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II_1 FACTORS, THEIR BIMODULES AND HYPERGROUPS

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ABSTRACT. In this paper, we introduce a notion that we call a hypergroup; this notion captures the natural algebraic structure possessed by the set of equivalence classes of irreducible bifinite bimodules over a II_1 factor. After developing some basic facts concerning bimodules over II_1 factors, we discuss abstract hypergroups. To make contact with the problem of what numbers can arise as index-values of subfactors of a given II_1 factor with trivial relative commutant, we define the notion of a dimension function on a hypergroup, and prove that every finite hypergroup admits a unique dimension function. we then give some nontrivial examples of hypergroups, some of which are related to the Jones subfactors of index $4\cos^2 \pi/(2n+1)$. In the last section, we study the hypergroup invariant corresponding to a bifinite module, which is used, among other things, to obtain a transparent proof of a strengthened version of what Ocneanu terms 'the crossed-product remembering the group.'

INTRODUCTION

In this paper, we introduce a notion that we call a hypergroup (although the term seems to have been used in the literature with somewhat differing definitions—see the opening remarks in §IV). Our hypergroup describes the natural algebraic structure possessed (with respect to tensor-products and contragredients) by the collection $\mathfrak{G}(N)$ of equivalence classes of irreducible bimodules over a II_1 factor N that are bifinite (in the sense of having finite left- and right-dimensions over N).

The first half of the paper is devoted to setting up the machinery. Although portions of this half can be gleaned from [C and P], these sections are included for the sake of completeness and setting up the notation and our models, and because there are some results here that are probably new—such, for instance, as the fact that for vectors in a bifinite bimodule, the notions of left- and right-boundedness are equivalent. The third section leads up to the fact that captures the contragredient axiom in the hypergroup.

The fourth section begins with the axiomatic definition of (our notion of) an abstract hypergroup. After developing some basic consequences of the axioms, we define the notion of a dimension function on a hypergroup and we prove the existence and uniqueness of such a function on a finite hypergroup. In case a finite hypergroup $\mathfrak{G}$ admits an outer action on a II_1 factor N (cf. Definition IV.7), and if $\alpha \to d_\alpha$ denotes the dimension function on $\mathfrak{G}$, it would follow that for each α in $\mathfrak{G}$, we can find a II_1 factor M_α containing a copy of

N as a subfactor with trivial relative commutant and index d_α^2. We exhibit some nontrivial examples of finite hypergroups; among them are n-element hypergroups $\mathcal{O}_n$ such that the value of the dimension function on the kth element is the 'Wenzl number' $(\sin k\pi/n+1)/(\sin \pi/n+1)$. It is our belief that every finite hypergroup admits an outer action on the hyperfinite II_1 factor R. If that were true, our examples would yield several old index numbers as well as several new ones such as $(1+\cos \pi/2n+1)^2$ and $(n+(n^2+4)^{1/2})^2/4$.

The final section concerns the 'hypergroup invariant' of a bifinite bimodule over a II_1 factor. We obtain a transparent proof of a fairly strong version—cf. Proposition V.3.—of what Ocneanu terms 'the extension remembering the group'. We finally describe the inclusion matrices governing the maps $T \to T \otimes_N \mathrm{id}_{\mathcal{H}}$, thus showing that the hypergroup invariant of the bimodule contains the data of the AF-algebra built from the spaces ${}_N\mathcal{L}_N(\mathcal{H}^n)$ of N-bilinear self-maps of the nth tensor power (over N) of $\mathcal{H}$; in particular, the hypergroup describes many of the 'reflection symmetries' possessed by the Bratteli diagram of the 'tower of the basic construction'.

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I. Preliminaries

The symbols N and M will always denote II_1 factors with separable preduals; Hilbert spaces will be assumed to be separable and will be denoted by such symbols as $\mathcal{H}$ and $\mathcal{K}$. The symbol tr will denote the unique faithful normal tracial state on any II_1 factor and the symbol $L^2(N)$ will denote the Hilbert space underlying the regular representation of N—i.e., the completion of N with respect to the norm $\|x\|_2 = \{\mathrm{tr}\, x^*x\}^{1/2}$. When convenient, we shall use the symbol $\|\ \|_\infty$ to denote the usual operator norm on N. The canonical antiunitary involution on $L^2(N)$—which restricts on N to the usual adjoint—will be denoted by J_N; as above, we shall regard N as a subset of $L^2(N)$.

1. **Definition.** (a) A left- (resp. right-) N-module is a Hilbert space $\mathcal{H}$ equipped with a normal *-homomorphism π of N (resp., π^0 of N^0, the opposite algebra of N) into $\mathcal{L}(\mathcal{H})$.

(b) An N-bimodule is a Hilbert space $\mathcal{H}$ equipped with normal *-homomorphisms π and π^0 of N and N^0 respectively into $\mathcal{L}(\mathcal{H})$ satisfying $\pi(N) \subset \pi^0(N^0)'$. □

Usually, when a left- (resp., right-) N-module $\mathcal{H}$ is given, we shall simply write $a \cdot \xi$ (resp., $\xi \cdot a$) for the action of a in N (resp., of a^0 in N^0, where $a \mapsto a^0$ denotes the natural anti-isomorphism of N on N^0) on the vector ξ in $\mathcal{H}$.

2. **Definition.** (a) A left (resp., right) N-module is said to be left- (resp., right-) finite if $\pi(N)'$ (resp., $\pi^0(N^0)'$) is also a finite von Neumann algebra (in which case, of course, it is automatically a II_1 factor).

(b) A bimodule is called bifinite if it is left- as well as right-finite. □

3. **Facts.** We gather together some well-known facts that may be found, for instance, in [J].

Let $\mathfrak{H}$ be a left-finite left N-module and let the action of N on $\mathfrak{H}$ be denoted by π.

(1) Let $\xi \in \mathfrak{H}$ be nonzero; let p_ξ and p'_ξ denote the projections onto $[\pi(N)'\xi]$ and $[\pi(N)\xi]$ respectively, where $[\mathscr{S}]$ denotes the closed subspace generated by the set $\mathscr{S}$; then ($p_\xi \in N$ and $p'_\xi \in N'$ and) the ratio $(\mathrm{tr}\, p_\xi / \mathrm{tr}\, p'_\xi)$ is a finite positive constant which is independent of the vector ξ. This constant is denoted by $\dim_N \mathfrak{H}$.

(2) Two left-finite N-modules $\mathfrak{H}$ and $\mathfrak{K}$ are equivalent—i.e., there exists a unitary operator $u : \mathfrak{H} \to \mathfrak{K}$ which is N-linear in the sense that $u(a \cdot \xi) = a \cdot u\xi$ for all a in N and ξ in $\mathfrak{H}$—if and only if $\dim_N \mathfrak{H} = \dim_N \mathfrak{K}$.

(3) $\dim_N \mathfrak{H} = (\dim_{\pi(N)'} \mathfrak{H})^{-1}$.

(4) If $p \in N$ is a projection, then $\dim_{pNp} p\mathfrak{H} = (\mathrm{tr}\, p)^{-1} \dim_N \mathfrak{H}$.

(5) If $p' \in N'$ is a projection, then $\dim_{p'N} p'\mathfrak{H} = (\mathrm{tr}\, p') \dim_N \mathfrak{H}$.

(6) If $\mathfrak{H} \otimes \mathbf{C}^n$ is made a left N-module in the natural way—via $\pi \otimes 1$—then $\dim_N \mathfrak{H} \otimes \mathbf{C}^n = n \cdot \dim_N \mathfrak{H}$.

(Clearly there is also a right-version of each of the above facts.)

4. *Notation.* We shall write $M_{m\times n}(A)$ for the set of $m \times n$ matrices whose entries come from A. For a Hilbert space $\mathfrak{H}$, we shall consider $M_{m\times n}(\mathfrak{H})$ as a Hilbert space with $\|\xi\|^2 = \sum \|\xi_{ij}\|^2$. As is customary, we shall write $M_n(\cdot)$ for $M_{n\times n}(\cdot)$. It is clear that $M_n(N)$ is a II_1 factor. When $\mathfrak{H}$ is a left- (resp., right-) N-module, matrix multiplication naturally induces a left $M_m(N)$-module structure (resp., a right $M_n(N)$-module structure) on $M_{m\times n}(\mathfrak{H})$.

We now consider matrices of nonintegral sizes. Suppose r, s are positive real numbers. Fix any integer n that is larger than both r and s. Select projections p and q in $M_n(N)$ such that $\mathrm{tr}\, p = r/n$ and $\mathrm{tr}\, q = s/n$. We then let $M_{r\times s}(A) = \{\xi \in M_n(A) : \xi = p\xi q\}$ where A denotes either N or $L^2(N)$. When $r = s$, we shall pick $p = q$ and we shall abbreviate $M_{r\times r}(\cdot)$ to $M_r(\cdot)$. It is clear that $M_r(N)$ is also a II_1 factor and that $M_{r\times s}(L^2(N))$ has a natural left $M_r(N)$-module structure and right $M_s(N)$-module structure. To be accurate, we should perhaps call the above "an $n-p-q$ model for $M_{r\times s}(A)$", but we may and do dispense with such subtleties for the following reasons: (i) if $M_i = p_i M_{n_i}(N) p_i$ where p_i is a projection in $M_{n_i}(N)$ with $\mathrm{tr}\, p_i = r/n_i$ for $i = 1, 2$, then there exists a partial isometry $u \in M_{n_1\times n_2}(N)$ such that the map $x \mapsto u^* x u$ defines a von Neumann algebra isomorphism of M_1 onto M_2; (ii) if $\mathfrak{H}_i$ is the $n_i - p_i - q_i$ model for $M_{r\times s}(L^2(N))$, then there exists partial isometries $u, v \in M_{n_1\times n_2}(N)$ such that $x \mapsto u^* x v$ defines a unitary isomorphism of $\mathfrak{H}_1$ onto $\mathfrak{H}_2$.

5. **Examples.** (a) $L^2(N)$ is naturally an N-bimodule; the vector 1 is cyclic for the left- as well as the right-actions of N; further, in this case, $\pi^0(N^0)$ agrees with $\pi(N)'$; it follows from the definitions that $\dim_N L^2(N) = 1$ and $\dim_{N^0} L^2(N) = 1$.

(b) As has already been noted, $M_{r\times s}(L^2(N))$ is a left $M_r(N)$-module; an easy application of the facts listed in §3 shows that $\dim_{M_r(N)} M_{r\times s}(L^2(N)) = s/r$.

6. **Finite extensions of N.** By an extension of N, we shall mean a II_1 factor M containing N as a subfactor. (To be precise, we must consider a pair (M, ρ) where ρ is a faithful normal unital *-homomorphism of N into M.)

We shall say the extension M is finite if $\dim_N L^2(M)$ is finite. In fact, this dimension is just the Jones index $[M : N]$ and it should be recalled that if $\mathfrak{H}$ is a left M-module, then $\mathfrak{H}$ is left M-finite if and only if $\mathfrak{H}$ is left N-finite and that $\dim_N \mathfrak{H} = [M : N] \cdot \dim_M \mathfrak{H}$.

Let $0 < r, s < \infty$ and let $\alpha : N \to M_r(N)$ be a unital normal *-homomorphism. Then $M_{r\times s}(L^2(N))$ is left-finite as an $M_r(N)$-module; it follows from the last paragraph that $M_{r\times s}(L^2(N))$ is left-finite as an N-module (where the action is given by $a \cdot \xi = \alpha(a)\xi$) if and only if $[M_r(N) : \alpha(N)] < \infty$. It further follows from the last paragraph that

$$\dim_{\alpha(N)} M_{r\times s}(L^2(N)) = [M_r(N) : \alpha(N)] \cdot s/r. \tag{6.1}$$

7. **Definitions and some notation.** If $\alpha : N \to M_r(N)$ is a unital normal *-homomorphism such that $[M_r(N) : \alpha(N)] < \infty$, we shall call such an α a 'cofinite morphism of N' and we shall write $\mathscr{L}(r \times s; \alpha)$ for $M_{r\times s}(L^2(N))$ viewed as a left N-module as in the last paragraph. Dually, we shall write $\mathscr{R}(s \times r; \alpha)$ for $M_{s\times r}(L^2(N))$ viewed as a right N-module via α. In conjunction, given a pair of cofinite morphisms $\alpha : N \to M_r(N)$ and $\beta : N \to M_s(N)$, then the Hilbert space $M_{r\times s}(L^2(N))$ acquires, naturally, the structure of an N-bimodule which we shall denote by $\mathscr{B}(r \times s; \alpha, \beta)$. Given a cofinite morphism $\alpha : N \to M_r(N)$, we shall write $d_\alpha = r$.

The symbols $\alpha, \beta, \gamma, \ldots$ will, in this paper, always denote cofinite morphisms. The symbol 1 will also be used, when the context is clear, to denote the identity automorphism, when viewed as a cofinite morphism—e.g., $d_1 = 1$. Finally, we shall consistently use the notation $\mathfrak{H}_\alpha = \mathscr{B}(1 \times d_\alpha; 1, \alpha)$. Thus, for instance, if α is an automorphism of N, then $\mathfrak{H}_\alpha$ is just $L^2(N)$ as a Hilbert space, while the actions are given by $a \cdot \xi \cdot b = a\xi\alpha(b)$.

Also, if $\mathfrak{H}$ and $\mathfrak{K}$ are N-bimodules, we shall write ${}_N\mathscr{L}(\mathfrak{H}, \mathfrak{K})$, $\mathscr{L}_N(\mathfrak{H}, \mathfrak{K})$ and ${}_N\mathscr{L}_N(\mathfrak{H}, \mathfrak{K})$ for the spaces of bounded operators from $\mathfrak{H}$ to $\mathfrak{K}$ which are, respectively, left-, right- and bi-N-linear.

8. **Lemma.** (i) *$T \in {}_N\mathscr{L}(\mathfrak{H}_\alpha)$ if and only if there exists a matrix $T^\sim \in M_{d_\alpha}(N)$ such that $T\xi = \xi T^\sim$ (matrix multiplication) for all ξ in $\mathfrak{H}_\alpha$;*

(ii) *$T \in {}_N\mathscr{L}_N(\mathfrak{H}_\alpha)$ if and only if there exists a matrix $T^\sim$ as in* (i) *above which further satisfies $T^\sim \in M_{d_\alpha}(N) \cap \alpha(N)'$.*

Proof. Assume that we are working with 'the $n-1-q$ model' of $M_{1\times d_\alpha}(L^2(N))$. Let $q^{(1)}, \ldots, q^{(n)}$ denote the rows of the matrix q and note that—due to the idempotence of q—each $q^{(j)}$ may be regarded as a vector in $\mathfrak{H}_\alpha$; in fact, note that

$$\xi = ((\xi_j)) \in \mathfrak{H}_\alpha \Leftrightarrow \xi = \xi q \Leftrightarrow \xi = \sum \xi_j \cdot q^{(j)}. \tag{8.1}$$

Let $Tq^{(i)} = ((t_{ij}))$; the proof of the assertion, with $T^\sim = ((t_{ij}))$, would be complete once we show that $T^\sim = qT^\sim q$ and that $t_{ij} \in N$ for all i and j. The first assertion follows from the definitions; as for the second, note that if

$x \in N$ and $i, j \leq n$, then

$$\|xt_{ij}\|_2^2 \leq \left(\sum_j \|xt_{ij}\|_2^2\right) \leq \|x \cdot Tq^{(i)}\|_2^2 = \|T(x \cdot q^{(i)})\|_2^2$$

$$\leq \|T\|^2 \cdot \left(\sum_j \|xq_{ij}\|_2^2\right) \leq \|T\|^2 \cdot \left(\sum_j \|q_{ij}\|_\infty^2\right) \|x\|_2^2;$$

it follows that each t_{ij} is a 'right-bounded vector' in $L^2(N)$ and must consequently belong to N; it is clear conversely that any T as above is necessarily left-N-linear.

As for the second assertion, suppose $T \in {}_N\mathscr{L}_N(L^2(N))$; it follows from (i) that $T\xi = \xi T^\sim$ for some $T^\sim$ in $M_{d_\alpha}(N)$; it is easily seen that $T(\xi \cdot a) = (T\xi) \cdot a$ for all $a \in N$ if and only if $\alpha(a)T^\sim = T^\sim\alpha(a)$ for all $a \in N$. The proof is complete. □

In anticipation of the next and further results to come, we make the following definition.

9. **Definition.** Two cofinite morphisms α and α' will be said to be outer equivalent if there exists a (partial isometry) u in $M_{d_\alpha \times d_{\alpha'}}(N)$ such that $\alpha'(x) = u^*\alpha(x)u$ for all x in N and $uu^* = \alpha(1)$. (Note that then, necessarily $d_\alpha = \mathrm{tr}\alpha(1) = \mathrm{tr}\alpha'(1) = d_{\alpha'}$.)

10. **Proposition.** *Every bifinite bimodule is equivalent to $\mathfrak{H}_\alpha$ for some cofinite morphism α; further under the above passage from bimodule to cofinite morphism, isomorphism of bimodules corresponds precisely to outer equivalence of cofinite morphisms.*

Proof. Let $d = \dim_N \mathfrak{H}$ where $\mathfrak{H}$ is the bimodule under consideration. In view of §3(2), we may assume that, as a Hilbert space, $\mathfrak{H} = M_{1\times d}(L^2(N))$. Deduce now from Lemma 8 that there exists a map $\alpha : N \to M_d(N)$ such that $\xi \cdot a = \xi\alpha(a)$ for all a in N. It is a matter of routine verification to see that α is indeed a unital *-homomorphism—whose normality is ensured by that of the right action—and it then follows that $\mathfrak{H}$ is just $\mathfrak{H}_\alpha$.

As for the second assertion, begin by noting that $\dim_N \mathfrak{H}_\alpha = d_\alpha$ (cf. §5.(2) and the definition of d_α). It follows that if $\mathfrak{H}_\alpha \simeq \mathfrak{H}_{\alpha'}$, then $d_\alpha = d_{\alpha'}$; hence we may assume that the underlying Hilbert space for $\mathfrak{H}_\alpha$ as well as $\mathfrak{H}_{\alpha'}$ is $M_{1\times d}(L^2(N))$. Suppose that $u^\sim$ is a unitary operator that is N-bilinear. Deduce from our reduction and Lemma 8 that there exists a unitary $u \in M_d(N)$ such that $u^\sim\xi = \xi u$ for all $\xi \in \mathfrak{H}$. The right N-linearity of $u^\sim$ ensures that

$$\xi\alpha(a)u = u^\sim(\xi \cdot a) = u^\sim\xi \cdot a = \xi u\alpha'(a);$$

the desired conclusion follows immediately. □

(We remark here that there is a right version of the above proposition, namely that each bifinite bimodule determines—uniquely up to outer automorphism—a cofinite morphism $\alpha^\#$ such that $\mathfrak{H} \simeq \mathscr{B}(d_{\alpha^\#} \times 1; \alpha^\#, 1)$; we do not prove this separately here since it will follow from general assertions about the contragredient that we shall later establish.)

We now wish to discuss some representation-theoretic aspects of the theory of bifinite bimodules. The starting point is the deduction from §8(ii)—and

the basic fact from Jones' theory of subfactors that a subfactor of finite index necessarily has a finite-dimensional relative commutant—that if $\mathcal{H}$ is a bifinite bimodule, then $A = {}_N\mathcal{L}_N(\mathcal{H})$ is a finite-dimensional C^*-algebra of operators. (If $\mathcal{H} \sim \mathcal{H}_\alpha$, then $A \simeq M_{d_\alpha}(N) \cap \alpha(N')$.) Note that there is a natural bijection between projections in A and sub-bimodules of $\mathcal{H}$ such that the Murray-von Neumann notion of equivalence of projections corresponds to equivalence of sub-bimodules. The following lemma must be obvious; its proof is omitted.

11. **Lemma.** (a) *The following conditions are equivalent*:
   (i) $\mathcal{H}_\alpha$ *is an irreducible bimodule*;
   (ii) $M_{d_\alpha}(N) \cap \alpha(N)' \simeq \mathbf{C}$;
(b) *The following conditions are equivalent*:
   (i) $\mathcal{H}_\alpha$ *is isotypical—i.e., any two nonzero submodules contain further nonzero sub-bimodules which are equivalent*;
   (ii) $M_{d_\alpha}(N) \cap \alpha(N)' \simeq M_m(\mathbf{C})$ *for some integer* m.

We shall say that a cofinite morphism α is irreducible or isotypical when $\mathcal{H}_\alpha$ has that property.

12. **Schur's lemma.** *If* α *is irreducible and* β *is arbitrary, then* ${}_N\mathcal{L}_N(\mathcal{H}_\alpha, \mathcal{H}_\beta)$ *consists only of scalar multiples of isometries.*

Proof. If $t \in {}_N\mathcal{L}_N(\mathcal{H}_\alpha, \mathcal{H}_\beta)$ has polar decomposition $t = u|t|$, it is easy to see that $u \in {}_N\mathcal{L}_N(\mathcal{H}_\alpha, \mathcal{H}_\beta)$ and $|t| \in {}_N\mathcal{L}_N(\mathcal{H}_\alpha)$ and consequently $|t|$ is a nonnegative scalar; it follows that u is either the zero operator or an isometry and the proof is complete.

13. **Theorem.** *Every bifinite bimodule* $\mathcal{H}$ *admits a decomposition.*

$$\mathcal{H} \simeq \bigoplus_{i=1}^{n} (\mathcal{H}_i \oplus \mathbf{C}^{m_i})$$

where each $\mathcal{H}_i$ *is an irreducible bimodule and* m_i *are integers. Such a decomposition is unique up to permutations and isomorphisms of the* $\mathcal{H}_i$.

Proof. Assume, without loss of generality that $\mathcal{H} = \mathcal{H}_\alpha$. As has already been noted, bifiniteness ensures the finite-dimensionality of $A = {}_N\mathcal{L}_N(\mathcal{H}_\alpha)$.

To prove existence of the decomposition, let $\{p_1, \ldots, p_k\}$ be the partition of 1 into minimal central projections of A; each p_i can be decomposed as $p_i = \sum_{j=1}^{m_i} q_j^i$, where the q_j^i are minimal projections in A. It follows fairly easily that $\mathcal{H} = \bigoplus_i (\bigoplus_j q_j^i \mathcal{H})$ is a decomposition of the desired sort.

Uniqueness follows from the fact that every partition of 1 into minimal projections must necessarily refine any partition of 1 into central projections. In fact, it is evident that the decomposition into isotypical summands is canonical, while the decomposition into irreducible summands is determined only up to conjugation by a unitary operator in A. □

We conclude this section with a brief discussion of the contragredient of a module.

14. **Definition.** Let $\mathcal{H}$ be a left- (resp., right-) N-module. By (a model of) the contragredient of $\mathcal{H}$, we shall mean any right- (resp., left-) N-module $\mathcal{H}^*$ for

which there exists an anti-unitary operator $J: \mathfrak{H} \to \mathfrak{H}^{\#}$ satisfying $J(a \cdot \xi) = (J\xi) \cdot a^*$ (resp., $J(\xi \cdot a) = a^* \cdot (J\xi)$) for all a in N and ξ in $\mathfrak{H}$.

If $\mathfrak{H}$ is an N-bimodule, and if $\mathfrak{H}^{\#}$ is any N-bimodule for which there exists an anti-unitary operator $J: \mathfrak{H} \to \mathfrak{H}^{\#}$ satisfying $J(a \cdot \xi \cdot b) = b^* \cdot J\xi \cdot a^*$ for all a, b in N and ξ in $\mathfrak{H}$, then the bimodule $\mathfrak{H}^{\#}$ will be called (a model for) the contragredient of the bimodule $\mathfrak{H}$. □

Note that the contragredient clearly exists and is determined up to unitary isomorphism (since the composite of two anti-unitaries is unitary).

15. **Lemma.** $\mathscr{B}(d_\alpha \times d_\beta; \alpha, \beta)^{\#} \simeq \mathscr{B}(d_\beta \times d_\alpha; \beta, \alpha)$.

Proof. Define $J: M_{d_\alpha \times d_\beta}(L^2(N)) \to M_{d_\beta \times d_\alpha}(L^2(N))$ by $(J\xi)_{ij} = J_N \xi_{ji}$; the fact that J_N establishes the self-contragredience of the bimodule $L^2(N)$ together with a routine computation involving simple matrix-multiplication, suffices to complete the verification that the above J does the required job. □

16. *Notation.* if α is a cofinite morphism of N, then, by (a model for) $\alpha^{\#}$, we shall mean any cofinite morphism of N such that $\mathfrak{H}_{\alpha^{\#}} \simeq (\mathfrak{H}_\alpha)^{\#}$.

Note that if $\mathfrak{H}$ is a left N-module, then $\dim_N \mathfrak{H} = \dim \mathfrak{H}_N^{\#}$; in particular $d_{\alpha^{\#}} = \dim \mathfrak{H}_{\alpha_N}$, and we see (cf. 6.1) that

$$d_\alpha(d_{\alpha^{\#}}) = [M_{d_\alpha}(N) : \alpha(N)] = [M_{d_{\alpha^{\#}}}(N) : \alpha^{\#}(N)],$$

the second equality following from the obvious fact that $(\mathfrak{H}^{\#})^{\#} \simeq \mathfrak{H}$. □

II. Bounded vectors

Henceforth, the symbols $\mathfrak{H}$ and $\mathfrak{K}$ will always denote bifinite N-bimodules and the symbols $\alpha, \beta, \gamma, \kappa, \rho$, and μ will be reserved for cofinite morphisms of N.

1. **Definition.** A vector ξ in $\mathfrak{H}$ will be said to be left- (resp., right-) bounded for (the right- (resp., left-) action of) N if there exists a constant C such that $\|\xi \cdot a\| \leq C\|a\|_2$ (resp., $\|a \cdot \xi\| \leq C\|a\|_2$) for all a in N.

2. **Lemma.** *Suppose $N \subset M$ are II_1 factors such that $[M : N] < \infty$. Let $\mathfrak{H}$ be a left M-module (so that it can also be regarded as a left N-module). The following conditions on a vector ξ in $\mathfrak{H}$ are equivalent:*

(i) *ξ is right-bounded for the left-action of M;*

(ii) *ξ is right-bounded for the left-action of N.*

Proof. Since (i) clearly implies (ii), assume (ii). As is well known—cf. [PP]—the assumption of finite index implies that there exists a finite set $\{\lambda_1, \ldots, \lambda_n\}$ in M and a positive constant K such that any element x in M has a representation of the form $x = \sum_1^n \lambda_i a_i$ where $a_i \in N$ and $\|a_i\|_2 \leq K\|x\|_2$ for each i. It then follows that

$$\begin{aligned} \|x \cdot \xi\| &\leq \sum_1^n \|\lambda_i\|_\infty \cdot \|a_i \cdot \xi\| \\ &\leq \sum_1^n \|\lambda_i\|_\infty \cdot C \cdot \|a_i\|_2 \\ &\leq C' \cdot \|x\|_2 \end{aligned}$$

where C' is an appropriate constant, thus completing the proof. □

3. **Lemma.** *The following conditions on a vector ξ in $M_{r\times s}(L^2(N))$ are equivalent:*

(i) *ξ is right-bounded for the left-action of $M_r(N)$;*

(ii) *$\xi \in M_{r\times s}(N)$.*

Proof. Since (ii) clearly implies (i), assume (i). We are, thus, given an integer $n \geq r, s$, and projections $p, q \in M_n(N)$ satisfying $\mathrm{tr}\, p = r/n$ and $\mathrm{tr}\, q = s/n$, and we are given that $\xi = p\xi q$ and that ξ is right-bounded for the left-action of $pM_n(N)p$ on $pM_n(L^2(N))q$.

Note, first that, for any $b \in M_n(N)$, the vector $\xi \cdot b$ is right-bounded for the left-action of $pM_n(N)p$ on $pM_n(L^2(N))$. Apply this to $b = up$, where u is an arbitrary unitary element of $M_n(N)$, to find that the vector $\xi \cdot up$ is right-bounded for the left-action of $pM_n(N)p$ on $pM_n(L^2(N))p$. However, $pM_n(L^2(N))p$ is the standard left $pM_n(N)p$-module and it follows that necessarily, $\xi \cdot up \in pM_n(N)p$.

It follows, in particular that $\xi \cdot upu^* \in M_n(N)$, and this is true for every unitary u in $M_n(N)$. On the other hand, we can clearly find unitary elements $u_1, \ldots, u_n$ in $M_n(N)$ such that $a = \sum_1^n u_i p u_i^*$ is an invertible element of $M_n(N)$. We then deduce from the above that $\xi \cdot a \in M_n(N)$, and deduce finally that $\xi = (\xi \cdot a) \cdot a^{-1} \in M_n(N)$; since $\xi = p \cdot \xi \cdot q$, we have $\xi \in pM_n(N)q$, and the proof of the lemma is complete. □

Obviously, there is a dual proposition valid for vectors that are left-bounded under an appropriate right-action. On the other hand, Lemma 2 implies that a vector ξ in $\mathscr{B}(d_\alpha \times d_\beta; \alpha, \beta)$ is left- (resp., right-) bounded for the right- (resp., left-) action of N if and only if it is left- (resp., right-) bounded for the right- (resp., left-) action of $M_{d_\beta}(N)$ (resp., $M_{d_\alpha}(N)$). Coupled with Lemma 3, we have

4. **Proposition.** *A vector ξ in a bifinite N-module $\mathfrak{H}$ is left-bounded if and only if it is right-bounded. If $\mathfrak{H} = \mathscr{B}(d_\alpha \times d_\beta; \alpha, \beta)$, this happens precisely when all the entries of the matrix ξ come from N.*

Proof. This follows easily from the preceding remark and §3.

5. *Notation.* Given a bifinite N-bimodule $\mathfrak{H}$, we may thus ignore the qualifying adjectives left and right and simply talk about bounded vectors. We shall denote the class of all bounded vectors in $\mathfrak{H}$ by $\mathfrak{H}_0$.

6. **Proposition.** *Let $\mathfrak{H}$ and $\mathfrak{K}$ denote bifinite N-bimodules.*

(i) *$\mathfrak{H}_0$ is stable under both the left- and right-actions of N;*

(ii) *$\mathfrak{H}_0$ is dense in $\mathfrak{H}$;*

(iii) *The association $T \to T|\mathfrak{H}_0$ sets up a bijection between operators T in ${}_N\mathscr{L}_N(\mathfrak{H}, \mathfrak{K})$ and the class ${}_N L_N(\mathfrak{H}_0, \mathfrak{K}_0)$ of linear maps S from $\mathfrak{H}_0$ into $\mathfrak{K}_0$ that are N-bilinear in the sense that $S(a \cdot \xi \cdot b) = a \cdot S\xi \cdot b$ for all a, b in N and ξ in $\mathfrak{H}_0$.*

Proof. (i) Let $a, b \in N$ and $\xi \in \mathfrak{H}_0$ be fixed; for arbitrary x in N, we have

$$\begin{aligned} \|x \cdot (a \cdot \xi \cdot b)\| &\leq \|b\|_\infty \cdot \|xa \cdot \xi\| \\ &\leq \|b\|_\infty \cdot C\|xa\|_2 \\ &\leq C\|b\|_\infty \|a\|_\infty \|x\|_2, \end{aligned}$$

thus establishing that $a \cdot \xi \cdot b \in \mathfrak{H}_0$.

(ii) This follows immediately from the obvious fact that $M_{1\times d}(N)$ is dense in $M_{1\times d}(L^2(N))$.

(iii) If $\xi \in \mathfrak{H}_0$, the fact that $T\xi \in \mathfrak{R}_0$ follows from the inequalities

$$\|a \cdot T\xi\| = \|T(a \cdot \xi)\| \leq \|T\| \cdot \|a \cdot \xi\| \leq C\|T\| \cdot \|a\|_2 ;$$

thus the restriction of T maps $\mathfrak{H}_0$ to $\mathfrak{R}_0$. Suppose conversely that we are given a linear map $S : \mathfrak{H}_0 \to \mathfrak{R}_0$ which is N-bilinear. We may, and do assume that $\mathfrak{H} = \mathfrak{H}_\alpha$ and that $\mathfrak{R} = \mathfrak{H}_\beta$ for some cofinite morphisms α and β of N. Fix a suitably large integer n and assume that $\alpha, \beta : N \to M_n(N)$ and satisfy $\mathrm{tr}\,\alpha(1) = d_\alpha/n$ and $\mathrm{tr}\,\beta(1) = d_\beta/n$. Let λ^i denote the ith row of the matrix $\alpha(1)$. Then, $\lambda^i \in M_{1\times d_\alpha}(N) = \mathfrak{H}_0$ and so $S\lambda^i = s^i \in \mathfrak{R}_0 = M_{1\times d_\beta}(N)$; suppose $s^i = (s_{i1}, \ldots, s_{in})$. Since $s^i \in \mathfrak{R}_0$, it follows that $s^i = s^i \cdot \beta(1)$ and that $s_{ij} \in N$; note now that if $\xi = (\xi_1, \ldots, \xi_n) \in (\mathfrak{H}_\alpha)_0$, then

$$S\xi = S\left(\sum_1^n \xi_i \cdot \lambda^i\right) = \sum \xi_i \cdot s^i = \xi S^\sim$$

where the last term denotes the matrix-'product' of the row-vector ξ and the matrix $S^\sim$ whose (i, j)th entry is s_{ij}. Simply define $T\xi = \xi S^\sim$ for ξ in $\mathfrak{H}$—noting that this makes sense since the entries s_{ij} belong to N, which acts from the right on $L^2(N)$—and note that T is a bounded operator from $\mathfrak{H}$ to $\mathfrak{R}$ which extends S and consequently inherits N-bilinearity from S. □

We now relate bifinite N-bimodules with the theory of Hilbert modules.

7. **Proposition.** *For any bifinite N-bilinear bimodule $\mathfrak{H}$, there exists a unique mapping $\mathfrak{H}_0 \times \mathfrak{H}_0 \mapsto N$, denoted $(\xi, \eta) \mapsto \langle \xi, \eta\rangle_N$ which satisfies the following relations, for arbitrary ξ, η, ζ in $\mathfrak{H}_0$ and a in N:*

(a) $\langle \xi, \xi\rangle_N$ *is a positive element of N that can be zero only if $\xi = 0$;*
(b) $\langle \xi, \eta\rangle_N = (\langle \eta, \xi\rangle_N)^*$;
(c) $\langle \xi + \eta, \zeta\rangle_N = \langle \xi, \zeta\rangle_N + \langle \eta, \zeta\rangle_N$;
(d) $\langle a \cdot \xi, \eta\rangle_N = a\langle \xi, \eta\rangle_N$;
(e) $\langle \xi \cdot a, \eta\rangle_N = \langle \xi, \eta \cdot a^*\rangle_N$;
(f) $\langle \xi, a \cdot \eta\rangle_N = \langle \xi, \eta\rangle_N a^*$; *and*
(g) $\langle \xi, \eta\rangle = \mathrm{tr}\langle \xi, \eta\rangle_N$.

Proof (*Existence*). Assume, with no loss of generality, that $\mathfrak{H} = \mathfrak{H}_\alpha$ where $\alpha: N \to M_{d_\alpha}(N)$ is a cofinite morphism. If $\xi = (\xi_i)$, $\eta = (\eta_i) \in (\mathfrak{H}_\alpha)_0$, define $\langle \xi, \eta\rangle_N = \sum \xi_i \eta_i^*$ $(= \xi\eta^*)$ where the product occurring in the parentheses is matrix multiplication. The verification of the above conditions is a painless triviality; the proof of (e), for instance, is: $\langle \xi \cdot a, \eta\rangle_N = (\xi\alpha(a))\eta^* = \xi\alpha(a^*)^*\eta^* = \xi(\eta\alpha(a^*))^* = \langle \xi, \eta \cdot a^*\rangle_N$.

(*Uniqueness*). Suppose $(\ ,\ )$ is another function from $(\mathfrak{H}_\alpha)_0 \times (\mathfrak{H}_\alpha)_0$ to N which satisfies conditions (a)–(g) above. Note, as before, that if we let λ^i denote the ith row of $\alpha(1)$, then $\lambda^i \in (\mathfrak{H}_\alpha)_0$; define $p_{ij} = (\lambda^i, \lambda^j)$; then, for any ξ, η in $(\mathfrak{H}_\alpha)_0$, we have

$$(\xi, \eta) = \left(\sum \xi_i \cdot \lambda^i, \sum \eta_j \cdot \lambda^j\right) = \sum\sum \xi_i p_{ij} \eta_j^* = \xi p \eta^*$$

where, of course, the last product is a matrix-product and p denotes the matrix with (i, j)th entry p_{ij}.

It follows from the identity in the last paragraph and the definition of p that

$$p_{ij} = (\lambda^i, \lambda^j) = \lambda^i p \lambda^{j*},$$

which, in turn, implies that $p = \alpha(1)p\alpha(1)$.

Appeal to condition (e) to deduce that, for all ξ, η in $(\mathfrak{H}_\alpha)_0$, we have

$$\xi\alpha(a)p\eta^* = (\xi \cdot a, \eta) = (\xi, \eta \cdot a^*) = \xi p(\eta\alpha(a^*))^* = \xi p\alpha(a)\eta^*;$$

by allowing ξ and η to range over λ^i and λ^j, we easily deduce that this must imply that $p\alpha(a) = \alpha(a)p$ for every a in N. Thus p is a projection in $M_{d_\alpha}(N) \cap \alpha(N)'$. Condition (g) can now be translated as saying that $\operatorname{tr}\xi p\eta^* = \langle \xi, \eta \rangle$; in other words the operator, which is defined on $\mathfrak{H}_\alpha$ as multiplication on the right by the matrix p, induces the same sesquilinear form as the identity operator; this last conclusion forces p to be the matrix $\alpha(1)$ (which is the matrix which corresponds to the identity operator on $\mathfrak{H}_\alpha$). Hence, if $\xi, \eta \in \mathfrak{H}_0$, then, $\langle \xi, \eta \rangle = \operatorname{tr}\xi\alpha(1)\eta^* = \operatorname{tr}\xi\eta^*$, and the proof is complete. □

It goes without saying that there is a corresponding left-handed version of the above result, which we state here since we will later need this form.

7 (left version). **Proposition.** *There exists a unique map from* $\mathfrak{H}_0 \times \mathfrak{H}_0$ *to* N, *denoted* $(\xi, \eta) \to {}_N\langle \xi, \eta \rangle$, *that satisfies the following conditions, for all* ξ, η, ζ *in* $\mathfrak{H}_0$ *and* a *in* N:

(a) ${}_N\langle \xi, \xi \rangle$ *is a positive element of* N *that can be zero only if* $\xi = 0$;

(b) ${}_N\langle \xi, \eta \rangle = ({}_N\langle \eta, \xi \rangle)^*$;

(c) ${}_N\langle \xi + \eta, \zeta \rangle = {}_N\langle \xi, \zeta \rangle + {}_N\langle \eta, \zeta \rangle$;

(d) ${}_N\langle a \cdot \xi, \eta \rangle = {}_N\langle \xi, a^* \cdot \eta \rangle$;

(e) ${}_N\langle \xi \cdot a, \eta \rangle = {}_N\langle \xi, \eta \rangle a$;

(f) ${}_N\langle \zeta, a \cdot \eta \rangle = a^*_N\langle \xi, \eta \rangle$; *and*

(g) $\operatorname{tr}\, {}_N\langle \xi, \eta \rangle = \langle \xi, \eta \rangle$.

Further, if $\mathfrak{H}^\#$ *is a contragredient of* $\mathfrak{H}$, *and if* $\xi \to \xi^\#$ *is the 'implementing' anti-unitary operator, then* ${}_N\langle \xi, \eta \rangle = \langle \eta^\#, \xi^\# \rangle_N$.

Proof. The reader should have no difficulty in verifying the proposition either by perfectly reflecting the proof of the right-handed version or by taking a cue from the final assertion of the proposition and converting the problem concerning the 'right-handed N-valued inner-product' on $\mathfrak{H}$ to the problem concerning the 'left-handed N-valued inner-product' on $\mathfrak{H}^\#$. In terms of the model, if $\mathfrak{H} = \mathfrak{H}_\alpha$, then ${}_N\langle \xi, \eta \rangle = \sum \eta_i^* \xi_i$. □

For convenience of later reference, we give here a suitable formulation of the Riesz representation theorem.

8. **Proposition.** (a) *Each* η *in* $\mathfrak{H}_0$ *induces a unique operator* T_η *in* ${}_N\mathscr{L}(\mathfrak{H}, L^2(N))$ *such that* $T_\eta\xi = \langle \xi, \eta \rangle_N$ *for all* ξ *in* $\mathfrak{H}_0$.

(b) *Conversely every operator in* ${}_N\mathscr{L}(\mathfrak{H}, L^2(N))$ *is of the form* T_η *for a unique* η *in* $\mathfrak{H}_0$.

Proof. Assume that $\mathfrak{H} = \mathfrak{H}_\alpha$ for some α. (a) Define $T_\eta\xi = \sum \xi_i \eta_i^*$ for ξ in $\mathfrak{H}$ and observe that T_η fits the bill.

(b) Conversely, if $T \in {}_N\mathscr{L}(\mathfrak{H}, L^2(N))$, note first that T maps $\mathfrak{H}_0$, the set of bounded vectors in $\mathfrak{H}$, into the set of bounded vectors in $L^2(N)$ which is just N. Let $a^i = T\lambda^i$, where as usual we write λ^i for the ith row of $\alpha(1)$; thus, $a^i \in N$ for all i, and if $\xi = (\xi_i) \in \mathfrak{H}_0$, we then have

$$T\xi = T\left(\sum \xi_i \cdot \lambda^i\right) = \sum \xi_i \cdot a^i;$$

write a for the column vector with ith entry a^i, define $\eta = a^*\alpha(1)$, and note that $\eta = \eta\alpha(1)$ so that $\eta \in \mathfrak{H}_0$; further, if $\xi \in \mathfrak{H}_0$, then

$$T\xi = \xi a = (\xi\alpha(1))a = \xi\eta^* = \langle \xi, \eta\rangle_N;$$

thus T agrees (on the dense subspace $\mathfrak{H}_0$, and consequently on all of $\mathfrak{H}$) with the bounded operator T_η. □

9. *Remarks.* Note that, dually, every element η of $\mathfrak{H}_0$ determines a unique operator ${}_\eta T^\sim$ in $L_N(\mathfrak{H}_0, N)$ such that ${}_\eta T^\sim \xi = {}_N\langle \xi, \eta\rangle$ for all ξ in $\mathfrak{H}_0$, which operator, in turn, extends uniquely to an element ${}_\eta T$ of $\mathscr{L}_N(\mathfrak{H}, L^2(N))$; and conversely, any operator $T^\sim$ in $L_N(\mathfrak{H}_0, N)$ extends uniquely to an operator T in ${}_N\mathscr{L}(\mathfrak{H}, L^2(N))$ which in turn is ${}_\eta T$ for a unique η in $\mathfrak{H}_0$.

III. Tensor products

Recall—cf. II.6(iii) and its proof—that if $\mathfrak{H}$ and $\mathfrak{K}$ are bifinite N-bimodules, the association $T \mapsto T|_{\mathfrak{H}_0}$ sets up a bijection between ${}_N\mathscr{L}_N(\mathfrak{H}, \mathfrak{K})$ and ${}_NL_N(\mathfrak{H}_0, \mathfrak{K}_0)$; further, if $\mathfrak{H} = \mathfrak{H}_\alpha$ and $\mathfrak{K} = \mathfrak{H}_\beta$, then every operator T in ${}_N\mathscr{L}_N(\mathfrak{H}_\alpha, \mathfrak{H}_\beta)$ is given by $T\xi = \xi T^\sim$ for a uniquely determined matrix $T^\sim$ in $M_{d_\alpha \times d_\beta}(N)$ that satisfies $\alpha(a)T^\sim = T^\sim\beta(a)$ for all a in N.

1. **Proposition.** *Given any two bifinite N-bimodules $\mathfrak{H}$ and $\mathfrak{K}$, there exists a bifinite N-bimodule, denoted by $\mathfrak{H} \otimes_N \mathfrak{K}$, satisfying*

(i) *There exists a surjective linear transformation from the algebraic tensor-product $\mathfrak{H}_0 \otimes \mathfrak{K}_0$ onto $(\mathfrak{H} \otimes_N \mathfrak{K})_0$—the image under which map of $\xi \otimes \eta$ we shall denote by $\xi \otimes_N \eta$—such that, for all $(\xi, \eta) \in \mathfrak{H}_0 \times \mathfrak{K}_0$ and $a, b \in N$, we have*

$$\xi \cdot a \otimes_N \eta = \xi \otimes_N a \cdot \eta \quad \text{and} \quad a \cdot \xi \otimes_N \eta \cdot b = a \cdot (\xi \otimes_N \eta) \cdot b;$$

(ii) *if S is any linear map from $\mathfrak{H}_0 \otimes \mathfrak{K}_0$ into $\mathfrak{M}_0$—where $\mathfrak{M}$ is any bifinite N-bimodule—satisfying $S(\xi \cdot a \otimes \eta) = S(\xi \otimes a \cdot \eta)$ and $S(a \cdot \xi \otimes \eta \cdot b) = a \cdot S(\xi \otimes \eta) \cdot b$, for arbitrary $a, b \in N$ and $(\xi, \eta) \in (\mathfrak{H}_0 \times \mathfrak{K}_0)$, then there exists a unique operator $S^\sim$ in ${}_N\mathscr{L}_N(\mathfrak{H} \otimes_N \mathfrak{K}, \mathfrak{M})$ such that*

$$S^\sim(\xi \otimes_N \eta) = S(\xi \otimes \eta) \quad \text{for all } (\xi, \eta) \in \mathfrak{H}_0 \times \mathfrak{K}_0.$$

Proof. Assume $\mathfrak{H} = \mathfrak{H}_\alpha$ and $\mathfrak{K} = \mathfrak{H}_\beta$, where $\alpha, \beta : N \to M_n(N)$.

Assertion: $\mathscr{B}(d_{\alpha^*} \times d_\beta; \alpha^\#, \beta)$ is a model for $\mathfrak{H}_\alpha \otimes_N \mathfrak{H}_\beta$.

Note, to start with, that $\mathfrak{H}_\alpha \simeq (\mathfrak{H}_{\alpha^*})^* \simeq \mathscr{B}(d_{\alpha^*} \times 1; \alpha^\#, 1)$. Let

$$(\xi_1, \ldots, \xi_n) = (\xi_j) = \xi \in \mathfrak{H}_\alpha \mapsto (\xi^{\sim i}) = \xi^\sim \in M_{d_{\alpha^*} \times 1}(L^2(N))$$

denote such an isomorphism. For $\xi \in (\mathfrak{H}_\alpha)_0$ and $\eta \in (\mathfrak{H}_\beta)_0$, define

$$\xi \otimes_N \eta = ((\xi^{\sim i}\eta_j)) \in M_n(N)$$

and note that

$$(\alpha^{\#}(1)(\xi \otimes_N \eta)\beta(1))_{ij} = \sum_{k,l} \alpha^{\#}_{ik}(1)\xi^{\sim k}\eta_l\beta_{lj}(1)$$
$$= \xi^{\sim i}\eta_j = (\xi \otimes_N \eta)_{ij},$$

so that $\xi \otimes_N \eta$ does indeed belong to $M_{d_{\alpha^*} \times d_\beta}(N)$.

The equations of (i) are easily seen to be verified; as for the assertion concerning surjectivity, begin by noting that the columns of $\alpha^{\#}(1)$ belong to $M_{d_{\alpha^*} \times 1}(N)$; thus we may find λ_j in $\mathfrak{H}_\alpha$ such that $\lambda_j^{\sim}$ is the jth column of $\alpha^{\#}(1)$. Also, let μ^i denote the ith row of $\beta(1)$. For $a \in N$, compute thus:

$$(\lambda_j \cdot a \otimes_N \mu^i)_{k,l} = (\lambda_j \cdot a)^{\sim k}(\mu^i)_l = (\lambda_j^{\sim} \cdot a)^k(\mu^i)_l = \alpha^{\#}_{kj}(1)a\beta_{il}(1);$$

also, if $\xi = (\xi_{kl}) \in M_{d_{\alpha^*} \times d_\beta}(N) = (\mathscr{B}(d_{\alpha^*} \times d_\beta; \alpha^{\#}, \beta))_0$, then,

$$\xi_{kl} = (\alpha^{\#}(1)\xi\beta(1))_{kl} = \sum_{ij} \alpha^{\#}_{ki}(1)\xi_{ij}\beta_{jl}(1);$$

and hence $\xi = \sum_{ij}(\lambda_j \cdot \xi_{ji}) \otimes_N \mu^i$, thereby establishing the asserted surjectivity and completing the proof of (i).

As for (ii), if S is given as in the proposition, simply define $S^{\sim}$ by $S^{\sim}\xi = \sum_{i,j} S((\lambda_j \cdot \xi_{ji}) \otimes \mu^i)$ for any ξ in $(\mathfrak{H}_\alpha \otimes_N \mathfrak{H}_\beta)_0 = M_{d_{\alpha^*} \times d_\beta}(N)$; it is easily verified that $S^{\sim} \in {}_N L_N((\mathfrak{H}_\alpha \otimes_N \mathfrak{H}_\beta)_0, \mathfrak{M}_0)$; it follows from the remark preceding the proposition that $S^{\sim}$ extends uniquely to an N-bilinear bounded operator from $\mathfrak{H}_\alpha \otimes_N \mathfrak{H}_\beta$ to $\mathfrak{M}$. This extension, that we shall continue to denote by $S^{\sim}$, clearly has all the desired features. □

It follows easily from the above proposition that the tensor-product $\mathfrak{H} \otimes_N \mathfrak{R}$ is uniquely defined up to isomorphism.

2. **Corollary.** $\dim_N(\mathfrak{H} \otimes_N \mathfrak{R}) = \dim_N \mathfrak{H} \cdot \dim_N \mathfrak{R}$; *and* $\dim(\mathfrak{H} \otimes_N \mathfrak{R})_N = \dim \mathfrak{H}_N \cdot \dim \mathfrak{R}_N$.

Proof.

$$\dim_N \mathscr{B}(d_{\alpha^*} \times d_\beta; \alpha^{\#}, \beta) = \dim_{\alpha^{\#}(N)} M_{d_{\alpha^*} \times d_\beta}(L^2(N))$$
$$= [M_{d_{\alpha^*}(N)} : \alpha^{\#}(N)] \cdot \dim M_{d_{\alpha^*}}(N) M_{d_{\alpha^*} \times d_\beta}(L^2(N))$$
$$= d_\alpha d_{\alpha^*} \cdot d_\beta / d_{\alpha^*}$$
$$= d_\alpha \cdot d_\beta,$$

thus proving the first identity. The second identity can be proved in exactly the same manner, or may be deduced from the first by considering contragredients and appealing to the next result. □

3. **Corollary.** $(\mathfrak{H} \otimes_N \mathfrak{R})^{\#} \simeq \mathfrak{R}^{\#} \otimes_N \mathfrak{H}^{\#}$.

Proof. Assume that $\mathfrak{H} = \mathfrak{H}_\alpha$ and $\mathfrak{R} = \mathfrak{H}_\beta$, and note that

$$(\mathfrak{H}_\alpha \otimes_N \mathfrak{H}_\beta)^{\#} \simeq (\mathscr{B}(d_{\alpha^*} \times d_\beta; \alpha^{\#}, \beta))^{\#}$$
$$\simeq \mathscr{B}(d_\beta \times d_{\alpha^*}; \beta, \alpha^{\#})$$
$$\simeq \mathfrak{H}_{\beta^*} \otimes_N \mathfrak{H}_{\alpha^*}. \quad \square$$

We now wish to give an alternative description to the tensor-product which brings to focus the relationship between the cofinite morphisms involved.

4. **Definition.** If $\alpha : N \to M_m(N)$ and $\beta : N \to M_n(N)$ are maps, we shall use the symbol $\alpha \otimes \beta$ for the map from N into $M_{mn}(N)$—where the rows and columns are indexed by $\{1, 2, \ldots, m\} \times \{1, 2, \ldots, n\}$—defined by

$$(\alpha \otimes \beta)_{ij,kl}(a) = \alpha_{ik}(\beta_{jl}(a)).$$

5. **Lemma.** *If α and β are faithful normal *-homomorphisms, so is $\alpha \otimes \beta$.*

Proof. This is an easy and routine verification; for instance, the verification of the adjoint condition runs as follows: $(\alpha \otimes \beta)_{ij,kl}(a^*) = \alpha_{ik}(\beta_{jl}(a^*)) = \alpha_{ik}(\beta_{lj}(a)^*) = \alpha_{ki}(\beta_{lj}(a))^* = (\alpha \otimes \beta)_{kl,ij}(a)^*$. □

We shall establish, by a somewhat indirect argument, that $\alpha \otimes \beta$ inherits cofiniteness from α and β; we proceed in several steps. Note, to start with, that

$$\begin{aligned}
\operatorname{tr}(\alpha \otimes \beta)(1) &= (mn)^{-1} \sum_{i,j} \operatorname{tr} \alpha_{ii}(\beta_{jj}(1)) \\
&= m^{-1} \sum_j \operatorname{tr}_{M_n(N)} \alpha(\beta_{jj}(1)) \\
&= m^{-1} \sum_j (\operatorname{tr} \alpha(1)) \operatorname{tr} \beta_{jj}(1) \quad \text{(by uniqueness of trace)} \\
&= (\operatorname{tr} \alpha(1)) \cdot (\operatorname{tr} \beta(1))
\end{aligned}$$

and hence, $\mathfrak{H}_{\alpha\otimes\beta}$ is a left-finite N-module with $d_{\alpha\otimes\beta} = d_\alpha d_\beta$.

6. **Proposition.** *If α and β are cofinite morphisms, then $\mathfrak{H}_{\alpha\otimes\beta}$ is a model for $\mathfrak{H}_\alpha \otimes_N \mathfrak{H}_\beta$ and in particular, $\alpha \otimes \beta$ is cofinite.*

Proof. To begin with, we recall the following facts concerning left-finite N-bimodules:

(a) If $\mathfrak{H}$ is left-finite, then $\mathfrak{H} \simeq \mathscr{B}(1 \times d_\gamma; 1, \gamma) = \mathfrak{H}_\gamma$ where $\gamma : N \to M_{d_\gamma}(N)$ is some normal unital *-homomorphism, which is cofinite precisely when $\mathfrak{H}$ is also right-finite.

(b) Arguing exactly as in the proof of Lemma I.8(i), we see that a vector in $\mathfrak{H}_\gamma$ is right-bounded for the left-action of N precisely when all its entries come from N; consequently, $M_{1\times d_\gamma}(N)$ is the set of bi-bounded vectors in $\mathfrak{H}$.

(c) Deduce from (b) above—and arguing exactly as in the proof of Lemma I.8.(i)—that if $\alpha, \beta : N \to M_n(N)$ are normal *-homomorphisms (not necessarily unital or cofinite), then a typical element T of ${}_N\mathscr{L}(\mathfrak{H}_\alpha, \mathfrak{H}_\beta)$ is induced by multiplication on the right by a matrix $T^\sim$ in $M_{d_\alpha \times d_\beta}(N)$; thus, $T\xi = \xi T^\sim$ $\forall \xi \in \mathfrak{H}_\alpha$.

(d) It follows from (c) above that if $\mathfrak{H}$ and $\mathfrak{K}$ are left-finite N-bimodules, and if we let $\mathfrak{H}_0$ and $\mathfrak{K}_0$ denote the collections of bi-bounded vectors in $\mathfrak{H}$ and $\mathfrak{K}$ respectively, then, the association $T \to T|_{\mathfrak{H}_0}$ sets up a bijection between ${}_N\mathscr{L}_N(\mathfrak{H}, \mathfrak{K})$ and the linear space ${}_NL_N(\mathfrak{H}_0, \mathfrak{K}_0)$ of N-bilinear maps from $\mathfrak{H}_0$ to $\mathfrak{K}_0$.

(e) It is easily deduced from (d) above that Proposition III.1(ii) remains valid even when $\mathfrak{M}$ is only required to be left-finite, provided that $\mathfrak{M}_0$ is interpreted as the set of bi-bounded vectors in $\mathfrak{M}$.

We complete the proof by showing that $\mathfrak{H}_{\alpha\otimes\beta}$ satisfies the conditions of Proposition III.1, when condition (ii) is modified as in (e) above, and by invoking the uniqueness imposed by that universality condition. We begin by defining $T : (\mathfrak{H}_\alpha)_0 \otimes (\mathfrak{H}_\beta)_0 \to (\mathfrak{H}_{\alpha\otimes\beta})_0$ $(= M_{1\times d_\alpha d_\beta}(N))$ by the prescription

$$[T(\xi\otimes\eta)]_{ij} = \sum_r \xi_r \alpha_{ri}(\eta_j);$$

we then verify as follows:

(i)
$$\begin{aligned}[T(\xi\otimes\eta)(\alpha\otimes\beta)(1)]_{ij} &= \sum_{k,l}(T(\xi\otimes\eta))_{kl}(\alpha\otimes\beta)_{kl,ij}(1)\\ &= \sum_{k,l,r}\xi_r\alpha_{rk}(\eta_l)a_{ki}(\beta_{lj}(1)) = \sum_{l,r}\xi_r\alpha_{ri}(\eta_l\beta_{lj}(1))\\ &= \sum_r \xi_r\alpha_{ri}\left(\sum_l \eta_l\beta_{lj}(1)\right) = \sum_r \xi_r\alpha_{ri}(\eta_j) = [T(\xi\otimes\eta)]_{ij};\end{aligned}$$

(ii)
$$\begin{aligned}[T(\xi\cdot a\otimes\eta)]_{ij} &= \sum_r (\xi\cdot a)_r\alpha_{ri}(\eta_j)\\ &= \sum_r (\xi\alpha(a))_r\alpha_{ri}(\eta_j)\\ &= \sum_{r,s}\xi_s\alpha_{sr}(a)\alpha_{ri}(\eta_j) = [\xi\alpha(a)\alpha(\eta_j)]_i\\ &= [\xi\alpha(a\eta_j)]_i = \sum_r \xi_r\alpha_{ri}((a\cdot\eta)_j) = [T(\xi\otimes a\cdot\eta)]_{ij};\end{aligned}$$

and

(iii)
$$\begin{aligned}[T(a\cdot\xi\otimes\eta\cdot b)]_{ij} &= \sum_r (a\cdot\xi)_r\alpha_{ri}((\eta\cdot b)_j)\\ &= \sum_r a\xi_r\alpha_{ri}\left(\sum_s \eta_s\beta_{sj}(b)\right)\\ &= \sum_{r,s,p} a\xi_r\alpha_{rp}(\eta_s)\alpha_{pi}(\beta_{sj}(b))\\ &= a\sum_{p,s}[T(\xi\otimes\eta)]_{ps}(\alpha\otimes\beta)_{ps,ij}(b)\\ &= [a\cdot T(\xi\otimes\eta)\cdot b]_{ij}.\end{aligned}$$

We now show that $T((\mathfrak{H}_\alpha)_0 \times (\mathfrak{H}_\beta)_0) = M_{1\times d_\alpha d_\beta}(N)$. As before, we let λ^i and μ^j denote, respectively, the ith row of $\alpha(1)$ and the jth row of $\beta(1)$; observe then that

$$\begin{aligned}[T(\lambda^r\otimes\mu^s)]_{ij} &= \sum_p \lambda^r_p\alpha_{pi}(\mu^s_j) = \sum_p \alpha_{rp}(1)\alpha_{pi}(\beta_{sj}(1))\\ &= \alpha_{ri}(\beta_{sj}(1)) = (\alpha\otimes\beta)_{rs,ij}(1);\end{aligned}$$

also,

$$\begin{aligned}\xi \in M_{1\times d_\alpha d_\beta}(N) &\Rightarrow \xi = \xi(\alpha\otimes\beta)(1)\\ &\Rightarrow \xi_{ij} = \sum_{r,s} \xi_{rs}(\alpha\otimes\beta)_{rs,ij}(1)\\ &\Rightarrow \xi = \sum_{r,s} \xi_{rs}\cdot T(\lambda^r\otimes\mu^s)\\ &= T\left(\sum_{r,s}\xi_{rs}\cdot\lambda^r\otimes\mu^s\right)\end{aligned}$$

and hence T does have the asserted range.

Suppose now that $\mathfrak{M}$ is a left-finite N-module and that $S : (\mathfrak{H}_\alpha)_0 \times (\mathfrak{H}_\beta)_0 \to \mathfrak{M}_0$—where, of course, $\mathfrak{M}_0$ denotes the set of bi-bounded vectors of $\mathfrak{M}$—is a linear map satisfying $S(\xi\cdot a\otimes\eta) = S(\xi\otimes a\cdot\eta)$ and $S(a\cdot\xi\otimes\eta\cdot b) = a\cdot S(\xi\otimes\eta)\cdot b$; simply define $S^\sim : M_{1\times d_\alpha d_\beta}(N) = (\mathfrak{H}_{\alpha\otimes\beta})_0 \to \mathfrak{M}_0$ by $S^\sim\xi = \sum_{i,j}\xi_{ij}\cdot S(\lambda^i\otimes\mu^j)$; an easy computation verifies that $S^\sim \in {}_N\mathcal{L}_N((\mathfrak{H}_{\alpha\otimes\beta})_0, \mathfrak{M}_0)$. By the observation (d) made in the first part of the proof, the operator $S^\sim$ extends uniquely to an operator $S^\wedge \in {}_N\mathscr{L}_N(\mathfrak{H}_{\alpha\otimes\beta}, \mathfrak{M})$. It is clear from the definitions that $S = S^\wedge\circ T$.

In particular, take $\mathfrak{M}$ to be the model of $\mathfrak{H}_\alpha\otimes_N\mathfrak{H}_\beta$ described in Proposition III.1—or any other model which satisfies the strengthened version of condition (ii) as described in comment (e) at the start of this proof. Deduce from the above that there exists a unique map $S \in {}_N\mathscr{L}_N(\mathfrak{H}_{\alpha\otimes\beta}, \mathfrak{M})$ such that

$$S(T(\xi\otimes\eta)) = \xi\otimes_N\eta \quad \text{for all } \xi \text{ in } (\mathfrak{H}_\alpha)_0 \text{ and } \eta \text{ in } (\mathfrak{H}_\beta)_0.$$

Now interchange the roles of $\mathfrak{M}$ and $\mathfrak{H}_{\alpha\otimes\beta}$ to find a bounded N-bilinear operator R from $\mathfrak{M}$ to $\mathfrak{H}_{\alpha\otimes\beta}$ such that $R(\xi\otimes_N\eta) = T(\xi\otimes\eta)$. It follows that $R\circ S = \mathrm{id}_{\mathfrak{H}_{\alpha\otimes\beta}}$, since $R\circ S$ fixes the range of T which has already been verified to be dense in $\mathfrak{H}_{\alpha\otimes\beta}$; similarly we can see that $S\circ R = \mathrm{id}_{\mathfrak{M}}$. Thus R and S are invertible operators; it is not hard to see that if $R = U|R|$ is the polar decomposition of R, then U is an N-bilinear unitary isomorphism of $(\mathfrak{M} =)\ \mathfrak{H}_\alpha\otimes_N\mathfrak{H}_\beta$ onto $\mathfrak{H}_{\alpha\otimes\beta}$. In particular, $\mathfrak{H}_{\alpha\otimes\beta}$ is right-finite, thereby establishing the cofiniteness of $\alpha\otimes\beta$ and completing the proof. □

7. **Corollary.** *If $\mathfrak{H}$, $\mathfrak{R}$ and $\mathfrak{M}$ are bifinite N-bimodules, then*

$$(\mathfrak{H}\otimes_N\mathfrak{R})\otimes_N\mathfrak{M} \simeq \mathfrak{H}\otimes_N(\mathfrak{R}\otimes_N\mathfrak{M}).$$

Proof. If α, β and γ are cofinite morphisms, then $(\alpha\otimes\beta)\otimes\gamma = \alpha\otimes(\beta\otimes\gamma)$. □

Before proceeding further, we identify the 'Hilbert-module' structure on the tensor-product.

8. **Proposition.** *If $\mathfrak{H}$ and $\mathfrak{R}$ are bifinite N-bimodules and if $\xi, \xi' \in \mathfrak{H}_0$ and $\eta, \eta' \in \mathfrak{R}_0$, then*

$$\langle\xi\otimes_N\eta, \xi'\otimes_N\eta'\rangle_N = \langle\xi\cdot\langle\eta, \eta'\rangle_N, \xi'\rangle_N$$

and

$${}_N\langle\xi\otimes_N\eta, \xi'\otimes_N\eta'\rangle = {}_N\langle{}_N\langle\xi, \xi'\rangle\cdot\eta, \eta'\rangle.$$

Proof. Assume that $\mathfrak{H} = \mathfrak{H}_\alpha$ and $\mathfrak{K} = \mathfrak{H}_\beta$; then we may take $\mathfrak{H} \otimes_N \mathfrak{K} = \mathfrak{H}_{\alpha\otimes\beta} = M_{1\times d_\alpha d_\beta}(N)$. Compute thus:

$$\begin{aligned}
\langle \xi \otimes_N \eta, \xi' \otimes_N \eta' \rangle_N &= \sum_{i,j} (\xi \otimes_N \eta)_{ij} (\xi' \otimes_N \eta')^*_{ij} \\
&= \sum_{i,j,p,q} \xi_p \alpha_{pi}(\eta_j)(\xi'_q \alpha_{qi}(\eta'_j))^* \\
&= \sum_{i,j,p,q} \xi_p \alpha_{pi}(\eta_j)\alpha_{iq}(\eta'^*_j)\xi'^*_q \\
&= \sum_{p,q} \xi_p \alpha_{pq}\left(\sum_j \eta_j \eta'^*_j\right)\xi'^*_q \\
&= \sum_q (\xi \cdot \langle \eta, \eta' \rangle_N)_q \xi'^*_q \\
&= \langle \xi \cdot \langle \eta, \eta' \rangle_N, \xi' \rangle_N .
\end{aligned}$$

A similar computation, using the fact that $(\xi \otimes_N \eta)^\#$ may be naturally identified with $(\eta^\# \otimes_N \xi^\#)$, yields the second identity. □

Using our first model for the tensor-product, it is seen that

$${}_N\mathscr{L}_N(L^2(N), \mathfrak{H}_{\alpha^\circ} \otimes_N \mathfrak{H}_\beta)$$

is isomorphic, as a vector space, to the space $\mathscr{I}(\alpha, \beta)$ of all those matrices $T \in M_{d_\alpha \times d_\beta}(N)$ which satisfy $\alpha(a)T = T\beta(a)$ for all a in N; the latter space has already been seen to be isomorphic to ${}_N\mathscr{L}_N(\mathfrak{H}_\alpha, \mathfrak{H}_\beta)$. Instead of employing the above reasoning (which has the disadvantage of being basis- or model-dependent), we shall use the underlying Hilbert-module structure to exhibit the above as well as other (natural) isomorphisms between spaces of intertwiners.

We shall consistently use the notation $\xi \to \xi^\#$ to denote the anti-unitary isomorphism of an N-bimodule onto its contragredient that satisfies $(a \cdot \xi \cdot b)^\# = b^* \cdot \xi^\# \cdot a^*$ for all vectors ξ and for all a, b in N.

9. **Proposition.** *Let $\mathfrak{H}, \mathfrak{K}$ and $\mathfrak{M}$ denote bifinite N-bimodules.*

(a) *The equation $(T^*\xi)^\# = T^\#\xi^\#$ sets up a vector-space isomorphism between ${}_N\mathscr{L}_N(\mathfrak{H}, \mathfrak{K})$ and ${}_N\mathscr{L}_N(\mathfrak{K}^\#, \mathfrak{H}^\#)$;*

(b) *The equation ${}_N(T\xi, \eta) = T^\sim(\eta^\# \otimes_N \xi)$ sets up a vector-space isomorphism of ${}_N\mathscr{L}_N(\mathfrak{H}, \mathfrak{K})$ onto ${}_N\mathscr{L}_N(\mathfrak{K}^\# \otimes_N \mathfrak{H}, L^2(N))$; and*

(c) *The equation $\langle T\xi, \eta \rangle_N = T^\sim(\xi \otimes_N \eta^\#)$ sets up a vector-space isomorphism between ${}_N\mathscr{L}_N(\mathfrak{H}, \mathfrak{K})$ and ${}_N\mathscr{L}_N(\mathfrak{H} \otimes_N \mathfrak{K}^\#, L^2(N))$.*

Proof. (a) Clearly, $T \in {}_N\mathscr{L}_N(\mathfrak{H}, \mathfrak{K}) \Rightarrow T^* \in {}_N\mathscr{L}_N(\mathfrak{K}, \mathfrak{H})$; so if $\eta \in \mathfrak{K}$, then

$$\begin{aligned}
T^\#(a \cdot \eta^\# \cdot b) &= T^\#[(b^* \cdot \eta \cdot a^*)^\#] = [T^*(b^* \cdot \eta \cdot a^*)]^\# \\
&= [b^* \cdot T^*\eta \cdot a^*]^\# = a \cdot T^\#\eta^\# \cdot b,
\end{aligned}$$

so that $T^\# \in {}_N\mathscr{L}_N(\mathfrak{K}^\#, \mathfrak{H}^\#)$. It is clear that $(T^\#)^\# = T$ and that the assignment $T \mapsto T^\#$ is linear; it follows that the assignment defines a vector-space isomorphism.

(b) Given $T \in {}_N\mathscr{L}_N(\mathfrak{H}, \mathfrak{K})$, just define $T' : \mathfrak{K}_0^\# \otimes \mathfrak{H}_0 \to N$ by $T'(\eta^\# \otimes \xi) = {}_N\langle T\xi, \eta\rangle$ and perform the necessary verifications thus:

$$\begin{aligned} T'(\eta^* \cdot a \otimes \xi) &= {}_N\langle T\xi, a^*\eta\rangle = {}_N\langle a \cdot T\xi, \eta\rangle \\ &= {}_N\langle T(a \cdot \xi), \eta\rangle = T'(\eta^\# \otimes a \cdot \xi); \end{aligned}$$

a similar computation shows that $T'(a \cdot \eta^\# \otimes \xi \cdot b) = a \cdot T'(\eta^\# \otimes \xi) \cdot b$; hence T' extends uniquely to a bounded N-bilinear operator $T^\sim$ defined on all of $\mathfrak{K}^\# \otimes_N \mathfrak{H}$; it is quite painless to verify that $T \mapsto T^\sim$ defines a linear map between the appropriate vector spaces. We now show that the map is onto, and since the vector spaces in question are finite-dimensional, that is all we have to do.

Suppose we are given $T^\sim \in {}_N\mathscr{L}_N(\mathfrak{K}^\# \otimes_N \mathfrak{H})$; fix ξ in $\mathfrak{H}_0$ and consider the map $\mathfrak{K}_0 \mapsto N$ given by $\eta \mapsto [T^\sim(\eta^\# \otimes_N \xi)]^*$; note that

$$\eta \cdot a \mapsto [T^\sim(a^* \cdot \eta^\# \otimes_N \xi)]^* = [T^\sim(a^* \cdot (\eta^\# \otimes_N \xi))]^* = T^\sim(\eta^\# \otimes_N \xi)^* a;$$

so the above map defines a member of $L_N(\mathfrak{K}_0, N)$. Deduce from Remark II.9. the existence of a unique vector, call it $T\xi$, in $\mathfrak{K}_0$ such that $[T^\sim(\eta^\# \otimes_N \xi)]^* = {}_N\langle \eta, T\xi\rangle$, or equivalently, ${}_N\langle T\xi, \eta\rangle = T^\sim(\eta^\# \otimes_N \xi)$, for all η in $\mathfrak{K}_0$. Observe now that

$$\begin{aligned} {}_N\langle T(a \cdot \xi \cdot b), \eta\rangle &= T^\sim(\eta^\# \otimes_N (a \cdot \xi \cdot b)) \\ &= T^\sim((a^* \cdot \eta)^\# \otimes_N (\xi \cdot b)) \\ &= T^\sim((a^* \cdot \eta)^\# \otimes_N \xi)b \\ &= {}_N\langle T\xi, a^* \cdot \eta\rangle b \\ &= {}_N\langle a \cdot T\xi, \eta\rangle b \\ &= {}_N\langle a \cdot T\xi \cdot b, \eta\rangle; \end{aligned}$$

the aforementioned uniqueness imposed on '$T\xi$' by its defining condition implies now that necessarily $T \in {}_NL_N(\mathfrak{H}_0, \mathfrak{K}_0)$; hence T extends uniquely to a bounded N-bilinear operator from all of $\mathfrak{H}$ to $\mathfrak{K}$, that we shall continue to denote by T. This establishes the desired surjectivity thus completing the proof of (b).

(c) This is proved in exactly as (b) above, except that ${}_N\langle\ ,\ \rangle$ is replaced throughout by $\langle\ ,\ \rangle_N$. □

10. **Corollary.** *If $\mathfrak{H}, \mathfrak{K}$ and $\mathfrak{M}$ are any bifinite bimodules, then the vector spaces ${}_N\mathscr{L}_N(\mathfrak{H} \otimes_N \mathfrak{K}, \mathfrak{M})$, ${}_N\mathscr{L}_N(\mathfrak{K}, \mathfrak{H}^\# \otimes_N \mathfrak{M})$ and ${}_N\mathscr{L}_N(\mathfrak{H}, \mathfrak{M} \otimes_N \mathfrak{K}^\#)$ are naturally isomorphic and consequently have the same dimension (over* **C**).

Proof. Two applications of (b) of the previous proposition yields

$$\begin{aligned} {}_N\mathscr{L}_N(\mathfrak{H} \otimes_N \mathfrak{K}, \mathfrak{M}) &\simeq {}_N\mathscr{L}_N(\mathfrak{M}^\# \otimes_N (\mathfrak{H} \otimes_N \mathfrak{K}), L^2(N)) \\ &\simeq {}_N\mathscr{L}_N((\mathfrak{H}^\# \otimes_N \mathfrak{M})^\# \otimes_N \mathfrak{K}, L^2(N)) \\ &\simeq {}_N\mathscr{L}_N(\mathfrak{K}, \mathfrak{H}^\# \otimes_N \mathfrak{M}); \end{aligned}$$

similarly, two applications of (c) of the previous proposition yields

$${}_N\mathscr{L}_N(\mathfrak{H} \otimes_N \mathfrak{K}, \mathfrak{M}) \simeq {}_N\mathscr{L}_N(\mathfrak{H}, \mathfrak{M} \otimes_N \mathfrak{K}^\#). \quad \square$$

IV. Hypergroups

The term hypergroup has been used in the literature with somewhat differing definitions (cf., for instance, [MP and Ro]); while the basic structure is almost identical in all cases, what differs is the amount of commutativity, finiteness and

other such features that is assumed. We present here the version that seems to be most natural in the context of bifinite bimodules over II_1 factors. In order to avoid conflicting with other definitions, we should probably call the object defined here by some amended version such as II_1-hypergroup or some such thing; however, we shall never refer to any other kind of a hypergroup in this paper, so we go ahead and make the following definition.

1. **Definition.** By a (discrete) hypergroup is meant a set $\mathfrak{G}$ equipped with a function $\mathfrak{G} \times \mathfrak{G} \times \mathfrak{G} \to \mathbf{Z}_+$ $(= \{0, 1, 2, \ldots\})$—which shall be denoted by $(\alpha, \beta, \gamma) \mapsto \langle \alpha \otimes \beta, \gamma \rangle$—that satisfies the following conditions:

(a) (local finiteness): for all α, β in $\mathfrak{G}$, $\langle \alpha \otimes \beta, \gamma \rangle \neq 0$ for only finitely many γ;

(b) (associativity): for all α, β, γ and κ in $\mathfrak{G}$,

$$\sum_{\lambda} \langle \alpha \otimes \beta, \lambda \rangle \langle \lambda \otimes \gamma, \kappa \rangle = \sum_{\lambda} \langle \alpha \otimes \lambda, \kappa \rangle \langle \beta \otimes \gamma, \lambda \rangle;$$

(c) (identity): there exists an element 1 in $\mathfrak{G}$ such that, for all $\alpha, \beta \in \mathfrak{G}$,

$$\langle 1 \otimes \alpha, \beta \rangle = \langle \alpha \otimes 1, \beta \rangle = \delta_{\alpha\beta};$$

(d) (contragredient): there exists a self-map of $\mathfrak{G}$, denoted by $\alpha \mapsto \alpha^{\#}$, such that for all $\alpha, \beta, \gamma \in \mathfrak{G}$, $\langle \alpha \otimes \beta, \gamma \rangle = \langle \alpha^{\#} \otimes \gamma, \beta \rangle$. □

Remark. The symbol appearing on the right of the identity axiom is the Kronecker delta symbol and will be used in the sequel without further comment.

The two sides of the equation in the associativity axiom should be thought of as the two different ways of computing '$\langle \alpha \otimes \beta \otimes \gamma, \kappa \rangle$'.

The contragredient axiom would be a little more natural if we thought of the right side as $\langle \beta, \alpha^{\#} \otimes \gamma \rangle$; the condition would be just an adjoint condition.

2. **Examples.** (a) *Groups*: if $\mathfrak{G}$ is a group, define $\langle \alpha \otimes \beta, \gamma \rangle$ to be 1 precisely when γ is the (group-) product $\alpha\beta$, and to be zero otherwise; also define $\alpha^{\#} = \alpha^{-1}$ and verify that this defines a hypergroup structure on $\mathfrak{G}$.

(b) *Duals of compact groups.* The dual object of a compact group, i.e., the collection of equivalence classes of irreducible representations of the compact group, has the structure of a hypergroup, with $\langle \pi \otimes \pi', \rho \rangle$ denoting the multiplicity with which the irreducible representation ρ features in the tensor-product $\pi \otimes \pi'$ of the irreducible representations π and π', 1 denoting the trivial 1-dimensional representation, and $\pi^{\#}$ denoting the contragredient of the representation π.

(c) *The hypergroup of a* II_1 *factor*: If N is a II_1 factor, let $\mathfrak{G}(N)$ denote the collection of isomorphism classes of irreducible (bifinite N-bimodules or equivalently) cofinite morphisms of N. If $\langle \alpha \otimes \beta, \gamma \rangle$ is interpreted as the multiplicity with which $\mathcal{H}_\gamma$ features in $\mathcal{H}_\alpha \otimes_N \mathcal{H}_\beta$, and if the contragredient is interpreted as the notion already introduced, then $\mathfrak{G}(N)$ becomes a hypergroup. The associativity follows from associativity of tensor-products (over N) and distributivity of tensor-products over direct sums; the validity of the contragredient axiom is an immediate consequence of Corollary III.10. (In fact, that result shows that, in $\mathfrak{G}(N)$, we also have $\langle \alpha \otimes \beta, \gamma \rangle = \langle \gamma \otimes \beta^{\#}, \alpha \rangle$; we shall soon see that this equality is valid in any hypergroup.) □

Given an abstract hypergroup $\mathfrak{G}$, we denote by $\mathbf{C}\mathfrak{G}$ the class of finitely supported complex-valued functions on $\mathfrak{G}$. The space $\mathbf{C}\mathfrak{G}$ comes equipped with a distinguished basis $\{f_\alpha : \alpha \in \mathfrak{G}\}$, given by $f_\alpha(\beta) = \delta_{\alpha\beta}$; thus, $f = \sum_\alpha f(\alpha) f_\alpha$ for all f in $\mathbf{C}\mathfrak{G}$. We make $\mathbf{C}\mathfrak{G}$ into an algebra by demanding that $f_\alpha * f_\beta(\gamma) = \langle \alpha \otimes \beta , \gamma \rangle$; more generally, for $f, g \in \mathbb{C}^{\mathfrak{G}}$, define $f * g(\gamma) = \sum_{\alpha\beta} f(\alpha) g(\beta) \langle \alpha \otimes \beta , \gamma \rangle$; it is easily seen that the associativity axiom is precisely what is needed to ensure that $\mathbf{C}\mathfrak{G}$ is an associative algebra with respect to the 'convolution' product defined above. Finally, we can make $\mathbf{C}\mathfrak{G}$ into a pre-Hilbert space by demanding that $\{f_\alpha : \alpha \in \mathfrak{G}\}$ is a (necessarily maximal) orthonormal set of vectors. We shall write V for the inner-product space so obtained.

Consider now the 'left-regular representation' of $\mathbf{C}\mathfrak{G}$: if $f \in \mathbf{C}\mathfrak{G}$, define the associated left-multiplication operator L_f on V defined by $L_f g = f * g$. Clearly, $f \mapsto L_f$ is an algebra-homomorphism of $\mathbf{C}\mathfrak{G}$ into $L(V)$ which is easily seen to be unital—i.e., $L_{f_1} = \mathrm{id}_V$—and faithful (since f_1 is an identity for $\mathbf{C}\mathfrak{G}$). Further, if we define $f^*(\alpha) = (f(\alpha^\#))^*$—where we write ζ^* for the conjugate of the complex number ζ—we see that the contragredient axiom is just what is needed to ensure that $\langle L_f g , h \rangle = \langle g , L_{f^*} h \rangle$ for f, g, h in $\mathbf{C}\mathfrak{G}$. Appealing now to the injectivity of the regular representation and to the fact that identities and adjoints are unique in operator algebras, the following proposition is seen to immediately follow, and we shall say nothing more about its proof.

3. **Proposition.** *Let $\mathfrak{G}$ be any hypergroup.*
(a) *The identity element 1 of $\mathfrak{G}$ is unique and $1^\# = 1$;*
(b) *$(\alpha^\#)^\# = \alpha$ for every α in $\mathfrak{G}$.* □

4. **Proposition.** *The equation $\tau(f) = f(1)$ defines a faithful positive trace on the involutive algebra $\mathbf{C}\mathfrak{G}$, for any hypergroup $\mathfrak{G}$.*

Proof. Clearly $\tau(f) = \langle L_f f_1 , f_1 \rangle$, and hence,

$$\begin{aligned} \tau(f^* * f) &= \langle L_{f^* * f} f_1 , f_1 \rangle = \langle L_{f^*} L_f f_1 , f_1 \rangle \\ &= \langle L_{f^*} f , f_1 \rangle = \langle f , L_f f_1 \rangle = \langle f , f \rangle , \end{aligned}$$

and hence the faithfulness and positivity of τ. Finally, the trace condition follows from

$$\begin{aligned} \tau(f * g) &= \langle f * g , f_1 \rangle = \langle g , f^* * f_1 \rangle = \langle g , f^* \rangle \\ &= \sum_\alpha g(\alpha) f(\alpha^\#) = \sum_\beta f(\beta) g(\beta^\#) = \tau(g * f) . \end{aligned}$$ □

5. *Notation.* If $\alpha_1, \ldots, \alpha_n, \kappa \in \mathfrak{G}$, where $\mathfrak{G}$ is some hypergroup, define $\langle \alpha_1 \otimes \cdots \otimes \alpha_n , \kappa \rangle = (f_{\alpha_1} * \cdots * f_{\alpha_n})(\kappa)$, noting that this definition agrees with the already existing notion when $n = 2$. □

6. **Proposition.** *If $\alpha, \beta, \gamma, \kappa, \alpha_1, \ldots, \alpha_n$ denote elements of an arbitrary hypergroup $\mathfrak{G}$, we have:*
(a) $\langle \alpha \otimes \beta , 1 \rangle = \delta_{\beta, \alpha^\#}$;
(b) $\langle \alpha \otimes \beta , \gamma \rangle = \langle \gamma \otimes \beta^\# , \alpha \rangle$;
(c) $\langle \alpha \otimes \beta , \gamma \rangle = \langle \beta^\# \otimes \alpha^\# , \gamma^\# \rangle$;
(d) *if $m \leq n$, then*

$$\langle \alpha_1 \otimes \cdots \otimes \alpha_n , \kappa \rangle = \sum_{\beta , \gamma} \langle \alpha_1 \otimes \cdots \otimes \alpha_m , \beta \rangle \langle \alpha_{m+1} \otimes \cdots \otimes \alpha_n , \gamma \rangle \langle \beta \otimes \gamma , \kappa \rangle .$$

Proof. (a) $\langle \alpha \otimes \beta , 1 \rangle = \langle \alpha^{\#} \otimes 1 , \beta \rangle = \delta_{\beta \alpha^{\#}}$;

(b)

$$\begin{aligned} \langle \alpha \otimes \beta , \gamma \rangle &= \tau(f_{\gamma^{\#}} * f_{\alpha} * f_{\beta}) = \tau(f_{\beta} * f_{\gamma^{\#}} * f_{\alpha}) \\ &= \langle f_{\gamma^{\#}} * f_{\alpha} , f_{\beta^{\#}} \rangle = \langle f_{\alpha} , f_{\gamma} * f_{\beta^{\#}} \rangle \\ &= \langle \gamma \otimes \beta^{\#} , \alpha \rangle^{*} = \langle \gamma \otimes \beta^{\#} , \alpha \rangle \end{aligned}$$

since the inner product in question is (integral and hence) real.

(c) $$\langle \alpha \otimes \beta , \gamma \rangle = \tau(f_{\alpha} * f_{\beta} * f_{\gamma^{\#}});$$

notice that if g in $\mathbf{C}\mathfrak{G}$ satisfies $g(1) \in \mathbf{R}$, then $\tau(g) = \tau(g^{*})$; hence,

$$\langle \alpha \otimes \beta , \gamma \rangle = \tau(f_{\gamma} * f_{\beta^{\#}} * f_{\alpha^{\#}}) = \langle \beta^{\#} \otimes \alpha^{\#} , \gamma^{\#} \rangle .$$

(d) Put $f = f_{\alpha_1} \otimes \cdots \otimes f_{\alpha_m}$, $g = f_{\alpha_{m-1}} * \cdots * f_{\alpha_n}$, and notice that

$$(f * g)(k) = \sum_{\beta , \gamma} f(\beta) g(\gamma) \langle \beta \otimes \gamma , k \rangle$$

by the definition of the convolution product. □

7. **Definitions.** (a) A subset $\mathscr{H}$ of a hypergroup $\mathfrak{G}$ is said to be a sub-hypergroup if $\mathscr{H}$ is closed under taking contragredients and 'products' in the sense that if $\alpha , \beta \in \mathscr{H}$, then $\alpha^{\#} \in \mathscr{H}$ and $\gamma \in \mathfrak{H}$ for any $\gamma \in \mathfrak{G}$ such that $\langle \alpha \otimes \beta , \gamma \rangle > 0$;

(b) A map $\pi : \mathfrak{G} \to \mathfrak{G}'$ between hypergroups is called a homomorphism if $\pi(1) = 1$ and $\langle \pi(\alpha) \otimes \pi(\beta) , \pi(\gamma) \rangle = \langle \alpha \otimes \beta , \gamma \rangle$ for all α , β , γ in $\mathfrak{G}$, and if $\pi(\mathfrak{G})$ is a sub-hypergroup of $\mathfrak{G}'$.

(c) An (outer equivalence class of an) action of a hypergroup $\mathfrak{G}$ on a II_1 factor N is a homomorphism of $\mathfrak{G}$ into $\mathfrak{G}(N)$. □

8. *Remarks.* (a) It is a consequence of Proposition IV.6(a) that homomorphisms of hypergroups preserve contragredients.

(b) If α is an automorphism of N, the associated bimodule $\mathfrak{H}_{\alpha}$ is just $L^2(N)$ with the actions given by $a \cdot \xi \cdot b = a \xi \alpha(b)$; clearly $\mathfrak{H}_{\alpha}$ is irreducible. Let $[\alpha]$ denote the element of $\mathfrak{G}(N)$ given by $\mathfrak{H}_{\alpha}$. If β is another automorphism and if β is also viewed as a cofinite morphism of N, it follows from the definition of the tensor-product of cofinite morphisms that $\alpha \otimes \beta = \alpha \circ \beta$. Since $[1]$ is the identity of $\mathfrak{G}(N)$, it follows that $\alpha \mapsto [\alpha]$ is a homomorphism of $\mathrm{Aut}(N)$, viewed as a hypergroup, into $\mathfrak{G}(N)$. (It is true, although it requires some proving, that any homomorphism of a finite group G into $\mathfrak{G}(N)$ factors through $\mathrm{Aut}(N)$. One proof uses Theorem 10 below.) □

9. **Definition.** A function $\alpha \mapsto d_{\alpha}$ from $\mathfrak{G}$ to $(0 , \infty)$ is called a dimension function for the hypergroup $\mathfrak{G}$ if, for all α , β , γ in $\mathfrak{G}$, we have

$$d_{\alpha} d_{\beta} = \sum_{\gamma} \langle \alpha \otimes \beta , \gamma \rangle d_{\gamma} . \quad \square$$

10. **Theorem.** *Every finite hypergroup admits a unique dimension function.*

Proof. Let $\mathfrak{G}$ denote a finite hypergroup. If V denotes the finite-dimensional inner-product space $\mathbf{C}\mathfrak{G}$ with orthonormal basis $\{f_\gamma : \gamma \in \mathfrak{G}\}$, we shall identify a linear operator T on V with the matrix $(t_{\alpha\beta})$ given by $t_{\alpha\beta} = (Tf_\beta, f_\alpha)$; thus the symbol L_α will be thought of as a matrix—with rows and columns indexed by $\mathfrak{G}$—whose entry in position (β, γ) is $\langle \alpha \otimes \gamma, \beta \rangle$. We shall also make use of an auxiliary matrix Λ defined by $\Lambda(\alpha, \gamma) = \sum_\beta \langle \alpha \otimes \beta, \gamma \rangle$. (The reason for considering this matrix will become clear later.) We begin by making the following

Assertion: (a) Λ *commutes with* L_α *for all* α *in* $\mathfrak{G}$;
(b) $L_\alpha L_\beta = \sum_\gamma \langle \alpha \otimes \beta, \gamma \rangle L_\gamma$ *for all* α, β *in* $\mathfrak{G}$; *and*
(c) $\Lambda(\alpha, \gamma) > 0$ *for all* α, γ *in* $\mathfrak{G}$.

Proof of the assertion. (a) For arbitrary α, β in $\mathfrak{G}$, compute thus:

$$\begin{aligned}
(\Lambda L_\alpha)(\beta, \gamma) &= \sum_\kappa \Lambda(\beta, \kappa) L_\alpha(\kappa, \gamma) \\
&= \sum_{\kappa, \mu} \langle \beta \otimes \mu, \kappa \rangle \langle \alpha \otimes \gamma, \kappa \rangle \\
&= \sum_{\kappa, \mu} \langle \beta \otimes \mu, \kappa \rangle \langle \alpha^* \otimes \kappa, \gamma \rangle \\
&= \sum_\mu \langle \alpha^\# \otimes \beta \otimes \mu, \gamma \rangle ; \\
(L_\alpha \Lambda)(\beta, \gamma) &= \sum_\kappa L_\alpha(\beta, \kappa) \Lambda(\kappa, \gamma) \\
&= \sum_{\kappa, \mu} \langle \alpha \otimes \kappa, \beta \rangle \langle \kappa \otimes \mu, \gamma \rangle \\
&= \sum_{\kappa, \mu} \langle \alpha^\# \otimes \beta, \kappa \rangle \langle \kappa \otimes \mu, \gamma \rangle \\
&= \sum_\mu \langle \alpha^\# \otimes \beta \otimes \mu, \gamma \rangle .
\end{aligned}$$

(b) This follows from $f_\alpha * f_\beta = \sum_\gamma \langle \alpha \otimes \beta, \gamma \rangle f_\gamma$, and the fact that $f \mapsto L_f$ is an algebra-homomorphism.

(c) If $\Lambda(\alpha, \gamma) = 0$, then necessarily $\langle \alpha \otimes \beta, \gamma \rangle = 0$ for all $\beta \in \mathfrak{G}$, or equivalently, $\langle \alpha^\# \otimes \gamma, \beta \rangle = 0$ for all β in $\mathfrak{G}$; this says that $f_{\alpha^*} * f_\gamma = 0$; hence, $f_\alpha * f_{\alpha^*} * f_\gamma * f_{\gamma^*} = 0$. Since functions of the form $f_\lambda * f_\mu$ have nonnegative values and since $f_\mu * f_{\mu^*}(1) = 1$ for all μ, we find that the conclusion of the previous sentence is untenable; this contradiction completes the proof of the assertion.

Existence. Since Λ is a square matrix with strictly positive entries, the Perron-Frobenius theorem asserts the existence of a positive eigenvalue λ of Λ with both geometric and algebraic multiplicity one, the corresponding eigenspace being spanned by a vector v with strictly positive entries. Since any operator that commutes with Λ must necessarily leave each eigenspace of Λ invariant, it follows from the previous sentence that v is an eigenvector of each nonnegative matrix L_α; consequently, v is also, necessarily, the Perron-Frobenius eigenvector of each L_α, and so $L_\alpha v = d_\alpha v$, say, for some d_α that is (strictly)

positive (since each row of L_α is nonzero—by (c) of the assertion above—and nonnegative, and since v is strictly positive). Now, deduce from (b) of the earlier assertion that

$$d_\alpha d_\beta v = L_\alpha L_\beta v = \sum_\gamma \langle \alpha \otimes \beta, \gamma \rangle L_\gamma v = \sum_\gamma \langle \alpha \otimes \beta, \gamma \rangle d_\gamma v,$$

thus showing that a choice of the dimension function for $\mathfrak{G}$ is given by the Perron-Frobenius eigenvalue of the associated operator in the left regular representation of $\mathfrak{G}$.

Uniqueness. If $\alpha \mapsto \tilde{d}_\alpha$ is a dimension function for $\mathfrak{G}$, let $\tilde{v}$ be the element of V whose αth coordinate is given by $\tilde{d}_\alpha$ and notice that $\tilde{v}$ is a vector with strictly positive coordinates which is an eigenvector—and hence the Perron-Frobenius eigenvector—of Λ. (Reason: $(\Lambda\tilde{v})(\alpha) = \sum_\gamma \Lambda(\alpha, \gamma)\tilde{d}_\gamma = \sum_{\beta,\gamma} \langle \alpha \otimes \beta, \gamma \rangle \tilde{d}_\gamma = \sum_\beta \tilde{d}_\alpha \tilde{d}_\beta = \tilde{\lambda}\tilde{v}(\alpha)$, where $\tilde{\lambda} = \sum_\beta \tilde{d}_\beta$.) It follows that $\tilde{v}$ is that positive multiple of v for which the l^1-norm is the Perron-Frobenius eigenvalue of Λ and consequently there exists at most one dimension function for $\mathfrak{G}$. □

11. *Remarks.* (i) We wish to emphasise some facts thrown to light by the proof of the previous theorem; with the preceding notation, we thus have:

(a) All the matrices L_α, as well as the matrix Λ, have a common Perron-Frobenius eigenvector;

(b) The naturally scaled version of this eigenvector has coordinate d_α at the αth place;

(c) The Perron-Frobenius eigenvalue of L_α is also d_α.

(2) If $\mathfrak{G}$ is a group G, a dimension function on G is easily seen to be nothing but a homomorphism of G into the multiplicative group $\mathbf{R}_+^\times$ of positive real numbers. For groups, the previous theorem is a consequence of the fact that the only finite subgroup of $\mathbf{R}_+^\times$ is (1); further, the group case shows that the previous theorem is false for infinite $\mathfrak{G}$; if $\mathfrak{G} = \mathbf{Z}$, such homomorphisms are determined by the image of 1 which can be any $\lambda > 0$; it is an interesting aside that for any $\lambda > 0$, there exists a homomorphism $\pi_\lambda : \mathbf{Z} \to \mathfrak{G}(R)$—where R denotes the hyperfinite II_1 factor—such that $d_{\pi_\lambda(1)} = \lambda$. (Reason: let α denote an isomorphism of N onto $M_\lambda(N)$ and notice that the surjectivity of α implies that $d_\alpha d_{\alpha^\#} = 1$ and hence that the desired π_λ may be defined by sending the integer n to $\alpha^{[n]}$ where $\alpha^{[n]}$ denotes the n-fold tensor power of (the irreducible cofinite morphism) α or $\alpha^\#$ according as n is positive or negative (and of course, sending 0 to 1).)

(3) If G is a finite hypergroup, then $\alpha \mapsto d_{\alpha^*}$ is also seen to be a dimension function, and hence $d_{\alpha^*} = d_\alpha$ for all α in G. □

We return now to II_1 factors, and make the following conjecture:

12. **Conjecture.** *Every finite hypergroup admits an action on the hyperfinite* II_1 *factor.* □

Our interest in the above conjecture is primarily motivated by the following considerations. Suppose a finite hypergroup $\mathfrak{G}$ admits a homomorphism $\alpha \mapsto \alpha'$ into $\mathfrak{G}(N)$, for some II_1 factor N; it is easily checked that $\alpha \mapsto d_{\alpha'}$ is a (and hence the) dimension function for $\mathfrak{G}$. Notice, on the other hand, that the irreducible cofinite morphism α' of N determines the II_1 factor $M_{d_{\alpha'}}(N)$

which contains (the copy of $\alpha'(N)$ of) N as a subfactor with trivial relative commutant and index equal to $(d_{\alpha'}d_{\alpha'^*}) = d_\alpha^2$, as a result of Theorem 10 and Remark 11(3) above. Thus, given a finite hypergroup $\mathfrak{G}$ which is known to admit an action on N, then each α in $\mathfrak{G}$ will give rise to a II_1 factor M_α that contains a copy of N as a subfactor with trivial relative commutant and index given by d_α^2.

We now pass to some nontrivial examples of finite hypergroups which are not groups or group-deals. In fact, all the examples we shall exhibit will correspond to finite 'cyclic' hypergroups that are 2-hypergroups in the sense that every element is self-contragredient. We begin by translating the problem of constructing finite hypergroups into one of constructing certain kinds of sets of nonnegative integral matrices.

13. **Proposition.** *There is an essentially one-to-one correspondence between finite hypergroups with cardinality n on the one hand, and sets $\{A_1, \ldots, A_n\} \subset M_n(\mathbf{Z}_+)$ satisfying the following conditions*:

(a) $A_1 = 1$, *the $n \times n$ identity matrix*;

(b) *the collection $\{A_i\}$ is linearly independent and selfadjoint—i.e., closed under the formation of transposes*;

(c) $A_iA_j = \sum_k A_i(k, j)A_k$ *for* $1 \leq i, j \leq n$.

Proof. Given a finite hypergroup $\mathfrak{G}$, order the elements as $\alpha_1 = 1, \alpha_2, \ldots, \alpha_n$ and let A_i denote the matrix representing the operator L_{α_i} with respect to the ordered orthonormal basis $\{f_{\alpha_1}, \ldots, f_{\alpha_n}\}$. The standard facts about hypergroups—developed earlier—show that the A_i's satisfy the conditions (a)–(c) of the proposition.

Suppose, conversely, that we are given $A_1, \ldots, A_n$ satisfying (a)–(c). Simply define $\langle \alpha_i \otimes \alpha_j, \alpha_k \rangle = A_i(k, j)$. We need to verify that $\{\alpha_1, \ldots, \alpha_n\}$ becomes the hypergroup $\mathfrak{G}$ if the 'product' is defined as above, if α_1 is taken as the identity, and if $\alpha_i^\#$ is defined as that unique element α_{i^*} for which $A_{i^*} = A_i^*$. Put $j = 1$ in (c) to deduce that $A_i = \sum_k A_i(k, 1)A_k$; it follows from the assumed linear independence of the A_i's that $\langle \alpha_i \otimes \alpha_1, \alpha_k \rangle = \delta_{ik}$; similarly, (a) in conjunction with (c) implies that $\langle \alpha_1 \otimes \alpha_j, \alpha_k \rangle = \delta_{jk}$, thus verifying the 'identity' axiom of a hypergroup. Associativity of matrix-multiplication, the definition of $\langle \alpha_i \otimes \alpha_j, \alpha_k \rangle$ and (c) imply that the associativity axiom is also satisfied. As for the contragredient axiom, note that $\langle \alpha_i^\# \otimes \alpha_k, \alpha_j \rangle = A_{i^*}(j, k) = A_i^*(j, k) = A_i(k, j) = \langle \alpha_i \otimes \alpha_j, \alpha_k \rangle$ and the proof is complete. □

14. **Example.** This is a sequence of 2-element hypergroups. Define $A_1^{(n)}$ to be the 2×2 identity matrix and let

$$A_2^{(n)} = \begin{bmatrix} 0 & 1 \\ 1 & n \end{bmatrix}.$$

These matrices satisfy the conditions of the proposition, and the corresponding hypergroup $\mathfrak{G}_n = \{1, \sigma_n\}$, where $f_{\sigma_n} * f_{\sigma_n} = f_1 + n f_{\sigma_n}$ in $\mathbf{C}\mathfrak{G}_n$; it is clear that the Perron-Frobenius eigenvalue (= operator-norm) of $A_2^{(n)}$—which is the same as d_{σ_n}—is equal to $\{n + (n^2+4)^{1/2}\}/2$. (It must be remarked that the case $n = 1$ of this example has been known to Ocneanu (cf. [O$_1$]); in fact, it follows from his description of his paragroup invariant for the inclusion $R \subset M$, where M is the II_1 factor constructed by Jones which has the hyperfinite II_1 factor R as a

subfactor with index $4\cos^2 \pi/5$, then $\mathcal{G}_1$ admits a homomorphism into $\mathcal{G}(R)$, that sends σ_1 to an element α of $\mathcal{G}(R)$ such that $L^2(M)$ is isomorphic, as an R-bimodule, to $\mathcal{H}_1 \oplus \mathcal{H}_\alpha$, where of course $\mathcal{H}_1$ denotes the 'trivial' bimodule $L^2(R)$.) □

15. **Example.** This is a sequence of examples indexed by a positive integer n. As a sample, consider the case $n = 4$; the matrices A_i are given by

$$A_1 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \qquad A_2 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix},$$

$$A_3 = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}, \qquad A_4 = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}.$$

In fact, the matrices A_1 and A_3 generate a 2-element hypergroup that is isomorphic to the hypergroup $\mathcal{G}_1$ of the preceding example.

For a general n, it is still the case that A_1 is the $n \times n$ identity matrix and that A_n is the matrix with 1's on the 'skew-diagonal' and 0's elsewhere; for a general $k \leq n$, form a diamond whose vertices are (at those entries of an $n \times n$ matrix that correspond to) $(1, k)$, $(k, 1)$, $(n, n-k+1)$ and $(n-k+1, n)$, and on each row, mark off every second entry of the matrix, starting and ending with the entries on the aforementioned diamond; finally, define A_k to be the matrix with 1's on the entries marked off as per the above prescription, and with 0's elsewhere. It is not hard to see that each A_k is a symmetric and entry-wise nonnegative matrix. A look at the first row of the A_k's is enough to see that the A_k's are linearly independent. A not very difficult, although somewhat tedious case-by-case, analysis reveals that the A_k's satisfy the condition (c) of Proposition 13 also and hence give rise to an n-element hypergroup. Let v denote the n-vector whose kth coordinate is

$$\{(\sin(k\pi/n+1))/(\sin(\pi/n+1))\};$$

it is an easy consequence of basic trigonometric identities that v, whose coordinates are clearly strictly positive, is an eigenvector for A_2 with eigenvalue $2\cos(\pi/n+1)$; thus v is the Perron-Frobenius eigenvector of A_2; it follows from Remark 11.1 that A_k has v as the Perron-Frobenius eigenvector with eigenvalue $\{\sin(k\pi/n+1)/\sin(\pi/n+1)\}$. Thus, if Conjecture 12 is proved, this example would say that the hyperfinite factor admits subfactors with trivial relative commutant and index equal to $\{\sin^2(k\pi/n+1)/\sin^2(\pi/n+1)\}$; this latter fact is known and is one of the significant conclusions of [W]. As has been remarked by Wenzl, these numbers include the Jones numbers—when $k = 2$. □

16. **Example.** For each n, there exists a unique collection $\{A_1, \ldots, A_n\}$ that satisfies the conditions of Proposition 13, such that A_2 is tridiagonal, has 1's on the sub- and super-diagonals, and such that the main diagonal of A_2 is given by $(0, 1, 1, \ldots, 1)$.

In fact, this hypergroup is isomorphic to the sub-hypergroup $\{\alpha_{2k-1} : 0 < k \leq n\}$ of the $2n$-element hypergroup $\{\alpha_1, \ldots, \alpha_{2n}\}$ discussed in the preceding example. This example is also known to Ocneanu. In fact, it follows from his description of the paragroup invariant for the inclusion $R \subset M$, where M is the II_1 factor constructed by Jones which contains the hyperfinite II_1 factor R as a subfactor with index $4\cos^2 \pi/(2n+1)$, that the hypergroup given by the previous paragraph admits an action on R, in such a way that, if α is the element of the hypergroup corresponding to A_2, then $L^2(M)$ is isomorphic, as an R-bimodule, to $\mathfrak{H}_1 \oplus \mathfrak{H}_n$. □

Before concluding this section, we pause to mention that we have only discussed examples of some cyclic 2-hypergroups (meaning every element is self-contragredient). In fact, there is a 5-element hypergroup of the above form, which does not appear in the lists of examples covered by Examples 15 and 16 (or even in the more general examples treated in [BS]), in which the value of the dimension function on the generator is $(1+\sqrt{12})/2$. The point that is being made is that if Conjecture 12 were to be proved, the above method, together with a better understanding of finite hypergroups, would yield a plethora of numbers that can arise as the index of a subfactor with trivial relative commutant.

V. The hypergroup of a bimodule

1. *Notation.* Throughout this section, the symbol $\mathfrak{H}$ will denote a fixed but arbitrary bifinite N-bimodule. Then $\mathfrak{H}$ uniquely determines—via Theorem I.13—a finite subset $\mathfrak{G}_1$ of $\mathfrak{G}(N)$ and a 'multiplicity' function $m_1 : \mathfrak{G}_1 \to \mathbf{N} = \{1, 2, 3, \ldots\}$ such that $\mathfrak{H} \simeq \bigoplus_{\alpha \in \mathfrak{G}_1} (\mathfrak{H}_\alpha \otimes \mathbf{C}^{m_1(\alpha)})$; for the sake of typographical convenience we shall write $\mathfrak{H} \sim (\mathfrak{G}_1, m_1)$ to signify that $\mathfrak{H}$ has the above decomposition. The sub-hypergroup $\mathfrak{G}$ of $\mathfrak{G}(N)$ that is generated by $\mathfrak{G}_1$ is clearly an (isomorphism-) invariant of $\mathfrak{H}$ and will be referred to as the hypergroup of $\mathfrak{H}$.

2. **Example** (The group case). Suppose a finite group G admits an outer action $t \to \alpha_t$ on N. Let M denote the crossed product $N \times_\alpha G$ and let $\mathfrak{H} = L^2(M)$. By definition, M contains the image of a unitary representation $t \to u_t$ of G satisfying $u_t a = \alpha_t(a) u_t$ for all a in N and t in G. Then, $\mathfrak{H} = \bigoplus_{t \in G} [N \cdot u_t] = \bigoplus_{t \in G} \mathfrak{H}_{\alpha_t}$ is a decomposition of $\mathfrak{H}$ into irreducible N-bimodules. Notice however that $\mathfrak{H}_{\alpha_t} = \mathscr{B}(1 \times 1 ; 1, \alpha_t)$ on the one hand, while on the other, we have $\mathfrak{H}_{\beta\otimes\gamma} = \mathfrak{H}_{\beta\circ\gamma}$ for any pair of automorphisms β and γ of N. It follows easily that—in view of the assumed outerness of the action—in this case, $\mathfrak{G} = \mathfrak{G}_1 = \{\alpha_t : t \in G\}$ and that $m_1(\alpha_t) = 1$ for all t. In other words, the hypergroup $\mathfrak{G}$ of $\mathfrak{H}$ and the group G are isomorphic as abstract hypergroups. This is an instance of what Ocneanu terms 'the crossed product remembering the group'; he has remarked—cf. [O$_2$]—that if G_1 and G_2 are groups acting outerly on the hyperfinite factor R, and if the extensions $M_i = R \times_{\alpha_i} G_i$ of R are isomorphic—meaning that there is a von Neumann algebra isomorphism of M_1 onto M_2 that fixes R, then G_1 and G_2 are isomorphic as groups. Our considerations yield something slightly stronger. For one thing, we do not need to assume hyperfiniteness. For another, even if one had started with a twisted crossed product of G, the hypergroup of the bimodule given by the extension would be identifiable with the group G. (Reason: the cocycle comes into play

only when two group elements are multiplied.) Since the statement is quite striking, we isolate it as the next proposition; of course, no more need be said about the proof.

3. **Proposition.** *For $i = 1, 2$, let α_i denote an outer action of a finite group G_i on any II_1 factor N, let σ_i $G \times G \to \mathbf{T}$ be a torus-valued 2-cocycle, and let M_i denote the twisted crossed product of N by G_i given by α_i and σ_i. Then the groups G_i are isomorphic if the N-bimodules $L^2(M_i)$ are isomorphic.* □

(Of course, the above statement can also be interpreted as stating that the hypergroup does not detect certain cohomological data and consequently cannot be expected to tell the whole story about the inclusion of one II_1 factor into another, as a subfactor of finite index. We have chosen to look at the positive side of things, since, as has already been stated, consideration of the hypergroup invariant of the bimodule yields a transparent proof of the preceding proposition.)

We turn next to tensor-products. Suppose $\mathcal{H}_i$, $i = 1, 2$, are bifinite N-bimodules with $\mathcal{H}_i \sim (\mathcal{G}_1^{(i)}, m_1^{(i)})$ as in §1 above. Thus $\mathcal{G}_1^{(i)}$ are finite subsets of $\mathcal{G}(N)$ and $m_1^{(i)}$ are nonnegative integer-valued elements of the hypergroup algebra $C\mathcal{G}(N)$ with support $\mathcal{G}_1^{(i)}$. It is not hard to see that

$$\mathcal{H}_1 \otimes \mathcal{H}_2 \sim (\mathcal{G}_1^{(1)} \cdot \mathcal{G}_1^{(2)}, m_1^{(1)} * m_1^{(2)})$$

where the second term is just convolution in the hypergroup algebra, and the product $\mathcal{K} \cdot \mathcal{L}$ of two subsets of a hypergroup $\mathcal{G}$ is defined naturally as the set of those γ in $\mathcal{G}$ for which there exist α in $\mathcal{K}$ and β in $\mathcal{L}$ such that $\langle \alpha \otimes \beta, \gamma \rangle > 0$. Thus, for instance,

$$m_1^{(1)} * m_1^{(2)}(\gamma) = \sum_{\alpha \in \mathcal{G}_1, \beta \in \mathcal{G}_2} m_1^{(1)}(\alpha) m_1^{(2)}(\beta) \langle \alpha \otimes \beta, \gamma \rangle .$$

We wish, in particular, to examine the invariant corresponding to the tensor powers of $\mathcal{H}$. The underlying structure becomes much more transparent if we make some mild assumptions on $\mathcal{H}$. We assume in the rest of this section that $\mathcal{H}$, apart from being required to be a bifinite N-bimodule, will be assumed to satisfy the following conditions:

(i) $\mathcal{H}$ is self-contragredient; and

(ii) $\mathcal{H}$ contains the trivial N-bimodule $\mathcal{H}_1 = L^2(N)$ as an N-submodule. (Notice that if M is a II_1 factor containing N as a subfactor of finite index—or more generally, if M is any finite von Neumann algebra containing N as a von Neumann subalgebra such that $\dim_N L^2(M, \mathrm{tr}) < \infty$—then the N-bimodule $L^2(M)$ satisfies the above conditions.)

4. **Proposition.** *Let $\mathcal{H} \sim (\mathcal{G}_1, m_1)$ and suppose $\mathcal{H}^n = \bigotimes^n \mathcal{H} \sim (\mathcal{G}_n, m_n)$; then*

(a) $1 \in \mathcal{G}_1 \subset \mathcal{G}_2 \subset \cdots \subset \mathcal{G}_n \subset \mathcal{G}_{n+1} \subset \cdots$;

(b) *each $\mathcal{G}_n$ is closed under taking contragredients;*

(c) $m_n(\gamma) = \sum_{\alpha \in \mathcal{G}_k} \sum_{\beta \in \mathcal{G}_l} m_k(\alpha) m_l(\beta) \cdot \langle \alpha \otimes \beta, \gamma \rangle$, *whenever $k, l \geq 0$ and $k + l = n$;*

(d) *for $\gamma \in \mathcal{G}(N)$, $\gamma \in \mathcal{G}_n$ if and only if $m_n(\gamma) > 0$;*

(e) $m_n(\alpha^\#) = m_n(\alpha)$ *for all α in $\mathcal{G}_n$.*

Proof. The assumptions (i) and (ii) on $\mathfrak{H}$ are clearly equivalent to the requirements '$1 \in \mathfrak{G}_1$' and '$\mathfrak{G}_1$ and m_1 are invariant under taking contragredients.' Since $\mathfrak{H}^n$ clearly inherits properties (i) and (ii) from $\mathfrak{H}$, the validity of (a), (b) and (e) follow. The assertions (c) and (d) follow from the remarks made above concerning the 'multiplicity function' associated with a tensor-product, and the associativity of the convolution product in any hypergroup algebra. □

5. **Lemma.** *With the notation as above, define* $\Lambda_n : \mathfrak{G}_n \times \mathfrak{G}_{n+1} \to \mathbf{Z}^+$ *by* $\Lambda_n(\alpha, \gamma) = \sum_{\beta \in \mathfrak{G}_1} m_1(\beta)\langle \alpha \otimes \beta, \gamma \rangle$. *Then*

(a) *if we think of* m_n *as a row-vector (of size* $1 \times \mathfrak{G}_n$*) and* Λ_n *as a matrix (of size* $\mathfrak{G}_n \times \mathfrak{G}_{n+1}$*), then* $m_{n+1} = m_n \Lambda_n$;

(b) $\Lambda_n(\alpha, \gamma) = \Lambda_{n+1}(\gamma, \alpha)$ $\forall \alpha \in \mathfrak{G}_n$ *and* $\gamma \in \mathfrak{G}_{n+1}$;

(c) $\Lambda_n(\alpha, \gamma) = \Lambda_m(\alpha, \gamma)$ $\forall \alpha \in \mathfrak{G}_n$ *and* $\gamma \in \mathfrak{G}_{n+1}$, $\forall m \geq n$.

Proof. (a) This follows easily from (c) of the previous proposition and the definition of Λ_n.

(b) Note to start with that the right side of the equality to be proved makes sense since $\mathfrak{G}_n \subset \mathfrak{G}_{n+2}$. We have

$$\begin{aligned} \Lambda_n(\alpha, \gamma) &= \sum_{\beta \in \mathfrak{G}_1} m_1(\beta)\langle \alpha \otimes \beta, \gamma \rangle \\ &= \sum_{\beta} m_1(\beta^\#)\langle \gamma \otimes \beta^\#, \alpha \rangle = \Lambda_{n+1}(\gamma, \alpha), \end{aligned}$$

thanks to (e) of the previous proposition.

(c) This is clear from the definition of the Λ_n's. □

6. **Proposition.** *Let* $k_n = \#(\mathfrak{G}_n \backslash \mathfrak{G}_{n-1})$ *(with* $\mathfrak{G}_0 = \varnothing$*). Then there exist symmetric matrices* $A_n \in M_{k_n}(\mathbf{Z}^+)$ *and rectangular matrices* $B_n \in M_{k_n \times k_{n+1}}(\mathbf{Z}^+)$ *such that the matrix* Λ_n *has the block decomposition*

$$\Lambda_n = \begin{bmatrix} A_1 & B_1 & 0 & 0 & \cdots & 0 & 0 & 0 \\ B_1^t & A_2 & B_2 & 0 & \cdots & 0 & 0 & 0 \\ 0 & B_2^t & A_3 & B_3 & \cdots & 0 & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdots & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & 0 & \cdots & B_{n-1}^t & A_n & B_n \end{bmatrix}$$

for all n, *where of course we write* C^t *for the transpose of the matrix* C.

Proof. Write $\Gamma_n = \mathfrak{G}_n \backslash \mathfrak{G}_{n-1}$ and express Λ_n in terms of the block-decomposition arising from the partition $\mathfrak{G}_n = \Gamma_1 \cup \cdots \cup \Gamma_n$. The desired conclusion follows from the previous lemma. For instance, (c) and (b) of the lemma show that $\Lambda_n(\alpha, \gamma) = \Lambda_n(\gamma, \alpha)$ for $\alpha, \gamma \in \mathfrak{G}_n$, whence the symmetry of the first n (blocks of) columns; on the other hand, it is clear that if $\alpha \in \mathfrak{G}_k$, $\beta \in \mathfrak{G}_l$, $\gamma \in \mathfrak{G}_m$ and if $m > k + l$, then $\langle \alpha \otimes \beta, \gamma \rangle = 0$, thereby explaining the tri-(block)-diagonality of Λ_n. □

(The preceding proof shows that it is the contragredient axiom for hypergroups that is responsible for much of the 'reflection' symmetries found in the Bratteli diagrams associated with the tower of the basic construction.) The following corollary is seen to follow easily from the last proposition. We continue with the preceding notation.

7. **Corollary.** *The following conditions are equivalent:*

(a) $\mathfrak{G}$ *is finite;*

(b) $\exists n$ *such that* $\mathfrak{G}_n = \mathfrak{G}_{n+1}$;

(c) $\exists n$ *such that* $\mathfrak{G} = \mathfrak{G}_n = \mathfrak{G}_{n+k}$ *for all* $k \geq 0$.

If these conditions are satisfied, then there exists a symmetric matrix Λ*—of size* $\#\mathfrak{G}$*—such that* $\Lambda_{n-k} = \Lambda$ *for all* $k \geq 0$ $(n \geq \#\mathfrak{G}$ *will suffice); further the matrix* Λ^{2n} *has strictly positive entries.*

Proof. Recall that $\{\mathfrak{G}_n\}$ is an increasing sequence of self-contragredient subsets of the hypergroup $\mathfrak{G}(N)$ and that $\mathfrak{G}_k \cdot \mathfrak{G}_l \subset \mathfrak{G}_{k+l}$. Consequently, the subhypergroup $\mathfrak{G}$ generated by $\mathfrak{G}_1$ is clearly the (increasing) union of the $\mathfrak{G}_n$'s. Since $\mathfrak{G}_1$ is finite, the equivalence of (a)–(c) is obvious.

If these conditions are satisfied, the last column in the block-decomposition of Λ_n given by the above proposition is nonexistent, and the resulting matrix, which is the same for all large n, is clearly symmetric. As for the last assertion, it suffices to show that $\Lambda^n(\alpha, 1) > 0$; we shall show, by induction, that more generally, it is true that $\Lambda^m(\alpha, 1) > 0$ if $\alpha \in \mathfrak{G}_m$. If $m = 1$, so $\alpha \in \mathfrak{G}_1$, hence $\Lambda(\alpha, 1) = \sum_{\beta \in \mathfrak{G}_1} m_1(\beta)\langle \alpha \otimes \beta, 1\rangle \geq 1 \cdot \langle \alpha \otimes \alpha^{\#}, 1\rangle > 0$. Suppose the assertion holds for m and suppose $\alpha \in \mathfrak{G}_{m+1}$; then there exists $\kappa \in \mathfrak{G}_m$ and $\beta \in \mathfrak{G}_1$ such that $\langle \kappa \otimes \beta, \alpha\rangle > 0$; hence $\langle \alpha \otimes \beta^{\#}, \kappa\rangle > 0$, whence it follows that $\Lambda(\alpha, \kappa) \geq m_1(\beta^{\#})\langle \alpha \otimes \beta^{*}, \kappa\rangle > 0$; on the other hand, we know from the induction hypothesis that $\Lambda^m(\kappa, 1) > 0$; the desired inequality follows from the obvious inequality: $\Lambda^{m+1}(\alpha, 1) \geq \Lambda(\alpha, \kappa)\Lambda^m(\kappa, 1)$.

Although the proof is complete, we would like to point out that an essentially identical reasoning shows that even if $\mathfrak{G}$ is not finite, if we let Z_n denote the square matrix obtained by restricting Λ_n to $\mathfrak{G}_n \times \mathfrak{G}_n$—or equivalently, by deleting the last column in the block-decomposition of Λ_n given by Proposition 6—then Z_n^{2n} has strictly positive entries. □

It turns out that there is a natural relationship between the matrices Λ_n and the tensor powers of $\mathfrak{H}$. We continue to work with the notation of Proposition 4.

8. **Proposition.** (a) ${}_N\mathscr{L}_N(\mathfrak{H}^n) \simeq \bigoplus_{\gamma \in \mathfrak{G}_n} M_{m_n(\gamma)}(\mathbf{C})$, *so that the minimal central projections of* ${}_N\mathscr{L}_N(\mathfrak{H}^n)$ *are parametrised by* G_n;

(b) *The inclusion of* ${}_N\mathscr{L}_N(\mathfrak{H}^n)$ *into* ${}_N\mathscr{L}_N(\mathfrak{H}^{n+1})$ *given by* $T \to T \otimes_N \mathrm{id}_{\mathfrak{H}}$ *is governed by the inclusion matrix* Λ_n *defined earlier.*

Proof. (a) By definition, we have $\mathfrak{H}^n \simeq \bigoplus_{\gamma \in \mathfrak{G}_n}(\mathfrak{H}_\gamma \otimes \mathbf{C}^{m_n(\gamma)})$; since distinct elements of $\mathfrak{G}_n$ yield inequivalent irreducible N-bimodules, and since irreducible N-bimodules admit only the scalars as bounded N-bilinear self-maps, it is clear that ${}_N\mathscr{L}_N(\mathfrak{H}^n) \simeq \bigoplus_{\gamma \in \mathfrak{G}_n} M_{m_n(\gamma)}(\mathbf{C})$.

(b) It is easily verified—cf. [C], for instance—that if $\mathfrak{M}$ and $\mathfrak{N}$ are bifinite N-bimodules and if $T \in {}_N\mathscr{L}_N(\mathfrak{M})$, there exists a unique operator $T \otimes_N \mathrm{id}_{\mathfrak{N}}$ in ${}_N\mathscr{L}_N(\mathfrak{M} \otimes \mathfrak{N})$ which sends $\xi \otimes_N \eta$ to $T\xi \otimes_N \eta$ whenever $\xi \in \mathfrak{M}_0$ and $\eta \in \mathfrak{N}_0$, and that, furthermore, the map $T \to T \otimes_N \mathrm{id}_{\mathfrak{N}}$ is a unital $*$-monomorphism of ${}_N\mathscr{L}_N(\mathfrak{M})$ into ${}_N\mathscr{L}_N(\mathfrak{M} \otimes_N \mathfrak{N})$. For typographical convenience, we shall write $T \otimes 1$ in the sequel for $T \otimes_N \mathrm{id}_{\mathfrak{H}}$ for T in ${}_N\mathscr{L}_N(\mathfrak{H}^m)$ for any m. If we temporarily let Λ_n' denote the inclusion matrix corresponding to the inclusion map $T \to T \otimes 1$, and if $\alpha \in \mathfrak{G}_n$ and $\gamma \in \mathfrak{G}_{n+1}$, then by definition $\Lambda_n'(\alpha, \gamma)$ is the maximum number of pairwise orthogonal minimal projections of ${}_N\mathscr{L}_N(\mathfrak{H}^{n+1})$

that are subordinate to $(p_\alpha \otimes 1)e_\gamma$, where p_α denotes any minimal projection of ${}_N\mathscr{L}_N(\mathfrak{H}^n)$ that belongs to the central summand corresponding to α, and e_γ denotes the minimal central projection of ${}_N\mathscr{L}_N(\mathfrak{H}^{n+1})$ corresponding to γ. A moment's thought shows that fixing a p_α amounts to fixing an isometry in ${}_N\mathscr{L}_N(\mathfrak{H}_\alpha, \mathfrak{H}^n)$, while finding pairwise orthogonal minimal projections of ${}_N\mathscr{L}_N(\mathfrak{H}^{n+1})$ that are all subordinate to e_γ amounts to finding isometries in ${}_N\mathscr{L}_N(\mathfrak{H}_\gamma, \mathfrak{H}^{n+1})$ with pairwise orthogonal ranges. Since $\operatorname{ran} T \subset \operatorname{ran} e_\gamma$ for any T in ${}_N\mathscr{L}_N(\mathfrak{H}_\gamma, \mathfrak{H}^{n+1})$ and since $(p_\alpha \otimes 1)(\mathfrak{H}^{n+1})$ is isomorphic as an N-bimodule to $(\mathfrak{H}_\alpha \otimes_N \mathfrak{H})$, it follows

$$\begin{aligned}
\Lambda_n'(\alpha, \gamma) &= \dim_{\mathbf{C}} {}_N\mathscr{L}_N(\mathfrak{H}_\gamma, e_\gamma(p_\alpha \otimes 1)(\mathfrak{H}^{n+1})) \\
&= \dim_{\mathbf{C}} {}_N\mathscr{L}_N(\mathfrak{H}_\gamma, \mathfrak{H}_\alpha \otimes_N \mathfrak{H}) \\
&= \dim_{\mathbf{C}} {}_N\mathscr{L}_N\left(\mathfrak{H}_\gamma, \bigoplus_{\beta \in \mathfrak{G}_1} (\mathfrak{H}_\alpha \otimes_N (\mathfrak{H}_\beta \otimes_{\mathbf{C}} \mathbf{C}^{m_1(\beta)})\right) \\
&= \sum_{\beta \in \mathfrak{G}_1} m_1(\beta)\langle \alpha \otimes \beta, \gamma \rangle \\
&= \Lambda_n(\alpha, \gamma), \quad \text{as desired.} \quad \square
\end{aligned}$$

9. *Remarks.* (1) The data about the tower $\{{}_N\mathscr{L}_N(\mathfrak{H}^n) : n \geq 1\}$ is completely contained in the abstract hypergroup $\mathfrak{G}$, its generating (finite) subset $\mathfrak{G}_1$ and the multiplicity function $m_1 : \mathfrak{G}_1 \to \mathbf{Z}^+$; the point being made is that $\mathfrak{G}$ need not be realized as a sub-hypergroup of $\mathfrak{G}(N)$—or equivalently, it is not necessary '*a priori*' to be given a faithful action of $\mathfrak{G}$ on N—in order to make sense of the above AF-algebra.

(2) In case the hypergroup $\mathfrak{G}$ of $\mathfrak{H}$ is finite, it follows from Corollary 7 that the tower $\{{}_N\mathscr{L}_N(\mathfrak{H}^n)\}$ admits a unique normalized tracial state; in particular, the completion of the above AF-algebra with respect to the trace is the hyperfinite II_1 factor.

(3) We have considered the 'left', rather than the 'right' inclusion $T \to 1 \otimes T$ in building the tower. It is more than likely that it is possible to imitate the work of Doplicher and Roberts—cf. [DR]—with the role of the group dual of a compact group being replaced by more general hypergroups. $\square$

We conclude with a list of questions (of varying levels of difficulty) which, for the sake of emphasis, we start with what was earlier stated as Conjecture IV.12.

VI. Some questions

(a) Does every finite hypergroup admit a faithful action on the hyperfinite II_1 factor, and if so, what is the extent of uniqueness of such an action?

(b) Is there a classification of finite abelian hypergroups? What about finite cyclic hypergroups? 2-hypergroups?

(c) What are some nontrivial examples of finite nonabelian hypergroups which are not groups?

(d) If α and α' are generators of finite hypergroups $\mathfrak{G}$ and $\mathfrak{G}'$, is it possible for $\mathfrak{G}$ and $\mathfrak{G}'$ to be nonisomorphic and yet admit faithful actions on a II_1 factor N in such a way that the cofinite morphisms corresponding to α and α' yield isomorphic extensions of N? (The reason for this question is that there

are instances—the cases $l = 0$ and $l = 1$, for instance—of nonisomorphic $\mathfrak{G}$ and $\mathfrak{G}'$ such that the generators have the same value for the dimension function.)

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