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INTERSECTION PROPERTIES OF BALLS IN TENSOR PRODUCTS OF SOME BANACH SPACES – II

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In this paper we study an intersection property of balls called the finite intersection property, for injective and projective tensor products of Banach spaces. It turns out that for Banach space E containing an isometric copy of c_0, $E \check{\otimes} F$ fails this property when F is infinite dimensional. A precise positive answer can be obtained when E is a space of continuous functions. Similary this intersection property will not be in general preserved by projective tensor products. By eastablishing some general theorems about proper M-ideals we conclude that several of the classical compact operator spaces fail this property.

INTRODUCTION

In this paper we continue the study of the intersection properties of balls in tensor product spaces[17] started in. The main objective of this paper is to study the finite intersection property (FIP for short) of balls introduced by Lindenstrauss[16] for the projective and injective tensor products of real Banach spaces with particular emphasis on the space of vector valued integrable and continuous functions.

According to Lindenstrauss, a real Banach space E has the FIP if for every family $\{B(a_i, r_i)\}_{i \in I}$ of closed balls in E such that any finite subcollection has non-empty intersection, one has $\bigcap_{i \in I} B(a_i, r_i) \neq \phi$. Since closed balls in a dual space E^* are also w^*-compact, one can see that any dual space has the FIP. It is fairly easy to see that if $F \subseteq E^*$ is the range of a norm one projection then F has the FIP. A long standing open question due to Lindenstrauss is to decide whether every Banach space that has the FIP is the range of a norm one projection when canonically embedded in its bidual[16] (page 60). It is well known[18] that a Banach space E is the range of a norm one projection in its bidual iff it is isometric to the range of a norm one projection in some dual space. Hence the question can be reformulated as "If E has the FIP, is it isometric to the range of a norm one projection in a dual space?." There have been only a couple of papers, both by Godefroy[10,11] that have dealt with this problem before.

Seizing on the role played by the space c_0 (the space of sequences converging to zero) in the isomorphic theory of tensor product spaces[5,9,19] and utilizing the observation that c_0 fails the FIP, in Section 1 of this paper we show that when E has an isometric copy of c_0, the space $E \overset{\vee}{\otimes} F$ has the FIP only when F is finite dimensional and when $E = C(K)$, the space of continuous functions on an infinite compact Hausdorff space, a more precise answer is obtained by showing that $C(K) \overset{\vee}{\otimes} F$ has FIP iff K is extremally disconnected. We show that the projective tensor product of two spaces with the FIP need not in general have the FIP. However it turns out that the l^1-direct product of any family of spaces with the FIP has the FIP.

In section 2, we point out a general phenomenon that explains why c_0 fails the FIP when we show that any Banach space that is a proper M-ideal (see section 2 for the definitions) fails the FIP. For a further special class (that includes c_0) we show that no equivalent FIP norm exists. As a consequence of these results we get that several classical spaces of compact operators fail the FIP. We conclude with some open problems.

Our notation and terminology is fairly standard. For a compact Hausdorff space K, and a real Banach space E, the space $C(K, E)$ denotes E valued continuous functions on K, equipped with the supremum norm and for any measure space (X, ϑ, μ), $L^1(\mu, E)$ stands for the space of E-valued Bochner integrable functions. We will be using the notation for tensor products and the basic results of Distel and Uhl[7].

1. Tensor Product Spaces

In this section we show that if E is a Banach space containing an isometric copy of c_0 then for any non atomic measure space (X, ϑ, μ), $L^1(\mu, E)$ fails the FIP. In the case of injective tensor products it turns out that $E \overset{\vee}{\otimes} F$ can have the FIP only when F is finite dimensional.

Lindenstrauss[16] showed that in c_0, the family $\{B(e_n, \frac{1}{2})\}$ of closed balls with centre at the nth coordinate vector, has empty intersection. It is easy see that any finite subcollection from the above family has non-empty intersection. Consequently c_0 fails the FIP.

In recent years, starting with the paper of Cembranos[5], many situations have been found where tensor product spaces contain isomorphic, complemented copies of c_0. Our first results point out that the corresponding 'isometric' results are rather easily obtained.

We shall be repeatedly making use of the following easy observation, noted in the introduction.

Let $F \subset E$ be a closed subspace and let $P\ E: \to F$ be an onto projection of norm one. If E has the FIP then so does F.

Theorem 1—Let E be a Banach space having an isometric copy of c_0 and let F be any infinite dimensional Banach space then the injective tensor product $E \check{\otimes} F$ fails the FIP.

PROOF : We shall show that c_0 is isometric to the range of a norm one projection in $E \check{\otimes} F$. Since c_0 fails the FIP, we can conclude then that $E \check{\otimes} F$ fails the FIP,

We adapt the arguments of Saab and Saab[19]-which in turn depend on the beautiful way Josefson – Nissenzweig theorem[6] has been applied by Cembranos[5] to get complemented copies of c_0.

Since F is infinite dimensional, using the Josefson-Nissenzweig theorem we get a sequence $z_n^* \in F^*$ such that $\|z_n^*\| = 1$ and $z_n^* \to 0$ in the w^*-topology.

Applying now the Bishop-Phelps theorem[14] we can get a sequence y_n^* such that $\|z_n^* - y_n^*\| \to 0$ and $\|y_n^*\| = y_n^*(y_n)$, where $y_n \in F$ and $\|y_n\| = 1$.

Clearly $y_n^* \to 0$ in the w^*-topology and $\lim \|y_n^*\| = 1$. Put $x_n^* = y_n^* / \|y_n^*\|$.

Now $\|x_n^*\| = 1 = x_n^*(y_n) = \|y_n\|$ and $x_n^* \to 0$ in the w^*-topology.

Let T denote the isometric embedding of c_0 in E. Put $a_n = T(e_n)$ and let $a_n^* \in E^*$ be a norm preserving extension of $e_n \circ T^{-1}$.

Clearly $a_n^*(a_m) = \delta_{nm}$.

Define

$$\Phi : c_0 \to E \check{\otimes} F \text{ by}$$

$$\Phi((\alpha_n)) = \Sigma \alpha_n (a_n \otimes y_n).$$

We claim that Φ is a well-defined linear isometry. Enough to show that for any n

$$\|\sum_{i=1}^{n} \alpha_i (a_i \otimes y_i)\| = \max_{1 \leqslant i \leqslant n} |\alpha_i|.$$

For $x^* \in E^*$, $y^* \in F^*$ with $\|x^*\| \leqslant 1$, $\|y^*\| \leqslant 1$

$$|\sum_{i=1}^{n} \alpha_i x^*(a_i) y^*(y_i)| \leqslant (\sum_{i=1}^{n} |x^*(a_i)|) \max_{1 \leqslant i \leqslant n} |\alpha_i|.$$

But by the choice of the a_i

$$\sum_{i=1}^{n} | x^* (a_i) | \leqslant \|x^*\| \leqslant 1.$$

Hence by the definition of the injective tensor product norm, we have

$$\| \sum_{i=1}^{n} \alpha_i (a_i \otimes y_i)\| \leqslant \max_{1 \leqslant i \leqslant n} | \alpha_i | .$$

Evaluating $\sum_{1}^{n} \alpha_i (a_i \otimes y_i)$ at a suitable choice of a_i^* and x_i^* one can easily see the reverse inequality. Φ is thus well-defined and clearly is a linear isometry.

Define $P : E \otimes F \to \Phi (c_0)$ by

$$P \left(\sum_{i=1}^{k} r_i \otimes s_i \right) = \sum_{i=1}^{k} \Phi ((a_n^* (r_i) \; x_n^* (s_i)).$$

Then as in Saab and Saab[19], one can verify that P is a norm decreasing, linear map and hence can be extended to a norm one projection from $E \check{\otimes} F$ onto $\Phi (c_0)$.

Remark 1 : Since E (or F) is isometric to the range of a norm one projection in $E \check{\otimes} F$, clearly for $E \check{\otimes} F$ to have the FIP, it is necessary for E (or F) to have the FIP.

The next result shows that one can obtain a more precise information when E is the space of continuous functions on a compact Hausdorff space K.

It is well-know[15] that for an infinite K, $C (K)$ contains an isometric copy of c_0.

Theorem 2—For a finite dimensional space F and infinite compact Hausdorff space K, $C (K) \check{\otimes} F = C (K, F)$ has the FIP iff K is extremally disconnected.

PROOF : Suppose $C (K, F)$ has the FIP. As remarked before this implies that $C (K)$ has the FIP. It is well-known that in the space $C (K)$ (Lacey[15]) any finite family of pairwise intersecting closed balls, have non-empty intersection. Combined with the FIP this means that any family of pairwise intersecting closed balls have non-empty intersection. Hence using results from section 11 of Lacey[15], one can conclude that K is extremally disconnected.

Conversely suppose that K is extremally disconnected. Appealing again to the results from section 11 of Lacey[15], we get that there is a projection P of norm one from $C (K)^{**}$ onto $C (K)$.

It is fairly easy to see that $P \otimes I$ is a norm one projection from $C(K)^{**} \check{\otimes} F$ onto $C(K) \check{\otimes} F$.

Since the former space is a dual space, it has the FIP. Hence $C(K,F)$ has the FIP.

Remark 2 : We have in fact showed that $C(K, F)$ has the FIP iff it is the range of a norm-one projection in a dual space. Also contained in the above proof is the fact that if E is the range of a norm one projection in a dual space then for any finite dimensional space F, $E \check{\otimes} F$ is the range of a norm one projection in a dual space.

Remark 3 : Let $K(E)$ denote the space of compact operators on a Banach space E. It is a long standing open problem to decide when does $K(E)$ occur as a dual space. Since $K(l^1) = l^1 \check{\otimes} l^\infty$ and $K(l^\infty) = l^\infty \check{\otimes} (l^\infty)^*$, we can deduce from the above theorems that $K(l^1)$ and $K(l^\infty)$ fail the FIP.

Our next result shows that the projective tensor product of two Banach spaces with the FIP can fail to have the FIP.

We shall first introduce some notation and then state without proof a Proposition due to Godefroy[10,11].

For any Banach space E, let

$$\mathscr{C}_E = \{g \in E^{**} : \|g + x\| \geqslant \|x\|, \ \forall\, x \in E\}.$$

Proposition—For a Banach space E the following statements are equivalent.

(1) E has the FIP.

(2) $E^{**} = E + \mathscr{C}_E$

(3) For any family $\{B(a_i, r_i)\}$ of closed balls in E^{**} with centres from E, $\cap\, B(a_i, r_i) \neq \phi$ in E^{**} implies $\cap\, B(a_i, r_i)$ has a point from E.

Note : (3) is what Godefroy[10] calls as Property □.

Lemma—If $\{E_i\}$ is any family of Banach spaces having the FIP, then their l^1 direct product $E = \otimes_1 E_i$ has the FIP.

PROOF : In view of the Proposition stated above, it is enough to show that $E^{**} = E + \mathscr{C}_E$.

If F denotes the c_0 direct product of the family E_i^*, then clearly $F \subset E^* = \oplus_\infty E_i^*$.

Let $F^\perp$ denote the annihilator of F in E^{**}. Harmand[12] has observed that

$$E^{**} = (\oplus_1 E_i^{**}) \oplus_1 F^\perp.$$

Now let $f \in E^{**}$ and write $f = (f_i) + g$ where $(f_i) \in \oplus_1 E_i^{**}$ and $g \in F^{\perp}$.

Since E_i has the FIP for each i, appealing again to the above proposition, we can write $f_i = a_i + g_i$ where $a_i \in E_i$ and $g_i \in \mathscr{C}_{E_i}$.

Since $\|a_i\| \leqslant \|f_i\|$, we have $\sum_i \|a_i\| < \infty$

and hence $\Sigma \|g_i\| < \infty$, so that $(a_i) \in E$ and $(g_i) \in \oplus_1 E_i^{**}$.

We conclude the proof by showing that

$$g + (g_i) \in \mathscr{C}_L.$$

Let $(b_i) \in E$

$$\|(b_i) - (g_i) - g\| = \|(b_i) - (g_i)\| + \|g\|$$

(since E^{**} is the l^1 direct sum of $\oplus_1 E_i^{**}$ and F).

$$= \sum_i \|b_i - g_i\| + \|g\|$$

$$\geqslant \sum_i \|b_i\| = \|(b_i)\|$$

since $g_i \in \mathscr{C}_{E_i}$.

Therefore $g + (g_i) \in \mathscr{C}_L$.

Theorem 3—For any non-atomic measure space (X, θ, μ) and for any Banach space E containing an isometric copy of c_0, $L^1(\mu, E)$ fails FIP.

PROOF : Using standard measure theoretic arguments, one can see that it is enough to prove this result when $X = [0, 1]$ and μ the Lebesgue measure.

If r_n denotes the Rademacher functions in $L^1([0, 1])$, then observations similar to the ones given by Emmanuele[9] show that the map $\Phi : c_0 \to L^1([0, 1], E)$ defined by $\Phi((\alpha_n)) = \Sigma \alpha_n r_n a_n$ where the a_n have been chosen as in Theorem 1, is a linear isometry.

Note that for any $f \in L^1([0, 1], E)$, the sequence of vectors $\int_0^1 r_n f \, d\mu \to 0$. Consequently

$$P : L^1([0, 1], E) \to \Phi(c_0) \text{ defined by } P(f) = \Phi((a_n^* (\int_0^1 r_n f \, d\mu))$$

where the a_n^* have been chosen as in Theorem 1, is a linear projection of norm one.

Hence $L^1([0, 1], E)$ fails the FIP.

Remark 4 : Since for any measure space (X, ϑ, μ), $L^1(\mu)$ (see Lindenstrauss[16]) has the FIP, by taking E as, say, l^∞, we can see that projective tensor product of two spaces with the FIP can fail the FIP.

Earlier we have proved[18] that if E has the Radon-Nikodým property and is the range of a norm one projection in a dual, then for any measure space (X, ϑ, μ), $L^1(\mu, E)$ is the range of a norm one projection in a dual. The above result also shows that our theorem is not true without the additional hypothesis of Radon-Nikodým property, answering a question raised in our earlier work[18].

It is well known that for a dual space E^*, $C(K, E^*)$ can be identified as the space of compact operators from E into $C(K)$, (see Dunford and Schwartz). In view of our results about the FIP and $C(K, E)$, it is natural to enquire as to when does the space of bounded operators $L(E, C(K))$ has the FIP? If one, uses the identification of $L(E, C(K))$ as the space of functions on K, that are continuous when E^* has the w^*-topology (denoted by $C(K, E^*, w^*)$), equipped with the supremum norm (see Dunford and Schwartz[8]), our next proposition completely answers this question.

Proposition 2 – For an infinite compact Hausdorff space K, $C(K, (E^*, w))$ has the FIP iff K is extremally disconnected.

PROOF : By fixing an $e_0 \in E$ and $e_0^* \in E^*$ such $\|e_0\| = e_0^*(e_0) = \|e_0^*\| = 1$, one can see that the map $f \to f . e_0^*$ is an isometric embedding of $C(K)$ into $C(K, (E^*, w^*))$. The association $F \to F \circ e$ defines a norm one projection from $C(K, (E^*, w^*))$ onto the image of $C(K)$. Hence if $C(K, (E^*, w^*))$ has the FIP, then from what we have proved before, it follows that K is extremally disconnected.

Conversely when K is extremally disconnected, since $C(K)$ is the range of a norm-one projection in $C(K)^{**}$, arguments similar to the ones indicated before, show that $C(K, (E^*, w^*)) = L(E, C(K))$ is the range of a norm-one projection when embedded in $L(E, C(K)^{**})$ and the latter space, as is well-known is a dual space. Hence $C(K, (E^*, w^*))$ has the FIP.

Remark 5 : As before we note that the proof implies that $C(K, (E, w^*)$ has the FIP iff it is the range of a norm one projection in its bidual.

Remark 6 : It may be worth remarking here that since Josefson-Nissenzweig theorem is equivalent to the existence of non-compact operator from E and an infinite compact set $\acute{K}$, $K(E, C(\acute{K}))$ is always a proper subspace of $L(E, C(\acute{K}))$.

2. FIP AND *M*-IDEALS

A closed subspace $F \subset E$ is said to be an M-ideal, if there is a projection P from E^* onto $F^{\perp}$ such that

$$\|f\| = \|Pf\| + \|f - Pf\| \text{ for all } f \in E^*.$$

This concept was introduced by Alfsen and. Effros and Behrends[1] monograph is now the source book for M-ideals and their structure theory.

Behrends and Harmand[2] have introduced the notion of a proper M-ideal, and according to them, a Banach space F is a proper M-ideal, if there is a Banach space E containing F (isometrically) such that F is an M-ideal in E, but not an M-summand. The containment of c_0 in l^{∞} is a typical example of this situation. Behrends and Harmand[2] proved that no dual space can be a proper M-ideal. We next prove a theorem which is more general than this and which in a way explains c_0 failing the FIP.

Theorem 1—If a Banach space F has the FIP then F can not be a proper M-ideal.

PROOF : Suppose F is a proper M-ideal. Then using Lemma 2.2 of Behrends and Harmand[2], we may assume that there is a Banach space E such that $E = F \oplus$ span $\{a_0\}$, and F is an M-ideal in E, but not an M-summand.

Consider in E the family of balls $\{B(a, \|a - a_0\|)\}_{a \in F}$. If we can show that $\bigcap_{a \in F} B(a, \|a - a_0\|) \neq \phi$ in F, then one can use this information as in Lemma 5.2 of Lindenstrauss[16], to get a projection of norm one from E onto F and then appeal to Proposition 2.1 of Behrends and Harmand[2] to get a contradiction.

To this end consider the family of closed balls $\{B(a, \|a - a_0\| + \epsilon)\}_{\substack{a \in F \\ \epsilon > 0}}$ in F. We shall show that any finite sub-collection from here has non-empty intersection.

Let $\{B(a_i, \|a_i - a_0\| + \epsilon_i)\}$ $1 \leqslant i \leqslant n$ be any finite family of balls considered in F.

Put $\epsilon = \min\{\epsilon_i\}$: Notice that in E, a_0 is common to all of them and each ball intersects F at least in a_i, so since F is an M-ideal, using the characterization of M-ideals interms of intersection properties of balls (see Behrends[1]), we conclude that $\bigcap_i B(a_i, \|a_i - a_0\| + \epsilon)$ non-empty in F. Consequently $\bigcap_i B(a_i, \|a_i - a_0\| + \epsilon_i)$ is non-empty in F. Suppose now F has the FIP, we can conclude that

$$\bigcap_{\substack{a \in F \\ \epsilon > 0}} B(a, \|a - a_0\| + \epsilon) = \bigcap_{a \in F} B(a, \|a - a_0\|)$$

in non-empty in F. Hence F can not have the FIP.

Remark : Any non-reflexive Banach space that is an M-ideal in its bidual (see Harmand and Lima[13]) is a proper M-ideal and hence any such space fails the FIP. Prime examples of this phenomenon are $K(l^p)$ $(1 < p < \infty)$ (see Harmand and Lima[13] and the references there). Our concluding result shows that these spaces can not even be renormed to have the FIP.

Theorem 2—If E is a non-reflexive Banach space such that E is an M-ideal in its bidual then no equivalent norm on E can have the FIP.

Proof : Since E is an M-ideal in its bidual, E is an Asplund space[13]. Hence every equivalent norm on E is Frechèt differentiable on a dense subset of E. Now using (1) of Theorem 3 of Godefroy[11] we see that if an equivalent norm has the FIP then it must be a dual norm. Now note that since E is non-reflexive, from Theorem 3.5 of Harmand and Lima[13] we get that this dual norm has an isomorphic copy of c_0 and hence from the Bessaga and Pełczyński theorem (Diestel[6]) we get that this equivalent dual norm has an isomorphic copy of l^∞. This contradicts the fact that E is an Asplund space. Hence no equivalent norm on E can have the FIP.

We conclude with some open problems.

(1) If E is a Banach space having the FIP and $F \subset E$ is a separable subspace, does there always exists a separable space G with the FIP such that $F \subset G \subset E$?.

I do not know how to do this even when $E = l^\infty$ and $F = c_0$. One should keep in mind here the well known fact that c_0 is not contained in (isomorphically) a separable complemented subspace of a dual space.

(2) If E has the FIP and F is finite dimensional does $E \check{\otimes} F$ have the FIP?

(3) If E has the Radon-Nikodým property and the FIP does $L^1(\lambda, E)$ have the FIP? (where λ is the Lebesgue measure on [0, 1]).

After completing the work on this paper we have realized that a forthcoming article of Cambern and Greim[4] deals with a situation similar to the one in Proposition 2, when they are looking at the question when is $C(K, (E^*, w^*))$ a dual space?.

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