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A note on calculating cost of two-dimensional warranty policy

D.K. Manna [a,*], Surajit Pal [a], Sagnik Sinha [b]

[a] SQC & OR Unit, Indian Statistical Institute, 110 Nelson Manickam Road, Chennai 600 029, India
[b] Department of Mathematics, Jadavpur University, Kolkata 700 032, India

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Abstract

We study the derivation of warranty cost under rectangular two-dimensional policy for both repairable as well as non-repairable product. Typically, two approaches are adopted, namely, one-dimensional (1D) and two-dimensional (2D). We show the difference in results obtained from the formulae under the two approaches through several examples. Merits of the formulae are then analyzed to identify the correct ones.

Keywords: 1D/2D approach; Renewal function; Non-homogeneous Poisson process

1. Introduction

Warranty is attracting significantly greater importance in the marketplace for consumer durables and automobiles. Customers seek redress or compensation in the event of early failure of product, and the manufacturers intend to have a larger market share. Towards this, warranty provides a common ground for both the parties. However, it is well known that warranty alone cannot address the issue without adequate product quality and reliability. This necessitates modeling and analysis of warranty policy in decision-making. In particular, knowledge of warranty cost is extremely important.

The description of a warranty policy involves a *compensation scheme* (repair, replacement, refund, etc.) that is being offered towards product failure within a *specified period* since the sale has taken place. Depending upon how this period is defined, policies are classified into one- and two-dimensional. One-dimensional policy is characterized by an interval (usually of age) as warranty period, with no limitation on usage of the product. A two-dimensional policy is represented by a region in two-dimensional plane, generally one dimension describing age and the other one usage. A typical example of two-dimensional policy for an automobile is as follows. The manufacturer agrees to repair or provide replacement for its failed parts free of cost for a maximum period of 5 years or a maximum usage of 50,000 miles, whichever occurs first, from the time of sale. Following the nomenclature of warranty, we refer to this as rectangular two-dimensional policy (see

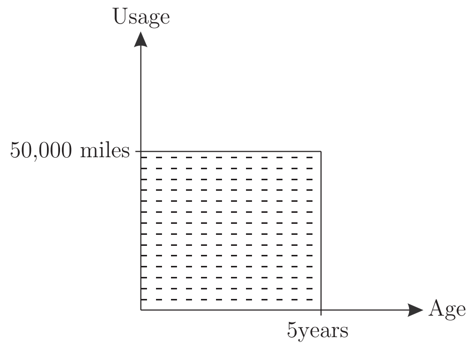

Fig. 1. Rectangular warranty region.

Fig. 1). For other variations of two-dimensional policies, see for example, Singpurwalla and Wilson (1993), Blischke and Murthy (1994), etc.

Referring to Fig. 1, observe that every failure within the shaded region attracts warranty service. It then follows that a failure of the product, under two-dimensional policy must be indexed by both age and usage. As a result, failure/cost modeling in such situations naturally involves bivariate failure model. However, two approaches have been adopted by researchers for warranty cost modeling, namely, (a) two-dimensional (2D): when the bivariate failure model is taken as it is, and (b) one-dimensional (1D): when the problem is reduced to univariate by incorporating the usage-rate suitably.

This note brings in focus the discrepancy in the formulae based on above two approaches for computation of warranty cost under both repairable and non-repairable cases.

The outline of this article is as follows. The modeling assumptions and notation are contained in Section 2. The formulae of warranty cost by the two approaches for both repairable as well as non-repairable product are presented in Section 3. We provide some examples in Section 4 in order to highlight the difference in results obtained by the two approaches. In Section 5, we study the merits of the formulae, and try to identify the correct ones. We then conclude in Section 6.

2. Assumptions and notation

We consider the warranty service (compensation scheme) to be free repair or replacement, which is referred to as FRW in the literature. For repairable product, we assume that a failure is always minimally repaired (see Baik, Murthy, & Jack, 2004; Lawless & Thiagarajah, 1996; etc.), that is, the corresponding repair helps restoring the condition of the product that prevailed immediately before the failure occurred. When a product is minimally repaired, only minor part/component may be replaced but not the whole product. We also assume that repair or replacement time is very small, and hence it can be ignored. For the sake of convenience, we shall denote both unit price as well as repair cost by c_r (constant). Note that they do not apply to the same product, namely, we talk of repair cost for repairable product only, and unit price for replacement of non-repairable product.

Rectangular warranty region is quite common in automobiles. Let us assume the same region for the product under consideration, with limits on age and usage be x_0 and y_0, respectively. Denote the first failure instance of a unit of the product by (X, Y), where $X =$ age and $Y =$ usage.

Let us write usage-rate as $R = Y/X$, and the age at failure for given usage-rate r be denoted by $X(r) = [X|R = r]$. Associated with (X, Y), we use the following notation:

$f_{X,Y}(x, y) =$ joint probability density function (pdf)
$F_{X,Y}(x, y) =$ distribution function
$S_{X,Y}(x, y) =$ survival function
$M_{X,Y}(x, y) =$ renewal function.

The mileage (usage) function is defined as the amount of usage at a given age of failure, and is denoted by $Y(x) = [Y|X = x]$. Besides, we also use the following notation concerned with any random variables U:

$f_U(u) =$ (marginal) pdf
$F_U(u) =$ distribution function
$M_U(u) =$ renewal function
$\mathrm{E}[U] =$ expectation of U.

3. Formulae for warranty cost

Warranty cost of a product, for a given policy, is measured by the expected cost of warranty service per unit of the product. In the following, we present the two approaches (1D and 2D) for the computation of warranty cost.

Denote the number of failures of a unit within the warranty region by $N(x_0, y_0)$, and the corresponding cost by $C(x_0, y_0)$. Therefore, we have $C(x_0, y_0) = c_r N(x_0, y_0)$. We are interested in the expected value of $C(x_0, y_0)$, namely, $E[C(x_0, y_0)]$, which is equal to $c_r E[N(x_0, y_0)]$. Consequently, the expression for $E[N(x_0, y_0)]$ will suffice.

We consider first the case of non-repairable product. The expressions for expected number of failures under the two approaches are as follows.

- *2D approach:* Following Hunter (1974, 1996) (also see Murthy, Iskandar, & Wilson, 1995), $N(x, y)$ is two-dimensional renewal counting process for x, $y > 0$, and

$$E[N(x_0, y_0)] = M_{X,Y}(x_0, y_0), \tag{1}$$

where $M_{X,Y}(x_0, y_0)$ is the two-dimensional renewal function, and is obtained as the solution of:

$$M_{X,Y}(x_0, y_0) = F_{X,Y}(x_0, y_0) + \int_{u=0}^{x_0} \int_{v=0}^{y_0} M_{X,Y}(x_0 - u, y_0 - v)\,\mathrm{d}F_{X,Y}(u, v).$$

It is nearly impossible to obtain $M_{X,Y}(.,.)$ analytically even for the simplest form of $F_{X,Y}(.,.)$. Iskandar (1991) provides a computational procedure, and tabulates selected $M_{X,Y}(.,.)$s for Beta Stacy and Downton's Bivariate Exponential distributions. Alternatively, one can obtain an estimate by the method of simulation.

- *1D approach:* According to Blischke and Murthy (1994) (also see Jack, Murthy, & Iskandar, 2003), with conditional on the usage-rate $R = r$ (see Fig. 2),

$$N((x_0, y_0)|r) = \begin{cases} N_{X(r)}(x_0) & \text{if } r < y_0/x_0 \\ N_{X(r)}(y_0/r) & \text{if } r \geqslant y_0/x_0, \end{cases} \tag{2}$$

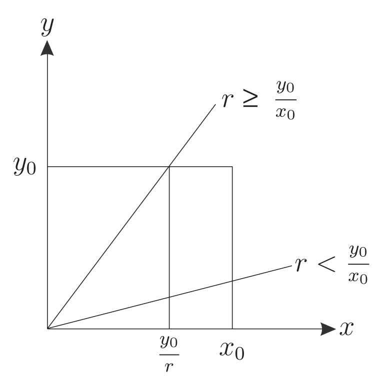

Fig. 2. Warranty coverage age with $R = r$.

where $N_{X(r)}(.)$ is one-dimensional renewal counting process associated with the distribution function $F_{X(r)}(.)$. This implies that

$$E[N((x_0,y_0)|r)] = \begin{cases} M_{X(r)}(x_0) & \text{if } r < y_0/x_0 \\ M_{X(r)}(y_0/r) & \text{if } r \geqslant y_0/x_0, \end{cases} \tag{3}$$

where $M_{X(r)}(.)$ is obtained as the solution of renewal equation:

$$M_{X(r)}(x) = F_{X(r)}(x) + \int_{u=0}^{x} M_{X(r)}(x-u)\,\mathrm{d}F_{X(r)}(u). \tag{4}$$

As a result, we have

$$E[N(x_0,y_0)] = \int_{r=0}^{y_0/x_0} M_{X(r)}(x_0)\,\mathrm{d}F_R(r) + \int_{r=y_0/x_0}^{\infty} M_{X(r)}(y_0/r)\,\mathrm{d}F_R(r). \tag{5}$$

It is again well known that $M_{X(r)}(.)$ can be obtained analytically only for a small class of distributions, for example, Exponential, Erlang, Uniform, etc. In general, one needs to use numerical procedure or adopt simulation. The numerical method of Xie (1989) is quite fast and very accurate.

We now turn to repairable product. It is strongly advocated (Ascher & Feigngold, 1984) that failure time modeling be done by non-homogeneous Poisson process (NHPP) with appropriate failure intensity function or rate of occurrence of failure (ROCOF). In the case of minimal repair, the conditional failure intensity function (as defined in Lawless & Thiagarajah, 1996) remains unaffected by each failure, and therefore ROCOF is taken as the hazard function of the first failure time.

Consequently, Baik et al. (2004) and Blischke and Murthy (1994) presume that $N(x,y)$ is NHPP in two dimensions for x, $y > 0$ with $\lambda_{X,Y}(.,.)$ as the intensity function, where $\lambda_{X,Y}(.,.)$ is hazard function of (X, Y). $E[N(x_0,y_0)]$ is then written under 2D approach as:

$$E[N(x_0,y_0)] = \int_{u=0}^{x_0}\int_{v=0}^{y_0} \lambda_{X,Y}(u,v)\,\mathrm{d}v\,\mathrm{d}u. \tag{6}$$

For 1D approach, they employ the similar arguments as in Eq. (2) and propose to compute $E[N(x_0,y_0)]$ by the formula:

$$E[N(x_0,y_0)] = \int_{r=0}^{y_0/x_0} \Lambda_{X(r)}(x_0)\,\mathrm{d}F_R(r) + \int_{r=y_0/x_0}^{\infty} \Lambda_{X(r)}(y_0/r)\,\mathrm{d}F_R(r), \tag{7}$$

where $N_{X(r)}(x)$ is assumed to NHPP for $x > 0$ with its intensity function as the hazard function of $X(r)$ so that the cumulative intensity function is given by $\Lambda_{X(r)}(u) = -\ln[1 - F_{X(r)}(u)]$.

While adopting the Eq. (7), under the assumption that first failure information is absent but $F_R(.)$ is known, researchers consider occurrence of failures according to Poisson process with suitable intensity function $\lambda_{X(r)}(.)$. For instance, Murthy and Blischke (1992) as well as Mitra and Patankar (2000) take $\lambda_{X(r)}(x) = \theta_0 + \theta_1 r + (\theta_2 + \theta_3 r)x$. A special case of this intensity function is proposed by Moskowitz and Chun (1994), but they do not exactly utilize the Eq. (7). We discuss this later in Section 5.

4. Some examples to compute $E[N(x_0,y_0)]$

In the following, we present some examples in order to illustrate the difference in $E[N(x_0,y_0)]$ values as computed by the two approaches: 1D and 2D. These examples are certainly not the exceptions. In fact, we strongly believe that it may be hard to find example where the difference, however small, is absent. This difference may be due to either use of incorrect formula or because of wrong interpretation.

Typically, the researchers have advanced those bivariate probability distributions for the analysis of two-dimensional warranty for which $E[Y(x)]$ is an increasing function of x; see Murthy et al. (1995) for instance. All our examples do possess this property.

4.1. Non-repairable case

Example 4.1.1. Suppose that $(X, Y) \sim$ Beta Stacy distribution. Murthy et al. (1995) propose its application in warranty analysis. The joint pdf is

$$f_{X,Y}(x,y) = \frac{(c/\phi)a^{-\alpha c}}{\Gamma(\alpha)B(\theta_1,\theta_2)} x^{\alpha c-\theta_1-\theta_2} \left(\frac{y}{\phi}\right)^{\theta_1-1} \left(x-\frac{y}{\phi}\right)^{\theta_2-1} \exp\left[-\left(\frac{x}{a}\right)^c\right]$$

for $x > 0$, $0 < y < \phi x$, and α, c, a, ϕ, θ_1, $\theta_2 > 0$. Observe that X and R are independent with respective pdfs as:

$$f_X(x) = \frac{ca^{-\alpha c}}{\Gamma(\alpha)} x^{\alpha c-1} \exp\left[-\left(\frac{x}{a}\right)^c\right], \quad x > 0,$$

$$f_R(r) = \frac{1}{\phi B(\theta_1,\theta_2)} \left(\frac{r}{\phi}\right)^{\theta_1-1} \left(1-\frac{r}{\phi}\right)^{\theta_2-1}, \quad 0 < r < \phi.$$

Consequently, the Eq. (5) becomes:

$$E[N(x_0,y_0)] = M_X(x_0)F_R(y_0/x_0) + \int_{r=y_0/x_0}^{\infty} M_X(y_0/r)\,\mathrm{d}F_R(r). \tag{8}$$

We obtain the values of $M_X(.)$ by the method of Xie (1989). With $\alpha = 1.9$, $c = 2.5$, $a = 0.2$, $\phi = 1.1$, $\theta_1 = 1.1$, and $\theta_2 = 1.1$, the values of $E[N(1, 0.3)]$ are given in Table 1. Iskandar (1991) tabulates the corresponding value as 1.8713. Based on the experience of numerical experimentation, we strongly feel that the difference in results between Iskandar (1991) and simulation (of 2D approach) is merely due to numerical error, but the difference in values obtained using the two approaches may not be just due to numerical error.

Example 4.1.2. Assume that the joint pdf of (X, Y) be

$$f_{X,Y}(x,y) = \frac{\lambda^\alpha}{\theta\Gamma(\alpha)} \left(\frac{y^\alpha}{x^{\alpha+1}}\right) \exp\left[-\left(\frac{1}{\theta}+\frac{\lambda}{x}\right)y\right]$$

for x, $y > 0$, and θ, λ, $\alpha > 0$. We then observe that $X(r) \sim$ Exponential(r/θ) and $R \sim$ Gamma(α, λ) with

$$f_{X(r)}(x) = \frac{r}{\theta}\mathrm{e}^{-rx/\theta}, \quad x > 0,$$

$$f_R(r) = \frac{\lambda^\alpha}{\Gamma(\alpha)}\mathrm{e}^{-\lambda r} r^{\alpha-1}, \quad r > 0,$$

$$\text{and } F_X(x) = 1 - \left(1+\frac{x}{\lambda\theta}\right)^{-\alpha}, \quad x > 0. \tag{9}$$

Let us now consider the special case: $y_0 = \infty$, that is, we are concerned with one-dimensional policy having no limitation on usage. Hence, the Eq. (5) turns out as

$$E[N(x_0,\infty)] = \frac{\alpha x_0}{\lambda\theta}. \tag{10}$$

On the other hand, the 2D approach (Eq. (1)) results in

$$E[N(x_0,\infty)] = M_{X,Y}(x_0,\infty) = M_X(x_0) \quad \text{(see Hunter, 1974)}. \tag{11}$$

Combining the Eqs. (10) and (11), we obtain $M_X(x_0) = \alpha x_0/(\lambda\theta)$. But one can easily verify that, with $F_X(x)$ as given in Eq. (9), $\alpha x_0/(\lambda\theta)$ is not solution of the renewal Eq. (4). This leads to a contradiction.

Table 1
Values of $E[N(x_0, y_0)]$

Example	(x_0, y_0)	1D approach	2D approach
4.1.1	(1, 0.3)	2.1312	Average = 1.9290* (Min. = 1.9188*, Max. = 1.9412*)
4.2.1	(1, 2)	0.9741	1.5186

* Estimate by method of simulation: $N_{ij}(x_0, y_0)$ = No. of renewals for jth unit in ith replication for $i = 1, \ldots, m$, $j = 1, \ldots, n$ where $m = 10$, $n = 10{,}000$; $M_i = (1/n)\sum_{j=1}^{n} N_{ij}(x_0, y_0)$; Average $= (1/m)\sum_{i=1}^{m} M_i$; Min. = $\min_i\{M_i\}$; Max. = $\max_i\{M_i\}$.

4.2. Repairable case

Recall that only minimal repair is under consideration. We refer to the Eqs. (6) and (7) for the two approaches – 2D and 1D, respectively.

Following Baik et al. (2004), we take $\lambda_{X,Y}(x,y) = f_{X,Y}(x,y)/S_{X,Y}(x,y)$. They observe that

$$\int_{u=0}^{x_0}\int_{v=0}^{y_0} \lambda_{X,Y}(u,v)\,\mathrm{d}v\,\mathrm{d}u \neq -\ln[S_{X,Y}(x_0,y_0)].$$

Example 4.2.1. Let $(X,Y) \sim$ Bivariate Lognormal$(\mu_1,\mu_2,\sigma_1^2,\sigma_2^2,\rho)$, that is,

$$f_{X,Y}(x,y) = \frac{1}{2\pi(\sigma_1 x)(\sigma_2 y)\sqrt{1-\rho^2}}\exp\left[-\frac{1}{2(1-\rho^2)}\left\{\left(\frac{\ln x-\mu_1}{\sigma_1}\right)^2 + \left(\frac{\ln y-\mu_2}{\sigma_2}\right)^2 -2\rho\left(\frac{\ln x-\mu_1}{\sigma_1}\right)\left(\frac{\ln y-\mu_2}{\sigma_2}\right)\right\}\right]$$

for x, $y > 0$, σ_1, $\sigma_2 > 0$, and $-1 < \rho < 1$.

We assume $\rho > 0$, which implies that $E[Y(x)] \uparrow$ x. Observe that $R \sim$ Lognormal$(\mu_2-\mu_1, \sigma_1^2,+\sigma_2^2-2\rho\sigma_1\sigma_2)$, $X(r) \sim$ Lognormal(μ_0,σ_0^2), where

$$\mu_0 = \mu_1 + \frac{\sigma_1(\rho\sigma_2-\sigma_1)}{\sigma_1^2+\sigma_2^2-2\rho\sigma_1\sigma_2}(\ln r + \mu_1 - \mu_2),$$
$$\sigma_0^2 = \frac{(1-\rho^2)\sigma_1^2\sigma_2^2}{\sigma_1^2+\sigma_2^2-2\rho\sigma_1\sigma_2},$$
$$\text{and } \Lambda_{X(r)}(u) = -\ln\left[1-\Phi\left(\frac{\ln u-\mu_0}{\sigma_0}\right)\right].$$

For $\mu_1 = -0.5$, $\mu_2 = -0.2$, $\sigma_1 = 1.0$, $\sigma_2 = 1.1$, and $\rho = 0.5$, the values of $E[N(1,2)]$ are given in Table 1.

5. Discussion

It is therefore established that discrepancy does exist between the two approaches. Under such circumstances, how does one go about the decision-making?

In non-repairable case, we have no doubt that 2D approach (formula (1)) produces the correct result since it is based on well developed renewal theory in two dimensions (see Hunter, 1974, 1996, for reference). The 1D approach through formula (5) is an attempt to express two-dimensional renewal function in one dimension. A correct version involving use-rate is as follows. Let (X_i, Y_i) for $i = 1, 2, \ldots$ be the failure instances. Clearly, each (X_i, Y_i) is independent and identically distributed with common joint pdf as $f_{X,Y}(.,.)$. Also, let $S_n^X = \sum_{i=1}^n X_i$, $S_n^Y = \sum_{i=1}^n Y_i$, and $R_n = S_n^Y/S_n^X$ for $n \geqslant 1$. Then, we have:

$$\begin{aligned}
M_{X,Y}(x_0,y_0) &= \sum_{n=1}^{\infty} P\left[S_n^X \leqslant x_0, S_n^Y \leqslant y_0\right] = \sum_{n=1}^{\infty}\int_{r=0}^{\infty} P\left[S_n^X \leqslant x_0, S_n^Y \leqslant y_0 | R_n = r\right] f_{R_n}(r)\,\mathrm{d}r \\
&= \sum_{n=1}^{\infty}\int_{r=0}^{\infty} P\left[S_n^X \leqslant x_0, S_n^X \leqslant \frac{y_0}{r} | R_n = r\right] f_{R_n}(r)\,\mathrm{d}r \\
&= \sum_{n=1}^{\infty}\int_{r=0}^{\infty} P\left[S_n^X \leqslant \min\left\{x_0, \frac{y_0}{r}\right\} | R_n = r\right] f_{R_n}(r)\,\mathrm{d}r \\
&= \sum_{n=1}^{\infty}\left[\int_{r=0}^{y_0/x_0} F_{S_n^X|R_n=r}(x_0) f_{R_n}(r)\,\mathrm{d}r + \int_{r=y_0/x_0}^{\infty} F_{S_n^X|R_n=r}\left(\frac{y_0}{r}\right) f_{R_n}(r)\,\mathrm{d}r\right].
\end{aligned} \tag{12}$$

On comparison of the formulae (5) and (12), we conclude that the former is at best an approximation. Therefore, we require the knowledge of error involved for its possible application and interpretation. On the other hand, one can explore the possibility of adopting the approach given by Moskowitz and Chun (1994) for repairable product. This is described below. With conditional $R = r$, let $P_1(r) = \{(x, rx)$: $x \in (0, x_0)$ for $r < y_0/x_0\}$ and $P_2(r) = \{(y/r, y)$: $y \in (0, y_0)$ for $r \geqslant y_0/x_0\}$ be the two line segments (refer to Fig. 2). If we denote the expected number of failures occurring on these line segments by $E_f[P_i(r)]$ for $i = 1$ and 2, then

$$E[N(x_0, y_0)] = \int_{r=0}^{y_0/x_0} E_f[P_1(r)]\,\mathrm{d}F_R(r) + \int_{r=y_0/x_0}^{\infty} E_f[P_2(r)]\,\mathrm{d}F_R(r). \tag{13}$$

With regard to 2D approach for repairable case, Baik, Murthy, and Jack (2006) report that formula (6) is false, and provide a revised formulation. One must observe that this new formulation fails to preserve the distribution of product use-rate that changes from failure to failure. This is a serious drawback to arrive at a meaningful value of $E[N(x_0, y_0)]$. As we understand, theory on *minimal repair in two dimensions* is possibly yet to be developed.

6. Conclusion

In this note, we consider the problem of calculating warranty cost with rectangular two-dimensional policy for both the types of products: non-repairable, and repairable with minimal repair. Specifically, we demonstrate through examples the discrepancy in the warranty cost formulae derived by the two approaches, namely, 1D and 2D. It is shown that the existing formula by 1D approach for non-repairable product is incorrect. For repairable product, the formulae under the two approaches are not comparable. In this case, the limitation of the formula by 2D approach has been stressed.

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