + k_{i-1} = 0 \text{ and } k_i \neq 0, \\ 0 & \text{otherwise.} \end{cases}$$

The operators S_i are bounded and they satisfy $S_j S_i = 0$ for all $i < j$. Moreover, $S_i^* S_i$ is the projection onto the subspace spanned by $\{\underline{z}^{\underline{k}} : k_1 + \cdots + k_{i-1} = 0\}$ and $S_i S_i^*$ is the projection onto the subspace spanned by $\{\underline{z}^{\underline{k}} : k_1 + \cdots + k_{i-1} = 0$ and $k_i \neq 0\}$. Consequently,

$$S_i^* S_i = \mathbb{1} - \sum_{j=1}^{i-1} S_j S_j^* \quad \text{for } 1 \leqslant i \leqslant d.$$

For $i \neq j$, $S_i^* S_j = 0$. Thus all the operators $S_i^* S_j$ can be written as linear combinations of $\mathbb{1}$ and $S_r S_r^*$, $r = 1, \ldots, d$.

Proof. Here monomials $\{\underline{z}^{\underline{k}}\}$ are orthonormal in $\mathcal{H}$. If the z_i's satisfy the assumption of the lemma, then $S_1 \underline{z}^{\underline{k}} = \underline{z}^{\underline{k}+\underline{e}_1}$ but for any $i > 1$, $S_i \underline{z}^{\underline{k}}$ is clearly 0 unless $k_1 + \cdots + k_{i-1} = 0$ holds in which case $S_i \underline{z}^{\underline{k}} = \underline{z}^{\underline{k}+\underline{e}_i}$. The boundedness of S_i and the q-commutativity are now clear.

Note that $S_i^* \underline{z}^{\underline{k}}$ is 0 if $k_i = 0$. If $k_i \neq 0$, then it is a multiple of $\underline{z}^{\underline{k}-\underline{e}_i}$. The multiplying factor is $\langle \underline{z}^{\underline{k}}, S_i \underline{z}^{\underline{k}-\underline{e}_i} \rangle / \left\|\underline{z}^{\underline{k}-\underline{e}_i}\right\|$. This ratio is 1 or 0 depending on whether $k_1 + \cdots + k_{i-1}$ is equal to zero or not. Thus we have the stated formula for S_i^*.

It is now clear that $S_i S_i^*$ is the projection onto the subspace spanned by $\{\underline{z}^{\underline{k}} : k_1 + \cdots + k_{i-1} = 0$ and $k_i \neq 0\}$. Thus $S_1 S_1^*, \ldots, S_{i-1} S_{i-1}^*$ is a family of orthogonal projections for any $1 < i \leqslant d$. Hence their sum will be the projection onto the direct sum of the ranges which is the subspace spanned by all $\underline{z}^{\underline{k}}$ with at least one of k_j being non-zero for $1 \leqslant j \leqslant i-1$ or in other words $k_1+\cdots+k_{i-1} \neq 0$.

On the other hand, $S_i^* S_i \underline{z}^{\underline{k}}$ is $\underline{z}^{\underline{k}}$ or 0 depending on whether $k_1+\cdots+k_{i-1} = 0$ or not. Thus this is the projection which is the orthogonal complement of $S_1S_1^* + \cdots + S_{i-1}S_{i-1}^*$. So we have

$$S_i^* S_i = \mathbb{1} - \sum_{j=1}^{i-1} S_j S_j^*.$$

Also note that $S_1S_1^* + \cdots + S_dS_d^*$ is the projection onto the subspace spanned by all $\underline{z}^{\underline{k}}$ such that $k_1+\cdots+k_d \neq 0$ and hence as before $\mathbb{1}-(S_1S_1^*+\cdots+S_dS_d^*) = E_0$.

That $S_i^* S_j$ is 0 for $i \neq j$ is again a straightforward computation on the monomials. ∎

4. MODEL THEORY: VON NEUMANN'S INEQUALITY AND DILATION

By an operator space we shall mean a vector subspace of $\mathcal{B}(\mathcal{L})$ where $\mathcal{L}$ is a Hilbert space. Given an operator space $\mathcal{E}$ and an algebra $\mathcal{A} \subseteq \mathcal{E}$, a completely positive map φ from $\mathcal{E}$ to $\mathcal{B}(\mathcal{K})$ for some Hilbert space $\mathcal{K}$ is called an $\mathcal{A}$-*morphism* if

$$\varphi(AX) = \varphi(A)\varphi(X), \qquad \text{for any } A \in \mathcal{A} \text{ and } X, AX \in \mathcal{E}.$$

The C^*-subalgebra of $\mathcal{B}(\mathcal{H})$ generated by $S_1, \ldots, S_d$ and $\mathbb{1}$ will be denoted by $\mathcal{T}_d^q$. If all the q_{ij} are same as q for some fixed complex number q, then we do not have to, a priori, include $\mathbb{1}$ in the C^*-algebra $\mathcal{T}_d^q$ because then it is easy to see that the operator $\sum S_i^* S_i$ is invertible in $\mathcal{B}(\mathcal{H})$. Since C^*-algebras are inverse closed, $\left(\sum S_i^* S_i\right)^{-1}$ is in the C^*-algebra generated by $S_1, \ldots, S_d$ and hence $\mathcal{T}_d^q$ is unital. It is not clear to us whether this is the situation in the general case. The subalgebra of $\mathcal{T}_d^q$ consisting of polynomials in $S_1, \ldots, S_d$ and $\mathbb{1}$ will be denoted by $\mathcal{A}$. The operator space $\overline{\operatorname{span}}\, \mathcal{A}\mathcal{A}^*$ will be denoted by $\mathcal{E}$. This is a subspace of $\mathcal{T}_d^q$. Of course, $\mathcal{A}$ and $\mathcal{E}$ also depend upon q_{ij}'s and d, but we suppress it in our notation.

Lemma 4.1. *All compact operators are in $\mathcal{E}$.*

Proof. We see from Lemma 3.5 that the one-dimensional projection E_0 onto the space of constant functions is in $\operatorname{span} \mathcal{A}\mathcal{A}^*$. Now given any two polynomials f and g, the operator $f(\underline{S})E_0(g(\underline{S}))^*$ is in $\operatorname{span} \mathcal{A}\mathcal{A}^*$. But note that this operator is nothing but the rank-one operator

$$\xi \to \langle \xi, g \rangle f.$$

As polynomials are dense in $\mathcal{H}$, all compact operators are in $\mathcal{E}$. ∎

Note that given any unital $\mathcal{A}$-morphism φ from $\mathcal{E}$ to $\mathcal{B}(\mathcal{K})$, where $\mathcal{K}$ is another Hilbert space, the tuple $(\varphi(S_1), \ldots, \varphi(S_d))$ is a q-commuting contractive tuple on $\mathcal{K}$. We also have a converse of this statement.

THEOREM 4.2. *If* $\underline{T} = (T_1, \dots, T_d)$ *is a* q*-commuting contractive tuple acting on a Hilbert space* $\mathcal{K}$*, then there is a unique unital* $\mathcal{A}$*-morphism* $\varphi : \mathcal{E} \to \mathcal{B}(\mathcal{K})$ *such that* $\varphi(S_j) = T_j$.

Proof. This result follows by standard methods. We may follow the approach of Arias and Popescu ([2], Example 3.3). This just involves observing that for q-commuting tuples the Poisson transform of Popescu lands in a subspace naturally isomorphic to $\mathcal{H} \otimes \mathcal{K}$, so that we can apply a quotienting procedure to the minimal isometric dilation in the full Fock space (see Section 8 of [15]). An alternative approach is to follow Arveson's methods ([5]) verbatim. (Actually the two methods are essentially same.) ∎

THEOREM 4.3. *If* $\underline{T} = (T_1, \dots, T_d)$ *is a* q*-commuting contractive tuple acting on a Hilbert space* $\mathcal{K}$*, then there is a unital* $\mathcal{A}$*-morphism* $\varphi : \mathcal{T}_d^q \to \mathcal{B}(\mathcal{K})$ *such that* $\varphi(S_j) = T_j$.

Proof. The existence of a completely positive map $\varphi : \mathcal{T}_d^q \to \mathcal{B}(\mathcal{K})$ is got by applying Arveson's extension theorem ([4]) to the φ obtained in Theorem 4.2. Of course, the extended map may not be unique. To see that any such extension is an $\mathcal{A}$-morphism, we consider a Stinespring dilation of φ. Thus we get a Hilbert space $\widehat{\mathcal{K}}$ containing $\mathcal{K}$ and a representation π of $\mathcal{T}_d^q$ on $\widehat{\mathcal{K}}$ such that

$$\varphi(X) = P_{\mathcal{K}} \pi(X) P_{\mathcal{K}} \quad \text{for } X \in \mathcal{T}_d^q,$$

where $P_{\mathcal{K}}$ is the projection onto $\mathcal{K}$. (We are identifying any operator $Z \in \mathcal{B}(\mathcal{K})$ with $P_{\mathcal{K}} Z P_{\mathcal{K}} \in \mathcal{B}(\widehat{\mathcal{K}})$.) Now the $\mathcal{A}$-morphism property follows in the following way:

$$\begin{aligned}
\varphi(S_i)\varphi(S_i^*) &= \varphi(S_i S_i^*) = P_{\mathcal{K}} \pi(S_i S_i^*) P_{\mathcal{K}} = P_{\mathcal{K}} \pi(S_i) \pi(S_i^*) P_{\mathcal{K}} \\
&= P_{\mathcal{K}} \pi(S_i)(P_{\mathcal{K}} + P_{\mathcal{K}}^{\perp})(P_{\mathcal{K}} + P_{\mathcal{K}}^{\perp}) \pi(S_i^*) P_{\mathcal{K}} \\
&= (P_{\mathcal{K}} \pi(S_i) P_{\mathcal{K}} + P_{\mathcal{K}} \pi(S_i) P_{\mathcal{K}}^{\perp})(P_{\mathcal{K}} \pi(S_i^*) P_{\mathcal{K}} + P_{\mathcal{K}}^{\perp} \pi(S_i^*) P_{\mathcal{K}}) \\
&= \varphi(S_i)\varphi(S_i^*) + (P_{\mathcal{K}} \pi(S_i) P_{\mathcal{K}}^{\perp})(P_{\mathcal{K}} \pi(S_i) P_{\mathcal{K}}^{\perp})^*.
\end{aligned}$$

Thus

$$P_{\mathcal{K}} \pi(S_i) P_{\mathcal{K}}^{\perp} = 0. \tag{4.1}$$

Let $f(z_1, \dots, z_d)$ be any given polynomial. Then it follows immediately from (4.1) that

$$P_{\mathcal{K}} \pi(f(\underline{S})) P_{\mathcal{K}}^{\perp} = 0.$$

Hence for any $X \in \mathcal{T}_d^q$,

$$\begin{aligned}
\varphi(f(\underline{S})X) &= P_{\mathcal{K}} \pi(f(\underline{S})X) P_{\mathcal{K}} = P_{\mathcal{K}} \pi(f(\underline{S})) \pi(X) P_{\mathcal{K}} \\
&= P_{\mathcal{K}} \pi(f(\underline{S}))(P_{\mathcal{K}} + P_{\mathcal{K}}^{\perp})(P_{\mathcal{K}} + P_{\mathcal{K}}^{\perp}) \pi(X) P_{\mathcal{K}} \\
&= (P_{\mathcal{K}} \pi(f(\underline{S})) P_{\mathcal{K}} + P_{\mathcal{K}} \pi(f(\underline{S})) P_{\mathcal{K}}^{\perp})(P_{\mathcal{K}} \pi(X) P_{\mathcal{K}} + P_{\mathcal{K}}^{\perp} \pi(X) P_{\mathcal{K}} \\
&= \varphi(f(\underline{S}))\varphi(X).
\end{aligned}$$

Thus φ has the $\mathcal{A}$-morphism property. ∎

COROLLARY 4.4. (von Neumann's inequality) *Let* $\underline{T} = (T_1, \ldots, T_d)$ *be any q-commuting contractive tuple acting on a Hilbert space* $\mathcal{K}$ *and* $\underline{S} = (S_1, \ldots, S_d)$ *be the q-commuting shift. Then for any polynomial f in d-variables,*

$$\|f(T_1, \ldots, T_d)\| \leqslant \|f(S_1, \ldots, S_d)\|.$$

Proof. Making use of the unital completely positive map φ of last theorem which maps $f(S_1, \ldots, S_d)$ to $f(T_1, \ldots, T_d)$, we have

$$\|f(T_1, \ldots T_d)\| = \|\varphi(f(S_1, \ldots, S_d))\| \leqslant \|\varphi\| \, \|f(S_1, \ldots, S_d)\| = \|f(S_1, \ldots, S_d)\|. \quad \blacksquare$$

The above theorems lead us to the following dilation theorem for any q-commuting contractive tuple $\underline{T}$ acting on some Hilbert space $\mathcal{K}$. We need some notation. If n is a positive integer or ∞ and $\mathcal{M}$ is a Hilbert space of dimension n, we shall mean by $n \cdot \underline{S}$, the operator tuple $(S_1 \otimes \mathbb{1}_{\mathcal{M}}, \ldots, S_d \otimes \mathbb{1}_{\mathcal{M}})$ acting on $\mathcal{H} \otimes \mathcal{M}$. In the next theorem, we are going to express $\underline{T}$ as a compression of a direct sum one of whose components might be absent. To assimilate this in a single notation, we make the convention that $n \cdot \underline{S}$ is absent if $n = 0$. Given a Hilbert space $\mathcal{N}$, and a representation β of $\mathcal{T}_d^q$ on $\mathcal{N}$, the operator tuple

$$\underline{A} \stackrel{\text{def}}{=} n \cdot \underline{S} \oplus \beta(\underline{S})$$

is clearly a q-commuting contractive tuple on $\widehat{\mathcal{K}} \stackrel{\text{def}}{=} (\mathcal{H} \otimes \mathcal{M}) \oplus \mathcal{N}$. Let $\mathcal{K}$ be a subspace of $\widehat{\mathcal{K}}$ such that $A_i^* \mathcal{K} \subseteq \mathcal{K}$ for all $i = 1, \ldots, d$. Such subspaces are called *co-invariant* with respect to the tuple $\underline{A}$. Consider the compression $\underline{T}$ of $\underline{A}$ to $\mathcal{K}$ as follows:

$$T_i \stackrel{\text{def}}{=} P_{\mathcal{K}} A_i | \mathcal{K}.$$

This $\underline{T}$ is clearly a q-commuting contractive tuple on $\mathcal{K}$ and moreover, for any polynomial $f(z_1, \ldots, z_d)$, $f(\underline{T})$ is the compression of $f(\underline{A})$ due to the co-invariance of $\mathcal{K}$ with respect to $\underline{A}$. We prove that every q-commuting contractive tuple has such a realization with β sending all compact operators to zero.

THEOREM 4.5. (Dilation) *Let* $\underline{T}$ *be any q-commuting contractive tuple acting on a separable Hilbert space* $\mathcal{K}$ *and* $\operatorname{rank} D_{\underline{T}} = n$ *(which is a non-negative integer or* ∞*). Then there is a separable Hilbert space* $\mathcal{M}$ *of dimension n, another separable Hilbert space* $\mathcal{N}$ *with a q-commuting tuple of operators* $\underline{Z} = (Z_1, \ldots, Z_d)$ *acting on it, satisfying* $Z_1 Z_1^* + \cdots + Z_d Z_d^* = \mathbb{1}$ *such that:*

(i) $\mathcal{K}$ *is contained in* $\widehat{\mathcal{K}} \stackrel{\text{def}}{=} (\mathcal{H} \otimes \mathcal{M}) \oplus \mathcal{N}$ *as a subspace and it is co-invariant under* $\underline{A} \stackrel{\text{def}}{=} n \cdot \underline{S} \oplus \underline{Z}$.

(ii) $\underline{T}$ *is the compression of* $\underline{A}$ *to* $\mathcal{K}$*, that is,* $\underline{T} = P_{\mathcal{K}} \underline{A} | \mathcal{K}$.

Proof. Let $\varphi, \widehat{\mathcal{K}}$ and π be as in Theorem 4.3 and its proof. Note that we may and do assume π to be a *minimal* Stinespring dilation. So

$$\widehat{\mathcal{K}} = \overline{\operatorname{span}}\{\pi(X)u : X \in \mathcal{T}_d^q \text{ and } u \in \mathcal{K}\}.$$

The C^*-algebra $\mathcal{T}_d^q$ is separable and hence the Hilbert space $\widehat{\mathcal{K}}$ is also separable. The tuple $(\pi(S_1), \ldots, \pi(S_d))$ is a dilation of $(T_1, \ldots, T_d)$ in the sense that for any polynomial f (in d non-commuting variables),

$$f(T_1, \ldots, T_d) = P_{\mathcal{K}} f(\pi(S_1), \ldots, \pi(S_d)) | \mathcal{K},$$

and $\mathcal{K}$ is a co-invariant subspace for $(\pi(S_1), \ldots, \pi(S_d))$ in view of (4.1).

Let us denote the set of all compact operators on $\mathcal{H}$ by $\mathcal{B}_0(\mathcal{H})$ (or just $\mathcal{B}_0$ when there is no chance of confusion). Since $\mathcal{T}_d^q$ contains $\mathcal{B}_0$, by standard theory of representations of C^*-algebras (see [10], Chapter I for example), the representation π decomposes as $\pi = \pi_0 \oplus \pi_1$, where $\pi_i : \mathcal{T}_d^q \to \mathcal{B}(\widehat{\mathcal{K}}_i)$ with π_0 being a non-degenerate representation of $\mathcal{B}_0$ on $\widehat{\mathcal{K}}_0$, π_1 being 0 on $\mathcal{B}_0$ and $\widehat{\mathcal{K}} = \widehat{\mathcal{K}}_0 \oplus \widehat{\mathcal{K}}_1$ (one of π_0 and π_1 could be absent too). Since the only non-degenerate representation of the C^*-algebra of compact operators is the identity representation with some multiplicity and since a represenation which is non-degenerate on an ideal, extends uniquely to the entire C^*-algebra, it follows that π_0 is just the identity representation with some multiplicity i.e., upto unitary isomorphism, $\widehat{\mathcal{K}}_0 = \mathcal{H} \otimes \mathcal{M}$ and $\pi_0(X) = X \otimes \mathbb{1}_{\mathcal{M}}$ for some Hilbert space $\mathcal{M}$. So if we take $\mathcal{N} = \widehat{\mathcal{K}}_1$, and $\pi_1(S_i) = Z_i$ then $(Z_1, \ldots, Z_d)$ is a q-commuting contractive tuple and (i), (ii) are satisfied. Moreover $\sum Z_i Z_i^* = \mathbb{1}$ as π_1 kills compact operators and $\mathbb{1} - \sum S_i S_i^*$ is compact.

It remains to prove that the multiplicity i.e., $\dim(\mathcal{M})$ is just the rank of $D_{\underline{T}}$. For this, note that $\dim \mathcal{M} = \dim(\text{ range } \pi_0(E))$ where E is any one-dimensional projection in $\mathcal{T}_d^q$. Taking $E = E_0$, the projection onto the constant functions, and making use of minimality of Stinespring representation, we have

$$\operatorname{range} \pi(E_0) = \{\pi(E_0)\xi : \xi \in \widehat{\mathcal{K}}\} = \overline{\operatorname{span}}\{\pi(E_0)\pi(X)u : X \in \mathcal{T}_d^q, u \in \mathcal{K}\}.$$

Then by Lemma 4.1 and its proof,

$$\begin{aligned}
\operatorname{range} \pi(E_0) &= \overline{\operatorname{span}}\{\pi(E_0)\pi(E_0 X)u : X \in \mathcal{T}_d^q, u \in \mathcal{K}\} \\
&= \overline{\operatorname{span}}\{\pi(E_0)\pi(X)u : X \in \mathcal{B}_0, u \in \mathcal{K}\} \\
&= \overline{\operatorname{span}}\{\pi(E_0)\pi(\underline{S}^{\underline{k}} E_0 (\underline{S}^{\underline{l}})^*)u : \text{all multi-indices } \underline{k}, \underline{l}, \text{ and } u \in \mathcal{K}\} \\
&= \overline{\operatorname{span}}\{\pi(E_0)\pi((\underline{S}^{\underline{l}})^*)u : \text{all multi-indices } \underline{k}, \underline{l}, \text{ and } u \in \mathcal{K}\}.
\end{aligned}$$

Now we define a unitary $U : \text{range } \pi(E_0) \to \overline{\operatorname{range}}\, D_{\underline{T}}$ by setting

$$U\pi(E_0)\pi((\underline{S}^{\underline{l}})^*)u = D_{\underline{T}}(\underline{T}^{\underline{l}})^* u$$

and extending linearly. Then U is isometric because for $u, v \in \mathcal{K}$ and all $\underline{k}$ and $\underline{l}$,

$$\begin{aligned}
\langle \pi(E_0)\pi((\underline{S}^{\underline{k}})^*)u, \pi(E_0)\pi((\underline{S}^{\underline{l}})^*)v\rangle &= \langle u, \pi(\underline{S}^{\underline{k}})\pi(E_0)\pi((\underline{S}^{\underline{l}})^*)v\rangle \\
&= \langle u, \underline{T}^{\underline{k}} D_{\underline{T}}^2 (\underline{T}^{\underline{l}})^* v\rangle = \langle D_{\underline{T}}(\underline{T}^{\underline{k}})^* u, D_{\underline{T}}(\underline{T}^{\underline{l}})^* v\rangle.
\end{aligned}$$

Taking $\underline{l} = 0$, it is clear that U is onto. This proves that $\overline{\operatorname{range}} D_{\underline{T}}$ and $\mathcal{M}$ have the same dimensions. ∎

As remarked before in the direct sum for $\widehat{\mathcal{K}}$ and $\underline{A}$ appearing in this theorem one of the summands could be absent; $\mathcal{M}$ and $n \cdot \underline{S}$ are absent iff $n = 0$, that is, iff $\sum T_i T_i^* = \mathbb{1}$. Just as in [5], it can be shown that $\mathcal{N}$ and $\underline{Z}$ is absent if and only if $P_{\underline{T}}^m(\mathbb{1}_{\mathcal{K}})$ converges to zero strongly as m tends to infinity where $P_{\underline{T}}$ is the completely positive map associated with $\underline{T}$ in Section 2. Arveson's computation of multiplicity n used the fact that if $q_{ij} \equiv 1$, then $\overline{\operatorname{span}}\, \mathcal{A}\mathcal{A}^*$ is a C^*-algebra which may not be the case in general, and so the proof given here had to be different.

We have a couple of remarks to make about some special values of the complex numbers q_{ij}. The first one of them is about what happens when they lie on the unit circle, i.e., $p_{ij} \equiv 1$.

Remark 4.6. In the case when $p_{ij} \equiv 1$, the C^*-algebra $\mathcal{T}_d^q$ is the same as the operator space $\mathcal{E} = \overline{\text{span}}\mathcal{A}\mathcal{A}^*$. This follows from Corollary 3.8, and Lemma 3.9 as by virtue of them, commutators $[S_i^*, S_i]$, $[S_i^*, S_j]_q$ with $i < j$, and $[S_i^*, S_j]_{\overline{q}}$ with $i > j$, are compact, and all compact operators are in $\mathcal{E}$ (Lemma 4.1).

Because of the above fact, Arveson's extension theorem is not needed in proving Theorem 4.3 and there is unique unital $\mathcal{A}$-morphism φ from $\mathcal{T}_d^q$ to $\mathcal{B}(\mathcal{K})$ carrying S_j to T_j. Moreover, as each S_i is essentially normal, in Theorem 4.5 the operators $Z_1, \ldots, Z_d$ are now normal.

Definition 4.7. A q-commuting operator tuple $(Z_1, \ldots, Z_d)$ is said to be a *q-spherical unitary* if each Z_i is normal, and $\sum Z_i Z_i^* = \mathbb{1}$. It is said to be a *spherical unitary* if each $q_{ij} = 1$.

The following is a generalization, from commutative to the q-commutative case, of a result of Athavale ([6], Proposition 2).

Theorem 4.8. *Suppose $(T_1, \ldots, T_d)$ is a q-commuting d-tuple of operators on a Hilbert space $\mathcal{K}$, with $|q_{ij}| = 1$ for all $i < j$, and $\sum T_i T_i^* = \mathbb{1}$. Then there exists a q-spherical unitary $(Z_1, \ldots, Z_d)$ acting on a Hilbert space $\widehat{\mathcal{K}}$ containing $\mathcal{K}$ as a subspace such that $\sum Z_i Z_i^* = \mathbb{1}$, and each Z_i^* is an extension of T_i^*, that is, $\mathcal{K}$ is co-invariant for Z_i and $T_i = P_{\mathcal{K}} Z_i|\mathcal{K}$. In particular each T_i^* is sub-normal.*

Proof. Immediate from Theorem 4.5 and Remark 4.6. ∎

At this stage perhaps it is worthwhile to study q-spherical unitaries. Of course, if all $q_{ij} = 1$ their structure is quite transparent due to Gelfand theory of commutative C^*-algebras.

Theorem 4.9. *Suppose $(Z_1, \ldots, Z_d)$ is a q-spherical unitary acting on a Hilbert space $\mathcal{K}$.*

(i) *If $|q_{kl}| \neq 1$ for some $k < l$, then $Z_k Z_l = Z_l Z_k = Z_k^* Z_l = Z_l Z_k^* = 0$. If $|q_{ij}| \neq 1$ for all $1 \leqslant i < j \leqslant d$, then $Z_i^* Z_i$ are projections orthogonal to each other such that $\sum Z_i^* Z_i = \mathbb{1}$.*

(ii) *If $|q_{ij}| = 1$ for all $i < j$, let $Z_i = U_i P_i$ be the unique polar decomposition of Z_i, such that U_i is a partial isometry, P_i is a positive operator and* $\ker U_i = \ker P_i = \ker Z_i$, *for $1 \leqslant i \leqslant d$. Then $(P_1, \ldots, P_d)$ is a (commuting) spherical unitary and $(U_1, \ldots, U_d)$ is a q-commuting tuple.*

Proof. We make repeated use of Fuglede-Putnam Theorem ([12], [17]). Recall that this theorem states that if M, N, B are bounded operators on a Hilbert space satisfying $MB = BN$, and if M, N are normal then $M^* B = BN^*$.

Consider $1 \leqslant i < j \leqslant d$. From q-commutativity we have

$$Z_j Z_i = q_{ij} Z_i Z_j. \tag{4.2}$$

Taking $M = Z_j$, $N = q_{ij} Z_j$, $B = Z_i$ in Fuglede-Putnam Theorem,

$$Z_j^* Z_i = \overline{q}_{ij} Z_i Z_j^*. \tag{4.3}$$

In this equation taking $M = \overline{q}_{ij} Z_i$, $N = Z_i$, $B = Z_j^*$ and once again applying Fuglede-Putnam Theorem we have

$$q_{ij} Z_i^* Z_j^* = Z_j^* Z_i^*.$$

Taking adjoints, $Z_i Z_j = \overline{q}_{ij} Z_j Z_i = |q_{ij}|^2 Z_i Z_j$.

So if $|q_{kl}| \neq 1$, $Z_k Z_l = 0$ and consequently $Z_l Z_k = 0$. Once again by Fuglede-Putnam Theorem $Z_k^* Z_l = Z_l Z_k^* = 0$. If $|q_{ij}| \neq 1$ for all i, j then $Z_k^* Z_k$ and $Z_l^* Z_l$ are commuting and orthogonal for $k \neq l$. The condition that $\sum Z_i^* Z_i = \mathbb{1}$, clearly forces them to be projections.

Here after assume all $|q_{ij}| = 1$, and $Z_i = U_i P_i$ is the polar decompostion of Z_i as in the hypothesis of the theorem. From (4.2) and (4.3), $(Z_j^* Z_j) Z_i = q_{ij} Z_j^* Z_i Z_j = |q_{ij}|^2 Z_i Z_j^* Z_j = Z_i (Z_j^* Z_j)$. Hence Z_i and $Z_j^* Z_j$ commute. As $P_j = (Z_j^* Z_j)^{\frac{1}{2}}$, Z_i and P_j commute and also Z_i^* and P_j commute. If follows that $(P_1, \dots, P_d)$ is a commuting tuple. Of course, $\sum P_i P_i^* = \sum P_i^2 = \sum Z_i^* Z_i = \mathbb{1}$.

Finally to prove that $(U_1, \dots, U_d)$ q-commutes, fix $1 \leqslant i < j \leqslant d$. As P_i and P_j commute, the Hilbert space $\mathcal{K}$ decomposes as $\mathcal{K} = \mathcal{K}_0 \oplus \mathcal{K}_1 \oplus \mathcal{K}_2 \oplus \mathcal{K}_3$, where

$$\begin{aligned}
\mathcal{K}_0 &= (\ker(P_i)) \cap (\ker(P_j)) = (\ker(U_i)) \cap (\ker(U_j)) \\
\mathcal{K}_1 &= (\ker(P_i)) \cap (\ker(P_j))^{\perp} = (\ker(U_i)) \cap (\overline{\text{range}}(P_j)) \\
\mathcal{K}_2 &= (\ker(P_i))^{\perp} \cap (\ker(P_j)) = (\overline{\text{range}}(P_i)) \cap (\ker(U_j)) \\
\mathcal{K}_3 &= (\ker(P_i))^{\perp} \cap (\ker(P_j))^{\perp} = \overline{\text{range}} P_i P_j.
\end{aligned}$$

Now for $x \in \mathcal{K}_0$, clearly $U_i U_j x = U_j U_i x = 0$. For x in $\ker(U_i)) \cap (\text{range}(P_j))$, $U_j U_i x = 0$. Also as $x = P_j y$ for some $y \in \mathcal{K}$,

$$P_i U_j x = P_i U_j P_j y = P_i Z_j y = Z_j P_i y = U_j P_j P_i y = U_j P_i x = 0.$$

But then as $\ker(U_i) = \ker(P_i)$, $U_i U_j x = 0$. By continuity, $U_i U_j x = U_j U_i x = 0$ for $x \in \mathcal{K}_1$. For similar reasons $U_i U_j$ and $U_j U_i$ are zero operators on $\mathcal{K}_3$. However, for x in range $(P_i P_j)$, as $x = P_i P_j y$ for some $y \in \mathcal{K}$,

$$\begin{aligned}
U_j U_i x &= U_j U_i P_i P_j y = U_j Z_i P_j y = U_j P_j Z_i y = Z_j Z_i y \\
&= q_{ij} Z_i Z_j y = q_{ij} U_i P_i U_j P_j y = q_{ij} U_i U_j x.
\end{aligned}$$

Thus $U_j U_i x = q_{ij} U_i U_j x$ for all $x \in \mathcal{K}$, and $1 \leqslant i < j \leqslant d$. ∎

Now we examine the $q_{ij} \equiv 0$ case.

REMARK 4.10. For $q_{ij} \equiv 0$ also, the C^*-algebra $\mathcal{T}_d^q$ is the same as the operator space $\mathcal{E}$. This follows from Lemma 3.10 where it was noticed that all operators $S_i^* S_j$ can be written in terms of $\mathbb{1}$ and the operators $S_r S_r^*$ for $r = 1, \dots, d$. So the $\mathcal{A}$-morphism φ of Theorem 4.3 is uniquely defined.

It is not clear to us whether $\overline{\text{span}}\, \mathcal{A}\mathcal{A}^* = \mathcal{T}_d^q$ holds for any other values of q_{ij} when $|q_{ij}|$ is not identically equal to one or zero for all $i < j$. However, it is clear that whenever that is the case we have uniqueness of φ in Theorem 4.3. Moreover, using the uniqueness of minimal Stinespring representation one can also make a uniqueness up to unitary equivalence statement in dilation theorem namely in Theorem 4.5.

5. THE OPERATOR SPACE SPANNED BY $\underline{S}$

Here we study operator spaces spanned by q-commuting, contractive d-tuples. The operator space generated by standard q-commuting shift $\underline{S}$ will be denoted by $\mathcal{S}_d^q$. So $\mathcal{S}_d^q$ is simply the linear span of $S_1, \ldots, S_d$. Let $\mathcal{S}$ be an operator space acting on $\mathcal{K}$ for some Hilbert space $\mathcal{K}$. Any tuple of operators $T_1, \ldots, T_d$ from $\mathcal{S}$, gives rise to two completely positive maps on $\mathcal{B}(\mathcal{K})$ as follows:

$$P_{\underline{T}}(X) = T_1 X T_1^* + \cdots + T_d X T_d^*, \quad X \in \mathcal{B}(\mathcal{K}),$$

and

$$Q_{\underline{T}}(X) = T_1^* X T_1 + \cdots + T_d^* X T_d, \quad X \in \mathcal{B}(\mathcal{K}).$$

Assume that $\mathcal{S}$ is an operator space spanned by component operators of a q-commuting contractive d-tuple. The *energy sequence* of the operator space $\mathcal{S}$ is defined to be the sequence of numbers $(n \geqslant 1)$:

$$E_n^q(\mathcal{S}) = \sup\{\|Q_{\underline{T}}{}^n\| : \underline{T} \text{ is a } q\text{-commuting contractive } d\text{-tuple from } \mathcal{S} \text{ and } \|P_{\underline{T}}\| \leqslant 1\}. \tag{5.1}$$

In the commutative case the above definition is the same as the one by Arveson in [5]. Arveson does not restrict to d-tuples but as he himself shows, a completely positive map determined by a tuple of arbitrary length can always be re-written in terms of a d-tuple, where d is the dimension of the operator space and this does not change any of the norms involved. Obviously unlike the commutative case linear combinations of q-commuting operators need not q-commute.

Lemma 5.1. *Given a d-dimensional operator space $\mathcal{S}$ and $E_n^q(\mathcal{S})$ as defined in* (5.1),

$$E_n^q(\mathcal{S}) \leqslant \binom{n+d-1}{n}.$$

Proof. The proof is similar to that of Proposition 7.5 in [5]. Suppose we have a q-commuting contractive tuple $\underline{T}$ and its associated maps $P_{\underline{T}}$ and $Q_{\underline{T}}$ with $\|P_{\underline{T}}\| \leqslant 1$. Then we also have $\|P_{\underline{T}}^n(\mathbb{1})\| = \|P_{\underline{T}}^n\| \leqslant 1$. Now note that

$$P_{\underline{T}}^n(\mathbb{1}) = \sum_{|\underline{k}|=n} \frac{1}{\|\underline{z}^{\underline{k}}\|^2} \underline{T}^{\underline{k}} (\underline{T}^{\underline{k}})^*.$$

This implies that

$$\left\| \frac{1}{\left\|\underline{z}^{\underline{k}}\right\|^2} (\underline{T}^{\underline{k}})^* \underline{T}^{\underline{k}} \right\| \leqslant 1 \quad \text{for all } \underline{k}.$$

Now we have

$$\|Q_{\underline{T}}^n\| = \|Q_{\underline{T}}^n(\mathbb{1})\| \leqslant \sum_{|\underline{k}|=n} \left\| \frac{1}{\left\|\underline{z}^{\underline{k}}\right\|^2} (\underline{T}^{\underline{k}})^* \underline{T}^{\underline{k}} \right\| \leqslant \binom{n+d-1}{n}$$

because there are $\binom{n+d-1}{n}$ terms in the sum. ∎

LEMMA 5.2. *Let $\underline{S}$ be the q-commuting shift and let $Q = Q_{\underline{S}}$ be the completely positive map*

$$Q(X) = S_1^* X S_1 + \cdots + S_d^* X S_d.$$

For the positive operator $Q^n(\mathbb{1})$, the monomial $\underline{z}^{\underline{l}}$ is an eigen-vector for any multi-index $\underline{l}$. If $q_{ij} \neq 0$ for all $i < j$, this operator attains its norm only at the constant function 1.

Proof. For any multi-index $\underline{l}$, a short computation reveals,

$$Q^n(\mathbb{1})\underline{z}^{\underline{l}} = \sum_{|\underline{k}|=n} \frac{1}{\|\underline{z}^{\underline{k}}\|^2} (\underline{S}^{\underline{k}})^* \underline{S}^{\underline{k}} \underline{z}^{\underline{l}}.$$

Thus every $\underline{z}^{\underline{l}}$ is an eigen-vector of the operator $Q^n(\mathbb{1})$. Moreover, for $\underline{l} = 0$ the eigen-value is $\binom{n+d-1}{n}$ because each term in the sum is 1 and the sum is over $\binom{n+d-1}{n}$ many terms. It is easy to see from the proof of Lemma 3.3 that for any non-zero multi-index $\underline{l}$, the summands are less than or equal to 1 with strict inequality in some cases. Hence

$$\|Q^n\| = \binom{n+d-1}{n}$$

is attained at the unique vector 1. ∎

We see in the next lemma of this section that $E_n^q(\mathcal{S}_d^q)$ is independent of q.

LEMMA 5.3. *The energy sequence for the operator space $\mathcal{S}_d^q$ is*

$$\binom{n+d-1}{n}.$$

Proof. Let $P = P_{\underline{S}}$ and $Q = Q_{\underline{S}}$ be the completely positive maps determined by the q-commuting shift. Then by Lemma 3.5, $\|P\| = 1$. And in the proof of the last lemma we found that,

$$\|Q^n\| = \binom{n+d-1}{n}.$$

Since we have already proved that the energy sequence can not be bigger than

$$\binom{n+d-1}{n},$$

we are done. ∎

The converse of the above lemma is contained in the following theorem.

THEOREM 5.4. *Assume $q_{ij} \neq 0$ for all $1 \leqslant i < j \leqslant d$. Let $\mathcal{S} \subseteq \mathcal{B}(\mathcal{K})$ be a d-dimensional operator space with $E_n^q(\mathcal{S}) = E_n^q(\mathcal{S}_d^q)$ for all $n \geqslant 1$. Let $\mathcal{C}$ be the C^*-algebra generated by $\mathcal{S}$ and the identity operator on $\mathcal{K}$. Then there is a representation π of $\mathcal{C}$ on $\mathcal{H}$ such that $\pi(\mathcal{S}) = \mathcal{S}_d^q$.*

Proof. The proof involves the construction of a state ρ with the property that $\langle f, g\rangle_{\mathcal{H}} = \rho((g(\underline{T}))^* f(\underline{T}))$. The consideration of the GNS space for ρ along with standard theory of boundary representations give the result. We omit it because it is similar to the proof of Theorem 7.7 in [5]. ∎

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