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An Inventory Model for Deteriorating Items and Stock-dependent Consumption Rate

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An order-level inventory model is developed for deteriorating items with uniform rate of production and stock-dependent demand. Shortages are allowed, and excess demand is backlogged. Results are illustrated with numerical examples.

Key words: deteriorating items, stock-dependent consumption rate

INTRODUCTION

In formulating inventory models, two facets of the problem have been of growing interest, one being the deterioration of items, the other being the variation in the demand rate. Among researchers considering inventory models for deteriorating items, Shah and Jaiswal[1] considered the rate of deterioration to be uniform, Covert and Philip[2] formulated an EOQ model for items with variable rate of deterioration, Misra[3] used a two-parameter Weibull distribution to fit the deterioration rate, and Deb and Chaudhuri[4] suggested a model with variable rate of deterioration allowing shortages to occur. Gupta and Vrat[5] considered a model of stock-dependent consumption rate.

In the present paper, an inventory model concerning a single item is suggested for deteriorating items with a variable rate of deterioration. In the proposed model, the rate of production is uniform, shortages are allowed, set-up cost is considered, and the demand rate is varying, being dependent on instantaneous inventory level. The total cost per unit time is calculated, and the model is illustrated with some numerical examples.

MODEL

A single-item deterministic order-level model for deteriorating items with uniform rate of production and stock-dependent consumption rate is presented under the following assumptions:

(a) T is the duration of a production cycle, where $T = t_1 + t_2 + t_3 + t_4$ (see Figure 1).

(b) The production rate K is uniform.

(c) The demand rate $R(t)$ is linearly dependent on the instantaneous stock level $Q(t)$, $R(t) = \alpha + \beta Q(t)$, where α and β are non-negative constants ($\alpha < K$).

(d) Shortages are allowed and backlogged.

(e) c_1 is the unit holding cost per unit time.

(f) c_2 is the unit shortage cost per unit time.

(g) c_3 is the unit cost price.

(h) $\theta(t)$ is the variable rate of deterioration. This implies that it should be restricted as follows:

$$0 < \theta(t) < 1 \quad \text{for } 0 \leqslant t \leqslant t_1 + t_2 .$$

(i) C is the total average cost for a production cycle.

(j) A is the set-up cost for each new cycle.

The production with a uniform rate and subsequently supply to consumers starts at a time $t = 0$. This continues up to a time t_1 when the stock level reaches S (see Figure 1). Then the production is stopped, and the stock level declines continuously, ultimately becoming zero at time $t_1 + t_2$. Over the period $(0, t_1 + t_2)$ there is deterioration of items. Now shortages are allowed to occur down to an inventory level $-P$ at time $t_1 + t_2 + t_3$. At this instant of time, fresh production and supply to consumers start to clear the entire backlog at time T. This entire cycle is repeated.

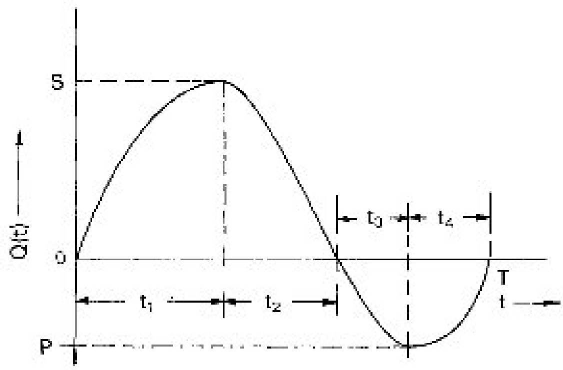

FIG. 1

It may be noted that the demand is always non-negative, even when the stock $Q(t)$ is negative. A sufficient condition for this is that $\alpha - \beta P \geqslant 0$, which is, however, automatically satisfied in this model. [see equation (13) below].

MATHEMATICAL ANALYSIS

Inventory level $Q(t)$

The inventory level $Q(t)$ at time $t(0 \leqslant t \leqslant T)$ satisfies the following differential equations:

$$Q'(t) + \theta(t)Q(t) = K - R(t), \quad 0 \leqslant t \leqslant t_1, \tag{1}$$

$$Q'(t) + \theta(t)Q(t) = -R(t), \quad t_1 \leqslant t \leqslant t_1 + t_2, \tag{2}$$

$$Q'(t) = -R(t), \quad t_1 + t_2 \leqslant t \leqslant t_1 + t_2 + t_3, \tag{3}$$

$$Q'(t) = K - R(t), \quad t_1 + t_2 + t_3 \leqslant t \leqslant t_1 + t_2 + t_3 + t_4, \tag{4}$$

and the conditions that

$$Q(t) = 0 \text{ at } t = 0,\ t_1 + t_2 \text{ and } T. \tag{5}$$

Also $Q(t)$ is continuous at $t = t_1, t_1 + t_2$ and $t_1 + t_2 + t_3$. Further,

$$S = Q(t_1) \quad \text{and} \quad P = -Q(t_1 + t_2 + t_3). \tag{6}$$

The solutions of equations (1)–(4) are given by

$$Q(t) = \begin{cases} (K - \alpha)\dfrac{F(t)}{f(t)}, \quad 0 \leqslant t \leqslant t_1, & (7) \\[2ex] \dfrac{\alpha[F(t_1 + t_2) - F(t)]}{f(t)}, \quad t_1 \leqslant t \leqslant t_1 + t_2, & (8) \\[2ex] -\dfrac{\alpha}{\beta}[1 - \exp\{\beta(t_1 + t_2 - t)\}], \quad t_1 + t_2 \leqslant t \leqslant t_1 + t_2 + t_3, & (9) \\[2ex] -\dfrac{K - \alpha}{\beta}[\exp\{\beta(T - t)\} - 1], \quad t_1 + t_2 + t_3 \leqslant t \leqslant T, & (10) \end{cases}$$

where

$$\left.\begin{aligned} f(t) &= \exp\left[\int_0^t \{\theta(u) + \beta\}\, du\right] \\ \text{and} \quad F(t) &= \int_0^t f(u)\, du. \end{aligned}\right\} \tag{11}$$

From (6) we derive

$$S = (K - \alpha)\,\frac{F(t_1)}{f(t_1)} = \alpha\,\frac{F(t_1 + t_2) - F(t_1)}{f(t_1)} \tag{12}$$

$$P = \frac{\alpha}{\beta}\,[1 - \exp(-\beta t_3)] = \frac{K - \alpha}{\beta}\,[\exp(\beta t_4) - 1]. \tag{13}$$

Thus t_1 and t_2 are related by the equation

$$\frac{F(t_1 + t_2)}{F(t_1)} = \frac{K}{\alpha}, \tag{14}$$

while t_3 and t_4 are related by the equation

$$(K - \alpha)\exp(\beta t_4) + \alpha\exp(-\beta t_3) - K = 0. \tag{15}$$

Total average cost

The total number of deteriorated items over $(0, T)$ is

$$\int_0^{t_1} \theta(t)Q(t)\,dt + \int_{t_1}^{t_1+t_2} \theta(t)Q(t)\,dt.$$

Using (1), (2) and (5), this becomes

$$(K - \alpha)t_1 - \beta\int_0^{t_1} Q(t)\,dt - \alpha t_2 - \beta\int_{t_1}^{t_1+t_2} Q(t)\,dt, \tag{16}$$

where $Q(t)$ is given by (7) and (8) over the ranges $(0, t_1)$ and $(t_1, t_1 + t_2)$ respectively.

The holding cost over the period $(0, T)$ is

$$c_1\left[\int_0^{t_1} Q(t)\,dt + \int_{t_1}^{t_1-t_2} Q(t)\,dt\right], \tag{17}$$

and the shortage cost can be rewritten, using (15), as

$$c_2\left[-\int_{t_1+t_2}^{t_1+t_2+t_3} Q(t)\,dt - \int_{t_1+t_2+t_3}^{T} Q(t)\,dt\right] = \frac{c_2}{\beta}\,\{\alpha t_3 + (\alpha - K)t_4\}. \tag{18}$$

The total cost X over the period $(0, T)$ is given by

$$X = A + c_3 K t_1 - c_3\alpha(t_1 + t_2) + (c_1 - \beta c_3)$$
$$\times\left[(K - \alpha)\int_0^{t_1} \frac{F(u)}{f(u)}\,du + \alpha F(t_1 + t_2)\int_{t_1}^{t_1-t_2} \frac{du}{f(u)} - \int_{t_1}^{t_1+t_2} \frac{F(u)}{f(u)}\,du\right] + \frac{c_2}{\beta}\,\{\alpha t_3 + (\alpha - K)t_4\}. \tag{19}$$

Hence the total average cost is

$$C(t_1, t_3) = \frac{X}{T}, \tag{20}$$

where on the right-hand side t_4 is related to t_3 through equation (15).

Cost-minimization criteria

The optimum values of t_1 and t_3 for minimum total average cost are the solutions of

$$C_{t_1} = 0 \qquad \text{and} \qquad C_{t_3} = 0 \tag{21}$$

provided $C_{t_1t_1} > 0$, $C_{t_3t_3} > 0$ and $C_{t_1t_1}C_{t_3t_3} - C_{t_1t_3}^2 > 0$ for these values of t_1 and t_3. Using these optimum values of t_1 and t_3, the optimal values of S, P, C, t_2, t_4 and T can, in theory, be calculated. However, because of the complexity and generality of the model, only numerical solutions can be found.

Equations (21) are equivalent to

$$T\frac{\partial X}{\partial t_1} = X\left(1 + \frac{dt_2}{dt_1}\right) \quad \text{and} \quad T\frac{\partial X}{\partial t_3} = X\left(1 + \frac{dt_4}{dt_3}\right). \tag{22}$$

The solutions for optimal values of t_1 and t_2 from equation (22) seem to be a formidable task for a general $\theta(t)$. Hence two simple cases of practical interest are considered separately: $\theta(t) = a$ (a constant)—i.e. the rate of deterioration is constant; and $\theta(t) = at$—i.e. the rate of deterioration increases linearly with time.

SPECIAL CASES

Let $\theta(t) = a$ (constant). From (11) we then obtain

$$\left.\begin{aligned} f(t) &= \exp\{(a+\beta)t\}, \\ F(t) &= \frac{\exp\{(a+\beta)t\} - 1}{a+\beta}. \end{aligned}\right\} \tag{23}$$

Equation (14) then gives

$$t_2 = \frac{1}{a+\beta}\ln\left[\frac{K}{\alpha} - \frac{K-\alpha}{\alpha}\exp\{-(a+\beta)t_1\}\right], \tag{24}$$

while (15) gives

$$t_4 = \frac{1}{\beta}\ln\left[\frac{K - \alpha\exp(-\beta t_3)}{K-\alpha}\right] \tag{25}$$

so that

$$\frac{dt_2}{dt_1} = \frac{K-\alpha}{\alpha}\exp[-(a+\beta)(t_1+t_2)] \tag{26}$$

and

$$\frac{dt_4}{dt_3} = \frac{\alpha}{K\exp(\beta t_3) - \alpha}. \tag{27}$$

Again, in this case

$$\begin{aligned} X = {} & A + c_3(K-\alpha)t_1 - c_3\alpha t_2 + (c_1 - \beta c_3) \\ & \times\left[\frac{K-\alpha}{a+\beta}\left\{t_1 - \frac{1-\exp\{-(a+\beta)t_1\}}{a+\beta}\right\} - \frac{\alpha}{(a+\beta)^2}\{1-\exp((a+\beta)t_2)\} - \frac{\alpha t_2}{a+\beta}\right] \\ & \frac{c_2}{\beta}\{\alpha t_3 - (K-\alpha)t_4\} \end{aligned} \tag{28}$$

so that

$$\frac{\partial X}{\partial t_1} = (K-\alpha)\left[c_3 + \frac{c_1 - c_3\beta}{a+\beta}\right][1 - \exp\{-(a+\beta)(t_1+t_2)\}] \tag{29}$$

and

$$\frac{\partial X}{\partial t_3} = \frac{K\alpha c_2}{\beta}\,\frac{\exp(\beta t_3) - 1}{K\exp(\beta t_3) - \alpha}. \tag{30}$$

Substituting the different expressions in (22), we obtain two simultaneous non-linear equations in optimal values of t_1 and t_3. The second equation is simple and can be rewritten as

$$X = Tc_3\frac{\alpha}{\beta}[1 - \exp(-\beta t_3)]. \tag{31}$$

These equations can be solved numerically for different values of the various parameters.

The case $\theta(t) = at$ (a is a constant) can be dealt with similarly. The details of the analytical calculations leading to two simultaneous non-linear equations are given in the Appendix.

NUMERICAL EXAMPLES

Let $A = £100$, $K = 250$ units per month, $\alpha = 100$ units per month, $c_1 = £1$ per unit per month, $c_2 = £10$ per unit per month, and $c_3 = £1$ per unit. The optimum values of t_1, t_2, t_3 and t_4, along with minimum total cost per month and optimum values of S and P, are calculated numerically for different values of a and β for the two cases where the rate of deterioration is constant or varies linearly with time. These are shown in Tables 1 and 2 respectively. It may be noted that the condition $0 < \theta(t) < 1$ for $0 \leqslant t \leqslant t_1^* + t_2^*$ is satisfied in this numerical example for different values of a.

TABLE 1. *Constant rate of deterioration; $\theta(t) = a$*

β	a	t_1^* in month	t_2^* in month	t_3^* in month	t_4^* in month	C^* in £	S^*	P^*
0.1	0.1	0.715	0.911	0.111	0.073	110.0	99.9	11.2
	0.2	0.706	0.839	0.115	0.076	114.5	95.4	11.6
	0.4	0.692	0.728	0.124	0.082	123.0	87.7	12.5
	0.6	0.682	0.644	0.131	0.086	130.0	81.3	13.2
	0.8	0.675	0.578	0.138	0.091	137.0	75.9	13.9
0.2	0.2	0.735	0.809	0.116	0.076	114.6	95.5	11.7
0.4		0.803	0.755	0.117	0.075	115.0	95.6	12.0
0.6		0.887	0.708	0.118	0.074	114.0	95.1	12.2
0.8		0.995	0.665	0.119	0.073	113.5	94.5	12.5

TABLE 2. *Variable rate of deterioration; $\theta(t) = at$*

β	a	t_1^* in month	t_2^* in month	t_3^* in month	t_4^* in month	C^* in £	S^*	P^*
0.1	0.1	0.678	0.887	0.110	0.073	108.0	96.8	11.1
	0.2	0.645	0.812	0.112	0.074	112.0	91.2	11.3
	0.4	0.600	0.713	0.118	0.078	118.0	83.3	11.9
	0.6	0.566	0.645	0.124	0.082	123.0	77.5	12.5
	0.8	0.541	0.596	0.128	0.084	127.0	73.1	12.9
0.2	0.1	0.667	0.783	0.113	0.074	112.0	91.0	11.4
0.4		0.708	0.731	0.113	0.073	111.0	89.4	11.6
0.6		0.740	0.683	0.114	0.072	110.0	86.2	11.8
0.8		0.759	0.639	0.113	0.070	108.0	81.6	11.8

DISCUSSION

The present stock-control model of deteriorating items with stock-dependent consumption rate is applicable to problems where a low stock level adversely affects the demand rate. The rate of deterioration is assumed first to be a constant and then to be an increasing linear function of time. The latter case would be more suitable for items which start deteriorating appreciably some time after they are produced and for which the rate of deterioration increases over time.

Keeping in mind that the objective of minimizing the total average cost is achieved through scheduling the different values of the subintervals t_1 to t_4, it is clearly demonstrated in Tables 1 and 2 that in the case of both constant rate as well as variable rate of deterioration, the system is fairly sensitive to the changes in the value of the parameter a as well as β. Tables 1 and 2 also demonstrate that for both constant and variable rate of deterioration, the minimum total average cost increases, as expected, with the rate of deterioration. However, in the present example, the above cost does not change much with the parameter β, and the reason is apparent from the nature of changes in the values of the subintervals, in this particular case. However, the system would be more sensitive to the changes in the value of β where α has a lower value.

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APPENDIX

When $\theta(t) = at$ (a is a constant), from (11) we obtain, in this case,

$$\left.\begin{aligned} f(t) &= \exp\left(\frac{at^2}{2} + \beta t\right), \\ F(t) &= \int_0^t \exp\left(\frac{au^2}{2} + \beta u\right) du. \end{aligned}\right\} \tag{A1}$$

The relation between t_1 and t_2 is now obtained from (14) as

$$\alpha \int_0^{t_1+t_2} \exp\left(\frac{au^2}{2} + \beta u\right) du = K \int_0^{t_1} \exp\left(\frac{au^2}{2} + \beta u\right) du, \tag{A2}$$

while the relation between t_3 and t_4 remains the same as in (15).

From (A2) we obtain

$$\frac{dt_2}{dt_1} = \frac{Kf(t_1)}{\alpha f(t_1 + t_2)} - 1, \tag{A3}$$

while dt_4/dt_3 remains the same as in (27), since the relation between t_3 and t_4 does not depend on $\theta(t)$.

Now from (19), differentiating both sides with respect to t_1, we get

$$\frac{\partial X}{\partial t_1} = c_3 K - c_3 \alpha\left(1 + \frac{dt_2}{dt_1}\right) + (c_1 - \beta c_3) \times \left[K \frac{F(t_1)}{f(t_1)} + \alpha f(t_1 + t_2)\left(1 + \frac{dt_2}{dt_1}\right) \int_{t_1}^{t_1 - t_2} \frac{du}{f(u)} - \alpha \frac{F(t_1 + t_2)}{f(t_1)}\right].$$

Using (A3), this can be rewritten as

$$\frac{\partial X}{\partial t_1} = c_3 K\left[1 - \frac{f(t_1)}{f(t_1 + t_2)}\right] + (c_1 - \beta c_3)\left[\frac{KF(t_1)}{f(t_1)} + Kf(t_1) \int_{t_1}^{t_1+t_2} \frac{du}{f(u)} - \alpha \frac{F(t_1 + t_2)}{f(t_1)}\right]. \tag{A4}$$

$\partial X/\partial t_3$, however, remains the same as in (30). Substituting from (A3) and (A4) in the first equation of (22), we obtain one non-linear equation, and the other non-linear equation is (31), from which set we would solve for the optimum values of t_1 and t_3.

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