Quaestiones Mathematicae **24**(2001), 225–235.

# BOUNDED SUBSETS AND WEAK REALCOMPACTNESS CONDITIONS

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Abstract. A subset $A$ of $X$ is bounded if every continuous real-valued function on $X$ is bounded on $A$. A completely regular Hausdorff space $X$ is said to have the *bz*-property if every bounded subset of $X$ is contained in a bounded zero subset of $X$. In this paper, we study the *bz*-property and its relation to other well known topological properties. We also introduce some new topological properties, all weaker than realcompactness, that are related to the *bz*-property. The origin of the *bz*-property lies in a measure-theoretic problem.

*Mathematics Subject Classification (2000):* 54C50, 54F99, 54G20.

*Key words:* Bounded set, zero set, (weak) *bz*-space, (weak) $\mu$-space, space of pseudo-countable type, unbounded point, untracable point.

**1. Introduction.** The topological spaces in this paper will always be completely regular and Hausdorff. If $X$ is a topological space, we write $C(X)$ for the ring of continuous real-valued functions on $X$ and $C^*(X)$ for the bounded functions in $C(X)$, and we write $\beta X$ (respectively $\upsilon X$) for the Stone-Čech compactification (respectively, the Hewitt realcompactification) of $X$. We denote $\beta X - X$, the *growth* of $X$, by $X^*$. A *zero-set* of $X$ is a set of the form $f^{\leftarrow}\{0\}$ for $f \in C(X)$. We denote the countable infinite cardinal by $\omega$, the first uncountable cardinal by $\omega_1$ and the cardinal number of the continuum by $\mathbf{c}$. If $\kappa$ is any ordinal, then $\kappa = \{\alpha : \alpha$ is an ordinal and $\alpha < \kappa\}$. $\mathbb{R}$ denotes the real numbers and $\mathbb{N}$ denotes the positive integers.

Let $X$ be a topological space. We denote by $C_k(X)$ the topological space with underlying set $C(X)$ equipped with the compact-open topology $k$ and note that

$C_k(X)$ is a locally convex space. A subset of a locally convex space is called a *barrel* if it is absolutely convex, absorbing and closed, and a locally convex space is called *barrelled* if every barrel is a neighborhood of $\mathbf{0}$. A space $X$ is a $\mu$-space if every bounded subset of $X$ has compact closure. A celebrated result of Nachbin and Shirota is that $C_k(X)$ is barrelled if and only if $X$ is a $\mu$-space (see [14]).

In view of this result, H. Buchwalter in [6] introduced another topology on $C(X)$ using bounded sets instead of compact sets. This topology, called the *bounded-open* topology and denoted by $C_b(X)$, is generated by the collection of seminorms $\{p_A : A$ is a bounded subset of $X\}$ where $p_A(f) = \sup\{|f(x)| : x \in A\}$ for $f \in C(X)$.

Also in [6], Buchwalter established the barrelledness of $C_b(X)$ and showed that the dual of $C_b(X)$ can be identified with a collection of measures on $X$. In 1982 Arhangelskiĭ in [1] considered independently the same topology on $C(X)$. In that paper, he made very clever use of this topology to obtain results about the relation between spaces $X$ and $Y$ having certain properties in common and $C_b(X)$ and $C_b(Y)$ being linearly homeomorphic. In [11], some topological properties of $C_b(X)$ were studied in a more general perspective, and the exact position occupied by Buchwalter's topology (i.e. the bounded-open topology) in the hierarchy of several topologies on $C(X)$ was determined. In [12], the dual of $C_b(X)$ was studied in order to have a better understanding of it in relation to the better-known duals of $C_k(X)$ and $C^*_\infty(X)$, where $C^*_\infty(X)$ is the topology of uniform convergence on $C^*(X)$.

One major goal of [12] was to find a nice measure-theoretic counterpart of the dual $\Lambda_b(X)$ of $C_b(X)$ in terms of measures on $X$. For normal $X$ [12, 5.4] gives such a result. For $X$ not necessarily normal, the best result hitherto known (at least to the authors) is Theorem 5.6 in [12]. For the sake of completeness we briefly state the theorem and some background below.

The set $\Lambda_b(X)$, the space of all continuous linear functionals on $C_b(X)$, may be considered in a natural way as a linear subspace of the Banach space $\Lambda_\infty(X)$, the dual space of $C^*_\infty(X)$. The norm of an element $\lambda$ in $\Lambda_\infty(X)$ is defined by $||\lambda||_* = \sup\{|\lambda(f)| : f \in C^*(X) \quad \text{and} \quad ||f||_\infty \leq 1\}$. An element $\lambda$ of $\Lambda_b(X)$ may be identified with $\lambda \circ i$ in $\Lambda_\infty(X)$ where $i : C^*(X) \hookrightarrow C(X)$ is the inclusion map. Let $Ba^*(X)$ stand for the algebra generated by the zero-sets of $X$. Let $BM(X)$ be the space of all finitely additive, bounded and zero-set regular measures defined on $Ba^*(X)$ equipped with the total variation norm and $BM_{bz}(X) = \{\mu \in BM(X) :$ $\mu$ has a support $A \subset X$, where $A$ is a bounded zero subset of $X\}$. The statement of the theorem follows.

Theorem 1.1. ([12, 5.6]) *Suppose every bounded subset of $X$ is contained in a bounded zero subset of $X$. Then the map*

$$F : (BM_{bz}(X), ||\cdot||) \longrightarrow (\Lambda_b(X), ||\cdot||_*)$$

*defined by $F(\mu)(f) = \int_X f d\mu$ is an isometric lattice isomorphism of $BM_{bz}(X)$ onto $\Lambda_b(X)$.*

Theorem 1.1 naturally raises the following problem: When is each bounded subset of $X$ contained in a bounded zero-set of $X$?

It is this problem that we discuss in the main body of the paper. But first we need some definitions and preliminary results.

The authors would like to thank J. Connor for some helpful comments.

## 2. Definitions and preliminaries.

The problem mentioned above gives rise to the following definition. We call a space in which every bounded set is contained in a bounded zero-set a *bz-space* and call this property of a space the *bz-property.* Clearly pseudocompact and perfectly normal spaces have the $bz$-property (and hence for perfectly normal spaces, Theorem 1.1 and [12, 5.4] coincide).

In the discussion below we are going to discuss concepts, some of which have been introduced before by other authors, often under a variety of names. We will give as many of these names as we know in the hope that the reader, who finds a familiar one, will be saved some time.

It should be noted, for example, that some authors (e.g. J. Trigos-Arrieta in [19]) call bounded sets "functionally bounded sets" while others (e.g. T. Isiwata in [10]) call them "relatively pseudocompact". Some prefer the expression "$C(X)$-bounded sets". The term "bounded subsets of $X$" stems from Buchwalter's work referred to above and thus will be used in this paper which has its origins in that work. (See [13] for a discussion of this topic. The authors thank O. Pavlov for calling [13] to our attention.)

It is clear that $\mu$-spaces play an important role in any theory involving bounded sets, and the theory of $bz$-spaces is no exception. A $\mu$-space is also known as a Nachbin-Shirota space (see [2]) or a hyperisocompact space (see [5]). Since the intersection of $\mu$-spaces is a $\mu$-space and since $\beta X$ is a $\mu$-space, there is a space $\mu X$ that is the smallest subspace of $\beta X$ which contains X and is a $\mu$- space. Thus $X$ is a $\mu$-space if and only if $X = \mu X$.

For any space $X$, we have $X \subseteq \mu X \subseteq \upsilon X \subseteq \beta X$ (and thus realcompact spaces are $\mu$-spaces). If $A$ is a bounded subset of $X$, then $cl_{\beta X} A \subseteq \mu X$. For details on $\mu X$, see [3], [6] and [22]. In the present work, we study the abovementioned problem; that is, we would like to determine which spaces have the $bz$-property. In addition we study properties weaker than the $bz$-property and the property of being a $\mu$-space and we study how other topological properties are related to these properties.

Recall that a space $X$ is of *pseudocountable (resp., countable) type* if each compact subset of $X$ is contained in a compact $G_\delta$-subset of $X$ (resp., a compact subset of $X$ with a countable base of neighborhoods).

Clearly a space of countable type is of pseudocountable type. Čech-complete spaces are of countable type and hence so are metrizable and locally compact spaces. See [20] for details on spaces of countable type.

Since compact $G_\delta$-sets in $X$ are bounded zero-sets of $X$, it is easy to see that a $\mu$-space $X$ is a $bz$-space if and only if $X$ is of pseudocountable type. More generally, spaces of pseudocountable type are related to $bz$- spaces by the following proposition, whose proof is found in [12].

Proposition 2.1. *Let $X$ be a space and suppose there is a space $Y$ of pseudocountable type with $\mu X \subseteq Y \subseteq \upsilon X$. Then $X$ is a bz-space.*

Proposition 2.1 leads to a sufficient condition for $X$ to be a $bz$-space. (A more general result is found in 4.2.)

Corollary 2.2. *Locally compact $\mu$-spaces are $bz$-spaces.*

Neither hypothesis in 2.2 is sufficient nor necessary, however, as we shall see in the sequel. Example 3.6(a) is a $\mu$-space and Example 4.1 is locally compact. Neither is a $bz$-space. Example 3.6(b) is a $bz$-space which is neither locally compact nor a $\mu$-space. It should also be noted that Example 3.6(b) shows that $bz$-spaces need not be of pseudocountable type.

Normality does not seem to be related to the $bz$-property. The space of Example 3.6(a) is Lindelöf, hence normal and realcompact, but does not have the $bz$-property. Also any pseudocompact non-normal space (e.g. the space $\Psi$ of [9, 5I]) is a $bz$-space that is not normal

Another concept that will be needed later is the following. A subset $A$ of $X$ is *well embedded* in $X$ if $A$ is completely separated in $X$ from every disjoint zero-set of $X$. It is well known that zero-sets of $X$ are well embedded in $X$.

A subset A of $X$ is *strongly bounded* if for every cozero-set neighborhood $P$ of $A$, every function in $C(P)$ is bounded on $A$. Strongly bounded sets are called strongly relatively pseudocompact in [5]. In [18] M. Tkachenko used "strongly bounded" in a meaning different from that of this paper.

Proposition 2.3. *If $A \subset X$, then the following are equivalent.*

1. *$A$ is strongly bounded.*
2. *$A$ is bounded and well-embedded in $X$.*
3. *For all $f \in C(X)$, $f^{\rightarrow}(A)$ is compact.*

*Proof.* (1)⇔(2). See Proposition 2.7 in [5].

(1)⇒(3). Let $A$ be strongly bounded and let $f : X \to \mathbb{R}$. Suppose $f^{\rightarrow}(A)$ is not compact. Since $A$ is bounded, there is $z \in (cl_{\mathbb{R}} f^{\rightarrow}(A)) - f^{\rightarrow}(A)$. Pick a sequence $\langle z_n : n \in \mathbb{N}\rangle$ in $f^{\rightarrow}(A)$ such that $z_n \to z$ and pick a sequence $\langle a_n : n \in \mathbb{N}\rangle$ such that $f(a_n) = z_n$. Now $P = \mathbb{R} - f^{\leftarrow}\{z\}$ is a cozero-set of $X$ containing $A$. Let $g = \frac{1}{|f-z|}$. Note that $g \in C(P)$ but $g$ is not bounded on $A$.

(3)⇒(2). Assume (3). Clearly $A$ is bounded. Let $Z(f)$ be a zero-set of $X$ with $A \cap Z(f) = \emptyset$. Then $|f|^{\rightarrow}(A) \cap \{0\} = \emptyset$. Since $|f|^{\rightarrow}(A)$ is compact, $|f|^{\rightarrow}(A) \subset [\epsilon, \infty)$ for some $\epsilon > 0$. Then $A \subset |f|^{\leftarrow}[\epsilon, \infty)$ and so $A$ is completely separated from $Z(f)$. We conclude that (2) holds. □

Note that by (2)⇒(1), bounded zero-sets are strongly bounded. Also note that in [8], (3) is called "$C$-compact".

A space $X$ is called a *weak $\mu$-space* if every strongly bounded subset is relatively compact. Weak $\mu$-spaces are also called strongly isocompact spaces (see [5]). Clearly $\mu$-spaces are weak $\mu$-spaces. The converse does not hold, however, as can be seen in [5, 3.9 (3)].

We call a space a *weak bz-space* if every strongly bounded subset is contained in a bounded zero-set. Clearly $bz$-spaces are weak $bz$-spaces. Again the converse does not hold. (See Example 3.6(c)). The next proposition is obvious.

Proposition 2.4. *If $X$ is a weak $\mu$-space and a $bz$-space, then $X$ is a $\mu$-space.*

**3. Generalized realcompactness conditions.** We first give some definitions of known weak realcompactness conditions that are related to $\mu$-spaces and to the $bz$-property. We say that a space $X$ is *p-realcompact* (resp. *nearly realcompact*) if every zero-set (resp. every open set) of $\beta X$ that meets $X^*$ meets $\beta X - \upsilon X$. Nearly realcompact spaces were defined in [4] and $p$-realcompact spaces were defined in [17]. The latter are called $p$-realcompact because they are precisely the spaces that have, in the presence of local compactness, growths that are $P'$-spaces (that is, all zero-sets are regular closed). In [17] it is shown that realcompact spaces are $p$-realcompact, and $p$-realcompact spaces are nearly realcompact, and that these are distinct classes of spaces.

For $f \in C^*(X)$ we let $f^\beta$ denote the extension of $f$ to $\beta X$.

The first proposition is essentially a consequence of [17, 2.11].

Proposition 3.1. *If $X$ is p-realcompact then for every zero-set $Z(f)$ of $X$ that is not compact, $Z(f^\beta)$ meets $\beta X - \upsilon X$ and hence $|X^* \cap Z(f^\beta)| \geq 2^c$.*

Thus every zero-set of a $p$-realcompact space is either compact or has a very large growth. We also have the following from [17, 2.6].

Proposition 3.2. *Weak $\mu$-spaces are p-realcompact.*

The converse does not hold. See Ex. 2.8(2) of [17].

We say that a function $f \in C(X)$ is *well separated* if whenever $Z(f) \subset P$ where $P$ is a cozero-set of $X$, there is $n \in \mathbb{N}$ such that $f^{\leftarrow}(-\frac{1}{n}, \frac{1}{n}) \subset P$. It was shown in [16, 4.1] that $f$ is a well separated function if and only if $cl_{\beta X} Z(f) = Z(f^\beta)$. (And hence a space $X$ is pseudocompact if and only if every function in $C^*(X)$ is well separated, see [16, 4.2].) Thus we have the following proposition.

Proposition 3.3. *If $X$ is p-realcompact, then for every non-compact bounded zero-set $Z(f)$ and for all well separated functions $g \in C^*(X)$, $Z(f) \neq Z(g)$.*

*Proof.* Let $Z(f)$ be a non-compact bounded zero-set of a space $X$. Now suppose there is a well separated function $g \in C^*(X)$ with $Z(f) = Z(g)$. Then $Z(g)$ is a non-compact bounded zero-set of a well separated function, and so by [17, 3.8], $X$ is not $p$-realcompact. □

Proposition 3.4. *The following are equivalent.*

1. *$X$ is a weak $\mu$-space and a weak $bz$-space.*

2. *$X$ is p-realcompact and every strongly bounded subset is contained in the bounded zero-set of a well separated function.*

*Proof.* (1)⇒(2). $X$ is $p$-realcompact by 3.2. Let $A$ be a strongly bounded subset of $X$. Since $X$ is a weak $bz$-space, there is a bounded zero-set $Z(f)$ with $A \subseteq Z(f)$. Since $X$ is a weak $\mu$-space, $Z(f)$ is compact. Let $h = \min(f, 1)$ and let, for all $n \in \mathbb{N}$, $h_n : \beta X \to [0,1]$ be such that $h_n = 0$ on $Z(f) = Z(h)$ and $h_n = 1$ on $\beta X - cl_{\beta X}(X - h^{\beta\leftarrow}[\frac{1}{n}, 1])$. Let $k = \sum_{n\in\mathbb{N}} \frac{h_n}{2^n}$ and let $g = k \restriction X$. It is easy to see that $g$ is well separated and that $Z(f) = Z(g)$.

(2)⇒(1). Let $A$ be strongly bounded and by (2) let $A \subset Z(f)$ where $Z(f)$ is bounded and $f$ is well separated. Since $X$ is $p$-realcompact, $Z(f)$ is compact by 3.3. Then $X$ is both a weak $bz$-space and a weak $\mu$-space. □

COROLLARY 3.5. *If a weak $\mu$-space $X$ is also a weak $bz$-space, then $X$ is of pseudocountable type.*

EXAMPLES 3.6. In this section we give several examples that distinguish between some of the concepts defined above.

(a) We first give an example of a realcompact space that does not have the $bz$-property.

Let $X$ be a discrete space and $|X| \geq \omega_1$ . Let $X' = X \cup \{\infty\}$ be the one-point Lindelöfization of $X$. (Thus $X'$ is also normal.)

Consider the singleton set $C = \{\infty\}$. Let $K$ be a bounded zero-set of $X'$ which contains $\infty$. Since $X'$ is realcompact, $K$ is compact and consequently $K$ must be finite.

Now, if in addition, $K$ is a zero-set, let $K = \bigcap_{n=1}^{\infty} U_n$ where each $U_n$ is open in $X'$. But then $X' \setminus \bigcap_{n=1}^{\infty} U_n = \bigcup_{n=1}^{\infty}(X' \setminus U_n)$. The right hand side is countable while the left hand side is uncountable. Hence the point $\infty$ is not contained in any bounded zero-set of $X'$.

(b) We are now going to show that a $bz$-space need not be a space of pseudocountable type. We describe Example 3.10.19 in [7].

For every set $X$ we set $[X]^\omega = \{A \subseteq X : |A| = \omega\}$. Let $f : [(\beta\mathbb{N})]^\omega \to \beta\mathbb{N}$ be such that for all $A \in [\beta\mathbb{N}]^\omega$, $f(A)$ is an accumulation point of $A$ in the space $\beta\mathbb{N}$. Let $X_0 = \mathbb{N}$ and for an ordinal $\alpha$ where $0 < \alpha < \omega_1$, let $X_\alpha = (\bigcup_{\beta<\alpha} X_\beta) \cup \{f(A) : A \in [\bigcup_{\beta<\alpha} X_\beta]^\omega\} \subset \beta\mathbb{N}$. Let $X = \bigcup_{\alpha<\omega_1} X_\alpha$ have the subspace topology derived from $\beta\mathbb{N}$.

Clearly every countably infinite subset of $X$ will be contained in some $X_\alpha$ and thus will have an accumulation point in $X$. Hence $X$ is countably compact and thus is a $bz$-space. We claim that $X$ is not of pseudocountable type. To establish this, let us first recall the following important properties of $\beta\mathbb{N}$. (We remind the reader that $\beta\mathbb{N} - \mathbb{N}$ is denoted by $\mathbb{N}^*$.)

(i) Every infinite closed subset of $\beta\mathbb{N}$ has cardinality $2^c$. ([9, 9.3])

(ii) Every clopen subset of $\mathbb{N}^*$ is of the form $(cl_{\beta\mathbb{N}}M) \cap \mathbb{N}^*$ where $M \in [\mathbb{N}]^\omega$. ([9, 6S(3)])

(iii) Every non-empty $G_\delta$-set in $\mathbb{N}^*$ has non-empty interior in $\mathbb{N}^*$ and thus contains a non-empty clopen subset of $\mathbb{N}^*$. ([9, 6S(8)])

Since $|X| \leq \mathbf{c}$, an immediate consequence of (a) is that every compact subset of $X$ is finite. If X is of pseudocountable type, then every finite subset of $X$ has to be contained in a finite $G_\delta$-subset of $X$. We show that this is not the case.

Let $p \in X - X_1$. If $\{p\}$ is contained in a compact (hence finite) $G_\delta$-set $G$, then $\{p\}$ itself is a $G_\delta$-set in $X$ and so $\{p\} = X \cap \bigcap_{n\in\mathbb{N}} H_n$ where each $H_n$ is clopen in $\beta\mathbb{N}$. But by (c), $(\bigcap_{n\in\mathbb{N}} H_n) - \mathbb{N}$ contains a non-empty clopen set which, by (b) is of the form $\mathbb{N}^* \cap cl_{\beta\mathbb{N}}M$ where $M \in [\mathbb{N}]^\omega$. Now $f(M) \in X_1 \cap \bigcap_{n\in\mathbb{N}} H_n$ and so $f(M) = p$. But $p \notin X_1$. This contradiction establishes the claim.

(c) We show that weak $bz$-spaces need not be $bz$-spaces. Let $T = (\omega_1 + 1) \times (\omega + 1) \setminus \{(\omega_1, \omega)\}$ be the Tychonoff plank. Let $T_i = \{(\alpha, \sigma, i) : (\alpha, \sigma) \in T\}, i = 1, 2$ be two distinct copies of $T$.

Let $Y$ be the quotient space obtained from the union of $T_1$ and $T_2$ on identifying $(\omega_1, n, 1)$ with $(\omega_1, n, 2)$ for each $n < \omega$. After this identification, the points $(\omega_1, n, 1)$ and $(\omega_1, n, 2)$ will be denoted by $(\omega_1, n)$, for all $n < \omega$.

A typical neighborhood of $(\omega_1, n) \in Y$ is of the form $\{(\omega_1, n)\} \cup \{(\gamma, n, 1) : \alpha < \gamma < \omega_1\} \cup \{(\delta, n, 2) : \beta < \delta < \omega_1\}$ for some $\alpha, \beta \in \omega_1$.

Let $X = Y \setminus \{(\alpha, \omega, 2) : \alpha < \omega_1\}$ be endowed with the subspace topology. $X$ is locally compact.

$X$ is not a $bz$-space. Consider the subset $B = \{(\omega_1, n) : n < \omega\}$ of $X$. $B$ is a bounded set. Let $Z(f)$ be a zero set containing $B$, where $f \in C(X)$. We can find $\alpha_1, \alpha_2 \in \omega_1$ such that the subset $\{(\gamma, n, 1) : \alpha_1 < \gamma < \omega_1, n < \omega\} \cup \{(\delta, n, 2) : \alpha_2 < \delta < \omega_1, n < \omega\}$ is contained in $Z(f)$.

Choose any non-limit ordinal $\beta$ with $\alpha_2 < \beta < \omega_1$ and consider the set $A = \{(\beta, n, 2) : n < \omega\}$. It can be seen that $A$ is an infinite discrete clopen subset of $X$ contained in $Z(f)$. Thus $Z(f)$ cannot be bounded. Consequently, no zero-set containing the bounded set $B$ is bounded.

Note that $B$ is not strongly bounded since it is not completely separated from $\omega_1 \times \{\omega\} \times \{1\}$. One can check that the strongly bounded subsets of $X$ are contained in bounded zero-sets. Hence $X$ is a weak $bz$-space.

**4. Bounded and traceable points.** In [6], Buchwalter asked whether

$$\mu X = \cup\{cl_{\beta X} B : B \text{ is a bounded subset of } X\},$$

and in [3], Blasco answered this question in the negative by producing the following counterexample.

EXAMPLE 4.1. In Question 2, page 181 in [3] by Blasco, there is an example of a locally compact space that is not a $bz$-space. The example we want is a subspace of this space. For completeness we describe Blasco's example and then show that it is not a $bz$-space..

Let $Y = (\omega_1+1)\times(\omega+1)$. If $\alpha < \omega_1$ is a limit ordinal, choose a strictly increasing sequence $\{r_n : n \geq 1\}$ of ordinals in $\alpha$ converging to $\alpha$. Define $\alpha_n = r_n + 1$, for all $n \geq 1$. Then $\{\alpha_n : n \geq 1\}$ is a strictly increasing sequence of non-limit ordinals from $\omega_1$ which converges to $\alpha$. For $m \geq 1$, define $A_{m,\alpha} = \{(\delta, m) : \alpha_m \leq \delta \leq \alpha\}$ and $U_{n,\alpha} = \{(\alpha, \omega)\} \bigcup (\bigcup_{m=n}^{\infty} A_{m,\alpha})$.

Let $\tau$ be the topology on $Y$ determined by the following neighborhood bases:

(1) For $(\eta, \sigma) \in Y \setminus \{(\alpha, \omega) : \alpha$ a limit ordinal and $\alpha < \omega_1\}$, the neighborhood base is the same as in the product topology of $Y$.

(2) For $(\alpha, \omega)$, where $\alpha < \omega_1$ is a limit ordinal, a base for the neighborhoods is the family $\{U_{n,\alpha} : n \geq 1\}$.

Let $X = Y \setminus (\{\omega_1, \omega)\}$. It has been established in [3] that $X$ is locally compact and is C-embedded in $Y$.

We show next that $X$ is not a *bz*-space. Consider the subset $B$ of $X$ given by $B = \{(\omega_1, n) : 1 \leq n < \omega\}$. Given any $f \in C(X)$, there exists $\gamma < \omega_1$ such that $f((\gamma, \omega)) = \lim_{n\to\infty} f((\omega_1, n))$. Thus $B$ is a bounded subset of $X$. Let $Z(g)$ be a zero-set containing $B$, where $g \in C(X)$. We claim that $Z(g)$ is not a bounded subset of $X$. Since $g((\omega_1, n)) = 0$ for all $n < \omega$, it is possible to find a $\gamma < \omega_1$ such that $g((\delta, n)) = 0$ for $\gamma \leq \delta \leq \omega_1$ and $n < \omega$; that is, $\{(\delta, n) : \gamma \leq \delta \leq \omega_1, 1 \leq n < \omega\} \subseteq Z(g) \subseteq X$.

Choose a limit ordinal $\alpha$ such that $\gamma < \alpha < \omega_1$ and choose a strictly increasing sequence $\{\gamma_n : n \in \mathbb{N}\}$ of non-limit ordinals such that $\gamma_n \nearrow \alpha$. Consider the subset $A = \{(\gamma_n, n) : n \in \mathbb{N}\}$ of $Z(g)$. It can be shown that $A$ is a clopen subset of $X$ which is an infinite discrete set. Consequently $Z(g)$ is not bounded. This proves that the bounded set $B$ cannot be contained in a bounded zero-set.

Now let $Z = Y \setminus \{(\omega_1, n) : 1 \leq n \leq \omega\}$where $Z$ has the subspace topology induced by $\tau$. Blasco notes that

1. $Z$ is locally compact.
2. $X = \bigcup\{cl_{\beta Z} B : B \in \mathcal{B}\}$ where $\mathcal{B}$ is the family of all closed bounded subsets of $Z$.
3. $Z \subsetneq X \subsetneq \mu Z$.

This example leads to the following definition. We say that a space $X$ is a *near $\mu$-space* if for any compact subset $A$ of $\mu X$ there exists a bounded subset $B$ of $X$ with $A \subseteq cl_{\mu X} B$. Clearly if a space is a near $\mu$-space, then Buchwalter's question is answered in the affirmative for that space. Obviously $\mu$-spaces are near $\mu$-spaces, and if $\mu X$ is a *bz*-space, then $X$ is a near $\mu$-space. Note that pseudocompact spaces are also near $\mu$-spaces. We also have the following proposition whose easy proof is omitted.

Proposition 4.2. *If $X$ is a near $\mu$-space, then $X$ is a bz-space if and only if $\mu X$ is of pseudocountable type.*

We call a point $x \in \mu X$ an *unbounded point* of $X$ if $x \notin \bigcup\{cl_{\mu X} B : B$ is bounded in $X\}$. Otherwise $x$ is called *bounded.* Clearly a near $\mu$-space has no unbounded points.

We call a point $x \in \mu X$ a *traceable point* of $X$ if some neighborhood of $x$ in $\mu X$ traces to a bounded subset of $X$; that is, if $x$ has a neighborhood $P$ in $\mu(X)$

such that $P \cap X$ is bounded. Otherwise $x$ is *untraceable*. Clearly unbounded points are untraceable, but the converse does not hold. Observe that in Example 3.6 (c), $\langle \omega_1, \omega \rangle \in cl_{\mu X}\{\langle \omega_1, n\rangle : n \in \omega\}$ is bounded and in $\mu X - X$ but no neighborhood of $\langle \omega_1, \omega \rangle$ traces to a bounded subset of $X$. Note that each point of $\mu Z - X$ is an unbounded point of $Z$.

We say that a space $X$ is *locally bounded at a point* $x \in X$ if $x$ has a bounded neighborhood, and $X$ is *locally bounded* if $X$ is locally bounded at each point.

Since open bounded subsets of $X$ have pseudocompact closures, $X$ is locally bounded at a point if and only if the point has a pseudocompact neighborhood. Also it is clear that a $\mu$-space is locally bounded if and only if it is locally compact. Hence the next proposition is immediate from Corollary 2.2.

Proposition 4.3. *If $X$ is a locally bounded $\mu$-space, then $X$ is a bz-space.*

Proposition 4.4. *Let $x \in \mu X$. The following are equivalent.*

(1) *$x$ is traceable.*

(2) *$\mu X$ is locally bounded at $x$.*

(3) *$\mu X$ is locally compact at $x$.*

(4) *$x \notin cl_{\beta X}(\beta X - \upsilon X)$.*

*Proof.* (1)⇒(2). If $x$ is traceable, the there is a neighborhood $U$ of $x$ in $\mu$X such that $U \cap X$ is bounded in $X$. Then by the denseness of $X$, $cl_{\beta X} U = cl_{\beta X}(U \cap X) \subseteq \mu X$ and hence $U$ is a bounded neighborhood of $x$ in $\mu X$.

(2)⇒(3). This follows from the fact that bounded subsets of a $\mu$-space are relatively compact.

(3)⇒(4). Let $K$ be a compact neighborhood of $x$ in $\mu X$. Then $int_{\mu X} K = G \cap \mu X$ where $G$ is open in $\beta X$. Since $cl_{\beta X} G = K \subseteq \mu X \subseteq \upsilon X$, $x \notin cl_{\beta X}(\beta X - \upsilon X)$.

(4)⇒ (1). Suppose $x$ is untraceable. Let $U$ be a closed neighborhood of $x$ in $\beta X$. By hypothesis $U \cap X$ is not bounded and so $cl_{\beta X}(U \cap X) = U \nsubseteq \upsilon X$ (see [5, 2.6]). Then (4) is false. □

Remark 4.5. By Proposition 4.4 and earlier remarks, a $\mu$-space is locally compact if and only if it is locally pseudocompact. In particular in a realcompact space all those points at which the space fails to be locally compact are untraceable points. For example, in the space of rationals equipped with the usual topology every point is untraceable.

In [4, 1.7] it was shown that a space $X$ is nearly realcompact if and only if no points of $\upsilon X - X$ have compact neighborhoods in $\upsilon X$. That result should be compared with 4.4 and the next proposition.

Proposition 4.6. *A space $X$ is nearly realcompact if and only if every point of $\mu X - X$ is untraceable.*

*Proof.* Suppose that $X$ is nearly realcompact and let $p \in \mu X - X$. Now $p \in cl_{\beta X}(\beta X - \upsilon X)$ since $X$ is nearly realcompact. Then $p$ is untraceable by Proposition 4.4.

Conversely, let $U$ be open in $\beta X$ with $p \in U \cap X^*$. There is an open set $G$ in $\beta X$ with $p \in G \subseteq cl_{\beta X} G \subseteq U$. Now if $p \in \mu X - X$, then $U \cap (\beta X - \upsilon X) \neq \emptyset$ by hypothesis, and so clearly we may assume that $p \in \upsilon X - \mu X$. But then $p \notin cl_{\beta X} B$ where $B$ is bounded in $X$, and so $G \cap X$ is not bounded in $X$. We conclude that $U$ meets $\beta X - \upsilon X$ which implies that $X$ is nearly realcompact. □

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*Received 15 August, 1999 and in revised form 23 February, 2001.*