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Completely Bounded Modules and Associated Extremal Problems

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In this paper we continue our study of certain finite dimensional Hilbert modules over the function algebra $\mathscr{A}(\Omega)$, $\Omega \subseteq \mathbb{C}^m$. We show that these modules are always completely bounded with the bound obtained as the matrix valued analogue of a certain scalar valued extremal problem. In particular, we obtain a necessary and sufficient condition for our module to be completely contractive. We produce a contractive module $\mathbb{C}^m_{\mathbf{N}}$ over $\mathscr{A}(\mathbb{B}^m)$ such that it is completely bounded with the complete bound equal to $\sqrt{m}$; that is, $\mathbb{C}^m_{\mathbf{N}}$ is not completely contractive.

INTRODUCTION

This is a continuation of our earlier work in [6]. We retain most of the notation from [6] and recall only a minimum of definitions and terminology, when necessary. For $\mathbf{v}$ in $\mathbb{C}^n$ and in $\mathbb{C}$, define the $(n+1)\times(n+1)$-matrix

$$N(\mathbf{v}, \lambda) = \begin{pmatrix} \lambda & \mathbf{v} \\ 0 & I_n \end{pmatrix}.$$

For $\mathbf{v}^i = (v_1^i, ..., v_n^i)$, $1 \leqslant i \leqslant m$, and $w = (w_1, ..., w_m)$ in a region Ω in $\mathbb{C}^m$, we consider the m-tuple of pairwise commuting operators

$$\mathbf{N} = (N_1, ..., N_m) = (N(\mathbf{v}^1, w_1), ..., N(\mathbf{v}^m, w_m)).$$

Here we study the bounded $\mathscr{A}(\Omega)$-module $\mathbb{C}^{n+1}_{\mathbf{N}}$ and determine when it is a completely bounded module.

1. $\mathbb{C}_N^{n+1}$ as a Completely Bounded Module over $\mathscr{A}(\Omega)$

In this section we assume that

(a) Ω is a bounded open neighbourhood of $\mathbf{0}$ in $\mathbb{C}^m$;

(b) Ω is convex and balanced;

(c) Ω admits a group of biholomorphic automorphisms, which acts transitively on Ω.

We note that (a), (b) implies Ω is polynomially convex [4, p. 67] and so by Oka's theorem [4, p. 84], $\mathscr{A}(\Omega)$ contains all functions holomorphic in a neighbourhood of $\bar{\Omega}$.

Following Arveson [1] and Douglas [2], we give the definition of a completely bounded $\mathscr{A}(\Omega)$-module.

For any function algebra A and an integer $k \geqslant 1$, let $\mathscr{M}_k(A) = \mathscr{A} \otimes \mathscr{M}_k(\mathbb{C})$ denote the algebra of $(k \times k)$-matrices with entries from $\mathscr{A}$. Here for $F = (f_{ij})$ in $\mathscr{M}_k(\mathscr{A})$, the norm $\|F\|$ of F is defined by

$$\|F\| = \operatorname{Sup}\{\|(f_{ij}(z))\| : z \in M\},$$

where M is the maximal ideal space for A. We note that for $\mathscr{A} = \mathscr{A}(\Omega)$, the maximal ideal space can be identified with [4, p. 67] and thus

$$\|F\| = \operatorname{Sup}\{\|(f_{ij}(z))\| : z \in \Omega\}.$$

1.1. Definition. If $\mathscr{H}$ *is a bounded Hilbert* $\mathscr{A}$-module, then $\mathscr{H} \otimes \mathbb{C}^k$ is a bounded $\mathscr{M}_k(A)$-module. For each k, let n_k denote the smallest bound for $\mathscr{H} \otimes \mathbb{C}^k$. The Hilbert $\mathscr{A}$-module is *completely bounded* if

$$n_\infty = \lim_{k \to \infty} n_k < \infty$$

and is *completely contractive* if $n_\infty \leqslant 1$.

Throughout this paper V will denote the $(m \times n)$-matrix whose rows $v^1, ..., v^m$ and we will write $\mathbf{v}_1, ..., \mathbf{v}_n$ for the columns of the matrix V. It was shown by the authors in [6, 2.2.4] that the map

$$\rho : \mathscr{P}(\Omega) \to L(\mathscr{C}^{n+1}),$$

$$\rho(p) = p(\mathbf{N}) = N(\nabla p(w) \cdot V, p(w))$$

extends continuously to $\operatorname{Hol}(\bar{\Omega})$. Indeed, we have

$$\rho(f) = f(\mathbf{N}) = N(\nabla f(w) \cdot V, f(w))$$

for all f in $\operatorname{Hol}(\bar{\Omega})$. It follows that the map $\rho \otimes I_k$: $\mathscr{M}_k(\mathscr{P}(\Omega)) \to$

$\mathcal{M}_k(\mathcal{L}(\mathbb{C}^{n+1}))$ extends continuously to $\mathcal{M}_k(\mathrm{Hol}(\bar{\Omega}))$ and we have (as shown in [6, 6.2.2])

$$(\rho \otimes I_k)(f_{ij}) = \begin{pmatrix} (f_{ij}(w)) & (D(f_{ij}))(w) \cdot (V \otimes I_k) \\ 0 & I_k \otimes (f_{ij}(w)) \end{pmatrix}.$$

Let X, Y be finite dimensional normed linear spaces and Ω be an open subset of X. A function $f: \Omega \subseteq X \to Y$ is said to be holomorphic if the Frechet derivative of f at w exists as a complex linear map from X to Y. Let $I = (i_1, ..., i_m)$ denote a multi-index of length $I = i_1 + \cdots + i_m$ and e_k denote the multi-index with a one in the kth position and zeros elsewhere. If $P: \Omega \to \mathcal{M}_k$ is a polynomial matrix valued function, i.e., $P(z) = (p_{ij}(z))$, where each p_{ij} is a polynomial function in m variables, then we can write

$$P(z) = \sum_I P_I (z - w)^I,$$

where each p_I is a scalar $(k \times k)$-matrix.

Now it is easy to verify that the derivative $DP(w)$ of p at w is

$$DP(w) = (p_{e_1}, ..., p_{e_m}),$$

which acts on a vector $\mathbf{v} = (v_1, ..., v_m)$ by

$$DP(w) \cdot \mathbf{v} = v_1 P_{e_1} + \cdots + v_m P_{e_m}.$$

Recall that $\mathscr{D}P(w)$ was defined in [6, 6.2.1] as

$$\left(\left(\frac{\partial}{\partial z_1} P \right)(w), ..., \left(\frac{\partial}{\partial z_m} P \right)(w) \right),$$

where

$$\left(\frac{\partial}{\partial z_l} P \right)(w) = \left(\frac{\partial}{\partial z_l} P_{ij} \right)(w).$$

Thus, it is easy to see that

$$(\mathscr{D}P)(w) \cdot (V \otimes I_k) = (DP(w) \cdot v_1, ..., DP(w) \cdot v_n).$$

Let $(X, \| \; \|_X)$ and $(Y, \| \; \|_Y)$ be normed linear spaces. By the operator norm for T in $L(X, \| \; \|_X; Y, \| \; \|_Y)$, we shall mean

$$\|T\|_X^Y = \mathrm{Sup}\{ \|Tx\|_Y : \|x\|_X \leqslant 1 \}.$$

As in [6], we choose a norm $\| \; \|_\Omega$ for $\mathbb{C}^m$ such that the unit ball of $\mathbb{C}^m$ with respect to this norm is Ω and write the corresponding normed linear space as $(\mathbb{C}^m, \|\ \|_\Omega)$. If no norms are mentioned for $\mathbb{C}^k$, it is understood to be the l_2-norm. We identify $\mathcal{M}_k$, the $(k\times k)$-matrices, with $\mathcal{L}(\mathbb{C}^k, \mathbb{C}^k)$ and the norm of such a matrix is the operator norm (with respect to the l_2-norm on $\mathbb{C}^k$) as above. By the same token, a linear transformation from $\mathcal{L}(X, Y)$ to $\mathcal{L}(X_1, Y_1)$ is an element of $\mathcal{L}(\mathcal{L}(X, Y), \mathcal{L}(X_1, Y_1))$ and possesses the operator norm.

1.2. Definition. For $w\in\Omega$, define

$$\mathbf{D}_{\mathcal{M}_k}\Omega(w)=\{DF(w)\in\mathcal{L}((\mathbb{C}^m, \|\ \|_\Omega); \mathcal{M}_k): F\in\mathcal{M}_k(\mathrm{Hol}(\bar{\Omega})),\ \|F\|\leqslant 1\}.$$

Of course, V determines a map $\rho_V:\mathcal{L}((\mathbb{C}^m, \|\ \|_\Omega); \mathcal{M}_k)\to(\mathcal{L}(\mathbb{C}^{kn}, \mathbb{C}^k))$ defined by

$$\rho_V(P_1, ..., P_m)=\left(\sum_{i=1}^{m} v_1^i P_i, ..., \sum_{i=1}^{m} v_n^i P_i\right)$$

and we set

$$M_\Omega^{\mathbb{C},k}(V, w)=\mathrm{Sup}\{\|\rho_V(T)\|_{\mathcal{L}(\mathbb{C}^{kn}, \mathbb{C}^k)}: T\in\mathbf{D}_{\mathcal{M}_k}\Omega(w)\}$$
$$M_\Omega^{\mathbb{C}}(V, w)=\mathrm{Sup}\{M^{\mathbb{C},k}(V, w): k\in\mathbb{N}\}.$$

1.3. *Remark.* Here we emphasize that for T in $\mathcal{L}(\mathbb{C}^m, \|\ \|_\Omega; \mathcal{M}_k)$ since $\|T\|_\Omega^{\mathcal{M}_k}=\mathrm{Sup}\{\|(T(z)\|_{\mathcal{M}_k}: z\in\Omega\}$, it follows that $\|T\|_\Omega^{\mathcal{M}_k}\leqslant 1$ is equivalent to saying that T maps Ω into the unit ball in $\mathcal{M}_k$.

The next lemma says that to determine when $\|\rho\otimes I_k\|\leqslant 1$, it is enough to consider those functions which vanish at a fixed but arbitrary point of Ω. However, to prove it we need the following result of Douglas, Muhly, and Pearcy [3, Proposition 2.2].

1.4. Lemma (DMP). *For $i=1, 2$ let T_i be a contraction on a Hilbert space $\mathcal{H}_i$ and let X be an operator mapping $\mathcal{H}_2$ into $\mathcal{H}_1$. A necessary and sufficient condition that the operator on $\mathcal{H}_1\oplus\mathcal{H}_2$ defined by the matrix $\left(\begin{smallmatrix} T_1 & X\\ 0 & T_2\end{smallmatrix}\right)$ be a contraction is that there exist a contraction C mapping $\mathcal{H}_2$ into $\mathcal{H}_1$ such that*

$$X=(I_{\mathcal{H}_1}-T_1T_1^*)^{1/2}\,C(I_{\mathcal{H}_2}-T_2^*T_2)^{1/2}.$$

We need some results about biholomorphic automorphisms of the unit ball in $\mathcal{M}_k$, which can be found in Harris [5, Theorem 2]. We collect the results we will need in the following proposition.

1.5. Proposition (Harris). *For each B in the unit ball $(\mathcal{M}_k)_1$ of $\mathcal{M}_k$, the Möbius transformation*

$$\varphi_B(A)=(I-BB^*)^{-1/2}(A+B)(I+B^*A)^{-1}(I-B^*B)^{1/2}$$

is a biholomorphic mapping of $(\mathcal{M}_k)_1$ *onto itself with* $\varphi_B(\mathbf{0}) = B$. *Moreover*,

$$\varphi_B^{-1} = \varphi_{-B}, \qquad \varphi_B(A)^* = \varphi_{B^*}(A^*), \qquad \|\varphi_B(A)\| \leqslant \varphi_{\|B\|}(\|A\|)$$

and

$$D\varphi_B(A)C = (I - BB^*)^{1/2}(I + AB^*)^{-1}C(I + B^*A)^{-1}(I - B^*B)^{1/2}$$

for A *in* $(\mathcal{M}_k)_1$ *and* C *in* $\mathcal{M}_k$.

Now, we prove

1.6. LEMMA. *If* $\|F(\mathbf{N})\| \leqslant 1$ *for all* F *in* $\mathcal{M}_k(\mathrm{hol}(\bar{\Omega}))$ *with* $\|F\| \leqslant 1$ *and* $F(w) = 0$, *then* $\|G(\mathbf{N})\| \leqslant 1$ *for all* G *in* $\mathcal{M}_k(\mathrm{Hol}(\bar{\Omega}))$ *with* $\|G\| \leqslant 1$.

Proof. Any G in $\mathcal{M}_k(\mathrm{Hol}(\bar{\Omega}))$ of norm less than or equal to one maps Ω into $(\mathcal{M}_k)_1$. In particular for w in Ω, $\|G(w)\| \leqslant 1$ and we can form the Möbius map $\varphi_{-G(w)}$ of $(\mathcal{M}_k)_1$. Consider the map $\varphi_{-G(w)} \circ G$, which maps w onto zero. Thus,

$$1 \geqslant \|\varphi_{-G(w)} \circ G(\mathbf{N})\| = \left\| \begin{pmatrix} \mathbf{0} & [D(\varphi_{-G(w)} \circ G)(w)] \cdot V \\ \mathbf{0} & \mathbf{0} \end{pmatrix} \right\|.$$

However,

$$[D(\varphi_{-G(w)} \circ G)(w)] \cdot V = ([(D\varphi_{-G(w)})(G(w))] \cdot [DG(w) \cdot \mathbf{v}_1], \ldots,$$
$$[(D\varphi_{-G(w)})(G(w))] \cdot [DG(w) \cdot \mathbf{v}_n]).$$

Let $R = (I - G(w)\,G(w)^*)^{-1/2}$ and $S = (I - G(w)\,G(w)^*)^{-1/2}$. Thus

$$[D(\varphi_{-G(w)} \circ G)(w)] \cdot V = (R(DG(w) \cdot \mathbf{v}_1)S, \ldots, R(DG(w) \cdot \mathbf{v}_n)S)$$
$$= R(DG(w) \cdot \mathbf{v}_1), \ldots, (DG(w) \cdot \mathbf{v}_n)) \begin{pmatrix} S & & \\ & \ddots & \\ & & S \end{pmatrix}.$$

We can apply Lemma 1.4 to conclude that

$$G(\mathbf{N}) = \left\| \begin{pmatrix} G(w) & DG(w) \cdot V \\ 0 & I_k \otimes G(w) \end{pmatrix} \right\|$$
$$= \left\| \begin{pmatrix} G(w) & DG(w) \cdot \mathbf{v}_1, \ldots, DG(w) \cdot \mathbf{v}_n \\ 0 & I_n \otimes G(w) \end{pmatrix} \right\| \leqslant 1,$$

which completes the proof of the lemma.

1.7. THEOREM. *$\mathbb{C}_{\mathbf{N}}^{m+1}$ is a completely contractive $\mathscr{A}(\Omega)$-module if and only if $M_{\Omega}^{\mathbb{C},k}(V, w) \leqslant 1$ for all k.*

The proof of this theorem is identical to that of Theorem 3.4 in [6].

With this lemma at our disposal, the proof of the following proposition becomes identical to that of Theorem 3.5 in [6].

1.8. PROPOSITION. *$\mathbb{C}_{\mathbf{N}}^{m+1}$ is a completely bounded $\mathscr{A}(\Omega)$-module with the bound $n_{\infty} = \max\{1, M_{\Omega}^{\mathbb{C}}(V, w)\}$. Further, if $M_{\Omega}^{\mathbb{C}}(V, w) > 1$ then there exists an invertible $(m+1) \times (m+1)$-matrix L such that $\|L\| \|L^{-1}\| = M_{\Omega}^{\mathbb{C}}(V, w)$ and $\mathbb{C}_{L\mathbf{N}L^{-1}}^{m+1}$ is a completely contractive $\mathscr{A}(\Omega)$-module.*

The following theorem is analogous to Theorem 4.1 in [6], where only scalar valued functions were considered.

1.9. THEOREM. *Let $w \in \Omega$ and θ_w be a biholomorphic automorphism of Ω such that $\theta_w(w) = \mathbf{0}$. Then,*

(a) $\mathbf{D}_{\mathscr{M}_k}\Omega(w) = \mathbf{D}_{\mathscr{M}_k}\Omega(\mathbf{0}) \cdot D\theta_w(w)$.

(b) $\mathbf{D}_{\mathscr{M}_k}\Omega(\mathbf{0}) = \{T \in \mathscr{L}(\mathbb{C}^m, \|\ \|_{\Omega}, \mathscr{M}_k) : \|\ \| \leqslant 1\}$.

(c) $M_{\Omega}^{\mathbb{C},k}(V, w) = M_{\Omega}^{\mathbb{C},k}(D\theta_w(w) \cdot V, \mathbf{0})$.

(d) $M_{\Omega}^{\mathbb{C},k}(V, \mathbf{0}) = \|\rho_V\|_{\mathscr{L}(\mathbb{C}^m, \|\ \|_{\Omega}; \mathscr{M}_k)}^{\mathscr{L}(\mathbb{C}^m, \mathbb{C}^k)}$.

Proof. Since the map $F \to F \circ \theta_w$ defines a bijection from $\{F \in \mathrm{Hol}(\bar{\Omega}) : \|F\| \leqslant 1$ and $F(\mathbf{0}) = 0\}$ to $\{F \in \mathrm{Hol}(\bar{\Omega}) : \|F\| \leqslant 1$ and $F(w) = 0\}$, (a) follows by the Chain rule.

To prove (b) first note that the Schwarz lemma as stated in Rudin [7, Theorem 8.12] actually applies to functions holomorphic from $\mathbb{C}^m$ to $\mathscr{M}_k$. Recall that $\mathbb{C}^m$ is given the norm $\|\cdot\|_{\Omega}$ with respect to which Ω becomes the unit ball and $\mathscr{M}_k$ has the usual uniform operator norm. Thus if F is in $\mathscr{M}_k(\mathrm{Hol}(\bar{\Omega}))$ with $\|F\| \leqslant 1$, then F must map Ω into $(\mathscr{M}_k)_1$ and the Schwarz lemma would guarantee that the linear operator $DF(\mathbf{0})$ maps Ω into $(\mathscr{M}_k)_1$. On the other hand if T is in $\mathscr{L}(\mathscr{C}^m, \|\ \|_{\Omega}; \mathscr{M}_k)$ and $\|T\| \leqslant 1$ then T automatically maps Ω into $(\mathscr{M}_k)_1$ and $T(\mathbf{0}) = \mathbf{0}$. Thus T lies in $\mathbf{D}_{\mathscr{M}_k}\Omega(\mathbf{0})$.

Part (c) follows from the definition of $M_{\Omega}^{\mathbb{C}}(V, w)$.

Part (d) is also immediate from the definition, once we note that

$$\begin{aligned}\|\rho_V\| &= \mathrm{Sup}\{\|\rho_V(T)\| : T \in \mathscr{L}(\mathbb{C}^m, \|\ \|_{\Omega}; \mathscr{M}_k), \|T\| \leqslant 1\} \\ &= \mathrm{Sup}\{\|\rho_V(T)\| : T \in \mathbf{D}_{\mathscr{M}_k}\Omega(\mathbf{0})\}.\end{aligned}$$

2. THE UNIT BALL, POLYDISK, AND SOME RELATED EXAMPLES

In this section, we explicitly compute $\|\rho_V\|$, when the domain under consideration is the unit ball in $\mathbb{C}^m$.

2.1. THEOREM. $M^{C}_{\mathbb{B}^m}(V, \mathbf{0}) = \|\rho_V\| = (\sum_{j=1}^{n} \|\mathbf{v}_j\|^2)^{1/2}$

Proof. Note that

$$M^{C,k}_{\mathbb{B}^m}(V, \mathbf{0})$$

$$= \operatorname{Sup}\{\|\rho_V(P_1, ..., P_m)\| : \|P_1 z_1 + \cdots + P_m z_m\| \leqslant 1 \text{ for all } (z_1, ..., z_m) \in \mathbb{B}^m\}$$

$$= \operatorname{Sup}\left\{\left(\left\|\sum_{j=1}^{n}\left(\sum_{k=1}^{m} P_k v_j^k\right)\left(\sum_{k=1}^{m} P_k v_j^k\right)^*\right\|\right)^{1/2} : (P_1, ..., P_m) \in \mathbf{D}_{\mathcal{M}_k}\mathbb{B}^n(\mathbf{0})\right\}$$

$$\leqslant \operatorname{Sup}\left\{\left(\sum_{j=1}^{n}\left\|\sum_{k=1}^{m} P_i v_j^k\right\|^2\right)^{1/2} : (P_1, ..., P_m) \in \mathbf{D}_{\mathcal{M}_k}\mathbb{B}^n(\mathbf{0})\right\}\left(\sum_{j=1}^{n}\|\mathbf{v}_j\|^2\right)^{1/2}$$

Since the bound for $M^{C,k}_{\mathbb{B}^m}(V, \mathbf{0})$ is independent of k, it follows that

$$M^{C}_{\mathbb{B}^m}(V, \mathbf{0}) = \left(\sum_{j=1}^{n}\|\mathbf{v}_j\|^2\right)^{1/2}$$

Now, Choosing $\mathbf{T} = (T_1, ..., T_m)$ with $T_k = e_{1k}$, where e_{1k} is the $(m \times m)$-matrix with 1 at the $(1, k)$ position and zeros elsewhere, it is trivially verified that $\|T(z)\| \leqslant 1$ for all z in $\mathbb{B}^m$. However,

$$\|\rho_V(T)\| = \left(\sum_{k=1}^{m} v_1^k T_k, ..., \sum_{k=1}^{m} v_n^k T_k\right)$$

$$= \left\|\begin{pmatrix}\mathbf{v}_1^t, \mathbf{v}_2^t, ..., \mathbf{v}_n^t \\ \text{——}\,\mathbf{0}\,\text{——}\end{pmatrix}\right\| = \left(\sum_{j=1}^{n}\|v_j^t\|^2\right)^{1/2}$$

COROLLARY. *If $\mathbb{C}^{n+1}_{\mathbf{N}}$ is a contractive module over $\mathscr{A}(\mathbb{B}^m)$ then it is a completely bounded module with bound at most $\sqrt{m}$.*

Proof. Assume without loss of generality that $\mathbf{N} = (N(\mathbf{v}^1, 0), ..., N(\mathbf{v}^m, 0)$. Recall that $\mathbb{C}^{m+1}_{\mathbf{N}}$ is contractive over $\mathscr{A}(\mathbb{B}^m)$ if and only if $\|V\| \leqslant 1$ [6, Theorem 4.1(d)]. However, by the preceding theorem it is completely contractive if and only if $\sum_{i,j=1}^{n} \|v_i^j\|^2 \leqslant 1$.

2.2. *The polydisk.* From [6], we know that $\mathbb{C}^{n+1}_{\mathbf{N}}$ is a contractive module over $\mathscr{A}(\mathbb{D}^m)$ if and only if $\max_{1 \leqslant k \leqslant m}\{\|v^k\|^2 \leqslant 1\}$. However, to answer the corresponding question about completely contractive modules, we need a rather exact description of those T in the unit ball of $\mathscr{L}(\mathbb{C}^m, \|\ \|_\Omega; \mathcal{M}_k)$, that is, $T: \mathbb{D}^m \to (\mathcal{M}_k)_1$, so that we can compute $\operatorname{Sup}\{\|\rho_V(T)\|_{\mathscr{L}(\mathbb{C}^{kn}, \mathbb{C}^k)}: T \in \mathbf{D}_{\mathcal{M}_k}\mathbb{D}^m(\mathbf{0})\}$. This at the moment seems to be a very difficult task. Of course, if we write $T: \mathbb{C}^m \to \mathcal{M}_k$ as $(T_1, ..., T_m)$ then $\|T_1\| + \cdots + \|T_m\| \leqslant 1$ implies $T: \mathbb{D}^m \to (\mathcal{M}_k)_1$.

However, the pair $((\begin{smallmatrix}1 & 0\\ 0 & 0\end{smallmatrix}), (\begin{smallmatrix}0 & 0\\ 0 & 1\end{smallmatrix}))$ which maps $\mathbb{D}^2$ into $(\mathcal{M}_2)_1$ with $\|T_1\| + \|T_2\| = 2$ shows that $\|T_1\| + \cdots + \|T_m\| \leqslant 1$ is not a necessary condition for T to map $\mathbb{D}^m$ into $\mathcal{M}_k$.

2.3. *A Family of Examples over the Ball Algebra.* Let $e_1, ..., e_m$ denote the usual basis in $\mathbb{C}^m$; set

$$\mathbf{N}_m = (N(\mathbf{0}, e_1), ..., N(\mathbf{0}, e_m)).$$

Thus, in this case $V = I_n$ and it follows that $\mathbb{C}_{\mathbf{N}}^{m+1}$ is a contractive module over the ball algebra [6, Theorem 4.1(d)]. However, $\mathbb{C}_{\mathbf{N}}^{m+1}$ is not a completely contractive module over $\mathscr{A}(\mathbb{B}^m)$. Indeed, Theorem 2.1, above, implies that

$$n_\infty(\mathbf{N}_m) = \sqrt{m}.$$

Thus,

$$n_\infty(\mathbf{N}_m) \to \infty \qquad \text{as} \quad m \to \infty$$

even though each $\mathbf{N}_m$ determines a completely contractive module. This example suggests that asymptotically it is possible to have a contractive module which is not even similar to a completely contractive module.

This family of examples perhaps should be compared to those of Varoupoulos [8].

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