Classical theorems of probability on Gelfand pairs - Khinchin theorems and Cramer theorem

P. Graczyk and C. R. E. Raja*

Abstract

We prove the Khinchin's Theorems for following Gelfand pairs (G, K) satisfying a condition (*): (a) G is connected; (b) G is almost connected and Ad (G/M) is almost algebraic for some compact normal subgroup M; (c) G admits a compact open normal subgroup; (d) (G, K) is symmetric and G is 2-root compact; (e) G is a Zariski-connected p-adic algebraic group; (f) compact extension of unipotent algebraic groups; (g) compact extension of connected nilpotent groups. In fact, condition (*) turns out to be necessary and sufficient for K-biinvariant measures on aforementioned Gelfand pairs to be Hungarian. We also prove that Cramer's theorem does not hold for a class of Gaussians on Compact Gelfand pairs.

KEY WORDS Locally compact groups, Lie groups, algebraic groups, Gelfand pairs, probability measures and factorization theorem, Khinchin's central limit theorem, limit theorem, Cramer theorem and anti-indecomposable measures, infinite divisibility and embedding.

1 Introduction

A classical theorem of Khinchin known as Khinchin factorization theorem which we would call Khinchin's first theorem says that any probability measure on $\mathbb{R}$ can be written as a countable product of indecomposable measures (possibly infinite) and a probability measure without indecomposable factors. Khinchin's factorization theorem was extended to all commutative Hausdorff metrizable groups by Ruzsa and Szekely (see [RS]). In [RS] Khinchin's factorization for measures on abelian Hausdorff groups is achieved by proving that the semigroup of probability measures on such groups form a first countable Hungarian semigroup. The notion of Hungarian semigroups was introduced by Ruzsa and Szekely and it was studied in [RS]. It is shown in [RS], that any element in a first countable Hungarian semigroup is a countable

*1991 Mathematics Subjects Classification:
primary; 22A20, 43A05, 60B15 and 60F17
secondary; 22E30, 22E35 and 43A85.

product of indecomposable elements (possibly infinite) and an anti-indecomposable element. It is shown in [R] that semigroup of K-biinvariant probability measures on real or p-adic reductive symmetric spaces is a Hungarian semigroup and hence the factorization theorem holds for such semigroups.

Another classical theorem of Khinchin which we would call Khinchin's second theorem says that any antiindecomposable measure on $\mathbb{R}$ is infinitely divisible. This result was extended to many other groups by various authors. In [RS], Khinchin's second theorem is also proved for anti-indecomposable measures on first countable abelian Hausdorff groups by showing that semigroup of measures on such groups form a normable Hungarian semigroup.

At this point we would like to note that Delphic semigroup is another approach to prove the Khinchin's Theorems for abelian semigroup. It has been proved in [G3] that the semigroup of measures on noncompact symmetric spaces form a Delphic semigroup but it can easily be seen that measures on compact symmetric spaces do not form a Delphic semigroup (see [G3] for definition of Delphic semigroup).

The study of probability questions on Gelfand pairs has been initiated by Letac in [Le] and by Heyer in [He1] and [He2] where the author proves the Khinchin type factorization result for some class of Gelfand pairs.

In this article we attempt to prove Khinchin's Theorems for measures on Gelfand pairs. In section 2 we introduce the concept of Gelfand pair and we also prove some preliminary results which are needed in the succeeding sections to prove Khinchin's Theorems. In the section 3 we prove results on factor compactness which are needed in proving Khinchin's Theorems. In section 4, we prove Khinchin's Theorems for connected Gelfand pairs. In sections 5 and 6, we prove Khinchin's Theorems for certain Gelfand pairs which include discrete groups and doubly transitive groups and p-adic algebraic groups.

One of the axioms of Hungarian semigroup is that the set of factors of an element is compact modulo the group of units. Some applications of this type of factor compactness in analysis and arithmetic of probability measures are limit theorems and embedding of infinitely divisible measures: see [S] and [Te] for more details on limit theorems on general locally compact groups. In section 7 we obtain limit theorems for measures on certain Gelfand pairs and we also obtain the embeddability of infinitely divisible meausres on certain Gelfand pairs: the embedding problem for general groups are studied by various authors (see [Mc]).

One more classical theorem of Khinchin which we would call Khinchin's third theorem says that infinitesimal limits are infinitely divisible. This result was extended by Ruzsa and Szekely to abelian metrizable groups such that the set of characters separates points of the groups by showing that the semigroup of probability measures on such groups form a stable normable Hungarian semigroup (see [RS]). In the section 8 we prove the normability which in turn proves second and third theorems of Khinchin for Gelfand pairs.

In section 9 we discuss Gaussian measures on compact Gelfand pairs and prove that Gaussian measures are not in the class of anti-indecomposable measures. This in particular implies that Gaussian measures on certain compact Gelfand pairs do not satisfy Cramér theorem: Cramér theorem says that Gaussian measures on reals have only Gaussian factors and Cramér theorem was generalized to abelian groups by various authors (see [Fe]) and to symmetric spaces of non-compact type by Graczyk (see [G2]). While proving this we obtain a class of measures which have indecomposable factors. In the last section we make some remarks on central limit theorems of Lindeberg-Feller type for probabilities on Gelfand pairs.

2 Preliminaries

Let G be a locally compact second countable group and K be a compact subgroup of G. Then we say that the pair (G, K) is a *Gelfand pair* if the convolution semigroup $\mathcal{P}_K(G)$ of all K-biinvariant probability measures on G is a commutative semigroup; see [BJR], [F], [GV] and [MV] for more on harmonic analysis on Gelfand pairs. For any probability measure μ, $S(\mu)$ denotes the support of μ and for any compact subgroup M of G, ω_M denotes the normalized Haar measure on M.

Examples **(1)** For any locally compact abelian group G and any compact subgroup K of G, (G, K) is Gelfand.

(2) Semigroup of probability measures on a real reductive group G that are K-biinvariant for a maximal compact subgroup K of G is commutative and hence the pair (G, K) is a Gelfand pair (see [R]).

(3) The semigroup of probability measures on the Euclidean motion group G that are $SO(n)$-biinvariant is commutative and hence $(G, SO(n))$ is a Gelfand pair.

Proposition 2.1 *Let G be a locally compact second countable group and K be a compact subgroup of G. Then the following are equivalent:*

1. *(G, K) is a Gelfand pair;*

2. *for any $x, y \in G$, $KxKyK = KyKxK$;*

3. *for any $x, y \in G$, $xy \in KyKxK$;*

4. *the algebra $L^1_K(G)$ of K-biinvariant integrable functions on G is a commutative algebra.*

Proof One may prove that (1) implies (2) by considering the K-biinvariant measures $\omega_K \delta_x \omega_K$ and $\omega_K \delta_y \omega_K$, for $x, y \in G$ and that (2) implies (3) is obvious.

We now prove (3) implies (4). We first prove that (3) implies G is unimodular. Let m be a left invariant Haar measure on G. Let U be a compact neighbourhood of e such that $KUK = U$. Then for $g \in G$,

$$\begin{aligned} m(Ug) &= \int \chi_U(xg)dx \\ &= \int \chi_E(gkx)dx \quad (k \text{ depends on } x) \\ &= \int \chi_U(x)dx \quad (ydx = dx, \;\; y \in G \text{ and } KUK = U) \\ &= m(U) \end{aligned}$$

This proves that G is unimodular. The rest of the proof of (3) implies (4) is quite similar to Theorem 1.12 of [BJR].

The implication (4) implies (1) follows from the existence of approximate identity sequence in $L^1_K(G)$ (see Lemma 1.6.8 of [GV] or Theorem 2.2.28 of of [BH]). □

Thus, the above result says that our definition of Gelfand pair agrees with the classical notion of Gelfand pair. We now prove that Gelfand pair property preserves quotients.

Proposition 2.2 *Let (G, K) be a Gelfand pair and H be a normal subgroup of G. Let $M = KH/K$. Then $(G/H, M)$ is also a Gelfand pair.*

Proof Let $x, y \in G$. Since (G, K) is a Gelfand pair, by Proposition 2.1, $xy \in KyKxK$, that is there exist k_1, k_2 and k_3 such that $xy = k_1yk_2xk_3$. This implies that $xHyH \in k_1HyHk_2HxHk_3H$. Again by Proposition 2.1, $(G/H, M)$ is also a Gelfand pair.

In this article we attempt to prove all three theorems of Khinchin for Gelfand pairs. This is achieved by applying the Hungarian semigroup theory: see [R] and [RS] for more details on Khinchin's theorems and Hungarian semigroups.

Let S be a commutative Hausdorff semigroup with identity e. Let $\sim$ be a relation defined on S for $x, y \in S$, by

$$x \sim y \Leftrightarrow x = ry \text{ and } y = sx$$

for some $r, s \in S$. Any two elements x and y of S are said to be *associates* if $x \sim y$. An element u of S is called *unit* of S if it is invertible in S. Let S^* be the quotient semigroup corresponding to the relation $\sim$ and $\phi: S \to S^*$ be the canonical quotient map (this notation is followed throughout the article). We say that the semigroup S is *Hungarian* if it satisfies the following properties:

(H-1) the set of associate pairs is a closed subset of $S \times S$;

(H-2) if x and y are associates, then $x = uy$ for some unit u in S;

(H-3) the set of divisors (factors) of any element in S^* is compact.

For any two subsets A and B of S and any $s, t \in S$, let us write $A_t \sim {}_s B$ if for any $a \in A$, there exists a $b \in B$ such that $a = sb$ and $b = ta$. A Hungarian semigroup S is called *uniformly Hungarian* if for any $s, t \in S$ and subsets A and B of S such that $A_s \sim {}_t B$ there exist units u and v in S such that $A_u \sim {}_v B$. The notion of uniformly Hungarian semigroup was introduced by A. Zempléni in [Z] to study the heredity of Hun and Hungarian semigroups.

A sequence (x_n) in a topological space X is said to be *relatively compact* or *bounded* if it is contained in a compact subset of X.

We first prove following elementary results that are needed in proving the main results. First of such results characterizes all units in the semigroup of probability measures on Gelfand pairs.

Proposition 2.3 *Let G be a locally compact group and K be a compact subgroup of G. Suppose λ and μ are K-biinvariant probability measures on G such that $\mu\lambda = \lambda\mu = \omega_K$. Then $\lambda = x\omega_K$ for some x in $N(K)$, the normalizer of K. Suppose (G, K) is a Gelfand pair and S is the semigroup of K-biinvariant probability measures on G. Then λ is a unit in S if and only if $\lambda = x\omega_K$ for some $x \in N(K)$.*

Proof Let λ and μ be K-biinvariant probability measures on G such that

$$\lambda * \mu = \omega_K = \mu * \lambda.$$

This implies that

$$S(\lambda)S(\mu) \cup S(\mu)S(\lambda) \subset K$$

and hence for any $g \in S(\mu)$, λg is a left K-invariant probability measure supported on K. Thus, $\lambda g = \omega_K$. Since $S(\lambda) \subset Kg^{-1}$, we have $\omega_K x = \omega_K g^{-1}$ for all $x \in S(\lambda)$. Thus,

$$\lambda = \omega_K x$$

for any $x \in S(\lambda)$. Similarly we may prove that

$$\lambda = x\omega_K$$

for all $x \in S(\lambda)$. This implies that

$$x\omega_K x^{-1} = \omega_K$$

and hence $x \in N(K)$. Second part of the proposition follows from the fact that any measure of the form $x\omega_K$, for $x \in N(K)$ is K-biinvariant and $x^{-1}\omega_K$ is the inverse of $x\omega_K$. □

The following lemma is very useful and used often in the sequel without even referring to it.

Lemma 2.1 *Let G be a locally compact group and K be a compact subgroup of G. Suppose (G, K) is a Gelfand pair. Then any compact subgroup M containing K is normalized by $N(K)$, the normalizer of K and (G, M) is also a Gelfand pair.*

Proof Let $x \in N(K)$. Then $x\omega_K$ and $\omega_K x^{-1}$ are K-biinvariant probability measures. Since (G, K) is a Gelfand pair, this implies that

$$\omega_{xMx^{-1}} = x\omega_K\omega_M\omega_K x^{-1} = \omega_M x\omega_K x^{-1} = \omega_M\omega_K = \omega_M.$$

Thus, $xMx^{-1} = M$. The second part of the theorem follows from the fact that M-biinvariant probability measures are also K-biinvariant. □

The next lemma determines when the semigroup of probability measures on a Gelfand pair satisfies (H-2).

Lemma 2.2 *Let G be a locally compact group and K be a compact subgroup of G. Let S be the semigroup of all K-biinvariant probability measures on G. Suppose (G, K) is a Gelfand pair. For any subgroup H of G, $N(H)$ denotes the normalizer of H in G. Then the following are equivalent:*

1. *(H-2) holds for S;*
2. *for every compact subgroup M of G containing K and $x \in G$ such that $xKx^{-1} \subset M$, we have $x \in N(K)M$;*
3. *for every compact subgroup M of G containing K, $N(M) = N(K)M$.*

Proof Suppose S satisfies (H-2). Let M be a compact subgroup of G containing K. Suppose $x \in G$ is such that

$$K \text{ and } xKx^{-1} \subset M. \qquad (i)$$

Consider

$$\lambda = \omega_M \text{ and } \mu = \omega_K\delta_x\omega_M. \qquad (ii)$$

Then λ and μ are in S. Let

$$\nu_1 = \omega_K\delta_x\omega_K \text{ and } \nu_2 = \omega_K\delta_{x^{-1}}\omega_K.$$

Then $\nu_1, \nu_2 \in S$ and by (i) we get that

$$\mu = \omega_K\delta_x\omega_M = \omega_K\delta_x\omega_K\omega_M = \nu_1\lambda$$

and

$$\lambda = \omega_M = \omega_K\omega_M = \omega_K\delta_{x^{-1}}\omega_K\delta_x\omega_M = \nu_2\mu.$$

Thus, λ and μ are associates. Then $\lambda = u\mu$ for some unit u in S. By Proposition 2.3,

$$\lambda = u\mu = g\omega_K\mu = g\mu$$

for some $g \in N(K)$. Thus, by substituting (ii), we have

$$\omega_K\delta_x\omega_M = g\omega_M$$

and hence $KxM = gM$ for some $g \in N(K)$. This implies that $x \in gM \subset N(K)M$. This proves that (1) implies (2).

Suppose (2) holds. Let M be a compact subgroup of G containing K. Then

$$xKx^{-1} \subset xMx^{-1} = M$$

for all $x \in N(M)$ and hence by assumption $x \in N(K)M$ for all $x \in N(M)$. This implies that $N(M) \subset N(K)M$. Since $K \subset M$, $N(K)$ normalizes M, that is $N(K) \subset N(M)$. Thus, $N(M) = N(K)M$. This proves that (2) implies (3).

Suppose for every compact subgroup M of G containing K, we have $N(M) = N(K)M$. We now prove that S satisfies (H-2). Let μ, λ, ν_1 and ν_2 be in S. Suppose

$$\mu = \nu_1\lambda \quad \text{and} \quad \lambda = \nu_2\mu.$$

Then

$$\mu = \nu_1\nu_2\mu$$

and hence by Theorem 1.2.7 of [He], $S(\nu_1)S(\nu_2) \subset \{g \in G \mid g\mu = \mu = \mu g\} = M$, say. Since μ is K-biinvariant, $K \subset M$. Replacing ν_i by $\omega_M * \nu_i$, for i=1,2, if necessary we may assume that $\nu_i * \omega_M = \nu_i$, for $i = 1, 2$. Then we have,

$$\nu_1\nu_2 = \omega_M = \nu_2\nu_1.$$

By Proposition 2.3, $\nu_i = x_i\omega_M$ for some $x_i \in N(M)$ for $i = 1, 2$. This implies that $x_i \in N(K)M = MN(K)$ for $i = 1, 2$. This implies that $\nu_i = g_i\omega_M$ for some $g_i \in N(K)$ for $i = 1, 2$. Thus, $\mu = g_1\lambda$ for $g_1 \in N(K)$. This proves that (3) implies (1). □

We say that a pair (G, K) consisting of a locally compact group G and a compact subgroup K of G *satisfies condition (*)* if (2) or (3) of Lemma 2.2 is satisfied. Thus, a Gelfand pair (G, K) satisfies condition (*) if and only if the semigroup of K-biinvariant probability measures on G satisfies (H-2). We will see that this condition plays a vital role in proving Khinchin's Theorems.

It is easy to see that when K is a maximal compact subgroup, condition (*) is satisfied. It is also easy to see that (G, K) satisfies condition (*) when G is a connected Lie group and K is a maximal torus which may be seen as follows: suppose M is a compact group containing K, then for $x \in N(M)$, $xKx^{-1} = mKm^{-1}$ for some $m \in M$ and hence $N(M) = N(K)M$. Also if there exists a compact group L contained in K

such that (G, L) satisfies condition (*), then (G, K) also satisfies condition (*) which follows from the equation that

$$N(M) = N(L)M = N(L)KM = N(K)M$$

for any compact subgroup M containing K.

We now prove that the Gelfand pair (G, K) satisfies the condition (*) when G/K is a compact Riemannian symmetric space. We first observe the following:

Proposition 2.4 *Let G/K be an irreducible Riemannian symmetric space. Then K is a maximal proper compact connected subgroup of G.*

Proof Let H be a compact connected subgroup of G containing K properly. Let $\mathcal{G}, \mathcal{K}$ and $\mathcal{H}$ be the Lie algebras of G, K and H respectively, with $\mathcal{G} = \mathcal{K} \oplus \mathcal{P}$. Since G/K is irreducible, Ad (K) acts irreducibly on the subspace $\mathcal{P}$ where Ad is the adjoint representation of G on its Lie algebra $\mathcal{G}$. Since H contains K, the subspace $\mathcal{H} \cap \mathcal{P}$ is a Ad (K)-invariant subspace of $\mathcal{P}$ and hence $\mathcal{H} \cap \mathcal{P} = (0)$ or $\mathcal{P}$. This implies that $H = K$ or $H = G$. This proves the proposition. □

Lemma 2.3 *Let G/K be a compact Riemannian symmetric space. Then the Gelfand pair (G, K) satisfies the condition (*).*

Proof Let (G, K) be a compact Riemannian symmetric pair. Let $\tilde{G}$ be the simply connected covering of G and $p: \tilde{G} \to G$ be the covering map of G. Let $\tilde{K} = p^{-1}(K)$. Let M be a compact subgroup of G containing K such that $xKx^{-1} \subset M$ for some $x \in G$. We now claim that $x \in N(K)M$. Let $\tilde{G}_1, \tilde{G}_2, \cdots, \tilde{G}_m$ be finite set of simple Lie subgroups of G_1 such that

$$G = \tilde{G}_1 \times \tilde{G}_2 \times \cdots \times \tilde{G}_m.$$

Now for each i, $1 \leq i \leq m$, there exists a compact subgroup $\tilde{K}_i$ of $\tilde{G}$ such that $\tilde{G}_i/\tilde{K}_i$ is a irreducible Riemannian symmetric space and

$$\tilde{K} = \tilde{K}_1 \times \tilde{K}_2 \times \cdots \times \tilde{K}_m.$$

Now let $\tilde{M} = p^{-1}(M)$. Then by Proposition 2.4, we have

$$\tilde{M}^0 = \tilde{K}_1 \times \cdots \times \tilde{K}_r \times \tilde{G}_{r+1} \times \cdots \times \tilde{G}_r$$

for some r, $0 \leq r \leq m$ where $\tilde{M}^0$ is the connected component of identity in M. Now let $y = (x_1, x_2, \cdots, x_m)$ be in $p^{-1}(x)$. Since $\tilde{K}$ is connected, we have that $y\tilde{K}y^{-1}$ and $\tilde{K}$ are contained in $\tilde{M}^0$. This implies that $x_i\tilde{K}_i x_i^{-1} \subset \tilde{K}_i$, for $1 \leq i \leq r$ and hence since $\tilde{K}$ is a connected Lie group, we have $x_i \in N(\tilde{K}_i)$, for $1 \leq i \leq r$. This implies that $y \in N(\tilde{K})\tilde{M}$ and hence $p^{-1}(x) \subset N(\tilde{K})\tilde{M}$. Thus, $x \in p(N(\tilde{K})\tilde{M}) = p(N(\tilde{K}))M \subset N(K)M$. This proves condition (*) for any compact Riemannian symmetric space. □

Remark The following gives an example of a Gelfand pair which does not satisfy the condition (*). Let $\mathbb{Q}_2$ be the additive group of 2-adic integers and $|\cdot|$ be the 2-adic norm on $\mathbb{Q}_2$. Let $L = \{x \in \mathbb{Q}_2 \mid |x| = 1\}$. Let K be the subgroup of automorphisms generated by the automorphism $x \mapsto -x$. Let G be the semidirect product of K and $\mathbb{Q}_2$. Let $x_0 \in \mathbb{Q}_2$ be such that $x_0 \notin L$ but $2x_0 \in L$. Then the semigroup of all K-biinvariant probability measures on G is isomorphic to the semigroup of all symmetric probability measures on $\mathbb{Q}_2$. Thus, (G, K) is a Gelfand pair. Let M be the compact subgroup of G generated by L and K. Then since $2x_0 \in L$, it is easy to see that x_0 normalizes M. Since $N(K) = K$ and $x_0 \notin M$, we get that $N(K)M = M$ is a proper subgroup of $N(M)$. Thus, (G, K) is a Gelfand pair which does not satisfy the condition (*).

We now present various types of Hungarian semigroups which are useful in proving the heredity of Hungarian semigroups and limit theorems. For any subset C of a semigroup P, let T_C be the set of all factors of elements of C. Let $\phi: S \to S^*$ denote the canonical quotient map. A Hungarian semigroup S is called *stable* if for every compact set C of S^*, T_C is compact. A Hungarian semigroup S is called *division compact* if for any two compact subsets C and L of S, the set $C/L = \{s \in S \mid \text{there exists a } l \in L, sl \in C\}$ is compact. It is shown in [RS] that the semigroup of all compact-regular probability measures on an abelian Hausdorff topological group G is a stable division compact Hungarian semigroup (see Chapter 3, Theorem 1.1 of [RS]).

A Hungarian semigroup S is called *strongly stable* if for any compact set C of S, there is a compact set L of S such that $\phi(T_C) = \phi(L)$. It should be noted that strongly stable Hungarian semigroups are stable. A. Zempléni introduced the notion of strongly stable Hungarian semigroups in [Ze]. It is shown in [Ze] that for a locally compact first countable abelian group G, the semigroup $\mathcal{P}(\mathcal{P}(\cdots(G)\cdots))$ is a strongly stable division compact uniformly Hungarian semigroup with Prohorov property. In [R], it is proved that $\mathcal{P}(\mathcal{P}(\cdots(S)\cdots))$ is a strongly stable division compact uniformly Hungarian semigroup with Prohorov property when S is the semigroup of K-biinvariant probability measures on a real reductive symmetric space. Here we prove a similar result for certain Gelfand pairs.

In order to achieve Khinchin's second and third Theorems, that is any anti-indecomposable measure or any infinitesimal limit is infinitely divisible, Ruzsa and Szekely introduced the concept of normable Hungarian semigroups. For any s in a Hungarian semigroup S, define $H(s)$ as the maximal idempotent factor of s in S (see 22.11 of [RS] for the existence of $H(s)$). A *normable Hungarian semigroup* is a Hungarian semigroup S satisfying the condition that for every $s \in S$ that is not an associate of an idempotent, there exists a map $\Delta_s: T_s \to [0, \infty)$ such that

$$\Delta_s(ab) = \Delta_s(a) + \Delta_s(b) \qquad (p)$$

for any $a, b \in T_s$ with $ab \in T_s$, $\Delta_s(s) > 0$ and Δ is continuous at $H(s)$ where T_s is the set of factors of s. Any map satisfying condition (p) is called a *partial homo-*

morphism. It is proved [RS], that the semigroup of probability measures on a locally compact abelian group is a normable Hungarian semigroup. Combining the Kendall homomorphism of [G] with the results in [R], we get that the semigroup of probability measures on a reductive symmetric space is a normable Hungarian semigroup. In the next sections we prove that the semigroup of K-biinvariant probability measures on Gelfand pairs is also normable.

Another application of normable stable Hungarian semigroup is the infinite divisibility of an infinitesimal limit, that is Khinchin's third Theorem. We will answer this question affirmatively in the section 8.

3 Factor compactness

The following lemma is an important tool in proving the factor compactness which is useful in establishing the strong stability and limit theorems: see [DM], [DR] and [M] for results on factor compactness for measures on general locally compact groups.

Lemma 3.1 *Let N be a connected nilpotent Lie group and A be a group acting on N by automorphisms such that the induced action on the Lie algebra of N is semisimple. Let X be a subset of N such that for any sequence (x_n) in X, the sequence*

$$(x_n\alpha(x_n^{-1}))$$

is relatively compact for every $\alpha \in A$. Then for each $x \in X$, there exists a_x and b_x such that

$$x = b_x a_x,$$

$\{b_x\}_{x\in X}$ is relatively compact and $\alpha(a_x) = a_x$ for all $x \in X$ and $\alpha \in A$. In other words, X is relatively compact in N/N^A where N^A denotes the group of all A-fixed points in N.

Proof Let $L(N)$ be the Lie algebra of N. We first consider the case when N is abelian. There is no loss of generality in assuming that N is a vector group. Let U be the subspace of $L(N)$ consisting of all $v \in L(N)$ such that

$$d\alpha(v) = v$$

for all $\alpha \in A$. Then there exists a A-invariant subspace W of $L(N)$ such that

$$L(N) = U \oplus W.$$

Now for each $x \in X$, there are a_x and b_x in the exponential image of U and W respectively, such that

$$x = b_x a_x.$$

Suppose $\{b_x\}_{x\in X}$ is not relatively compact, then there exists a sequence (x_n) in X such that

$$b_n = b_{x_n} \to \infty.$$

Let $Y_n \in W$ be such that

$$\exp(Y_n) = b_n.$$

Since exp is a diffeomorphism, we have

$$Y_n \to \infty$$

and $(d\alpha(Y_n) - Y_n)$ is relatively compact for all $\alpha \in A$ and hence

$$d\alpha(\frac{Y_n}{||Y_n||}) - \frac{Y_n}{||Y_n||} \to 0$$

for all $\alpha \in A$ where $||\cdot||$ is the Euclidean norm on $L(N)$. By passing to a subsequence, if necessary we may assume that

$$\frac{Y_n}{||Y_n||} \to Y$$

and hence Y is a nonzero vector in W such that

$$d\alpha(Y) = Y$$

for all $\alpha \in A$. This is a contradiction. This proves that $\{b_x\}_{x\in X}$ is relatively compact.

We now consider the general case. The rest of the proof is based on induction on dimension of $L(N)$. Suppose dimension of $L(N)$ is one, the result follows from the abelian case. Now let Z be the center of N and $L(Z)$ be the Lie algebra of Z. Since the action of A on $L(N)$ is semisimple, there exists a A-invariant subspace W of $L(N)$ such that

$$L(N) = L(Z) \oplus W.$$

Since N is nilpotent, $Z(N)$ is of positive dimension, now applying induction hypothesis to $N/Z(N)$ yields that for each $x \in X$, there are a_x, b_x and z_x such that

$$x = b_x a_x z_x$$

where $a_x Z$ is fixed by all elements of A, z_x is in Z for all x and $\{b_x\}_{x\in X}$ is a relatively compact subset of N. Let exp be the exponential map of $L(N)$ into N. Since N is a connected nilpotent Lie group, by Theorem 3.6.1 of [V], exp is an onto map. Since $L(N) = L(Z) + W$, for each $x \in X$, there exists a $v_x \in L(Z)$ and $w_x \in W$ such that

$$\exp(v_x + w_x) = a_x$$

and hence, since v_x belongs to the center of the Lie algebra, by Corollary 2.13.3 of [V], we have

$$\exp(w_x)\exp(v_x) = a_x$$

for all $x \in X$. Thus, for each $x \in X$, replacing a_x by $a_x \exp(-v_x)$, we may assume that

$$a_x \in \exp(W)$$

for all $x \in X$.

We now claim that a_x is fixed by all elements of A. Let $w_x \in W$ be such that $\exp(w_x) = a_x$. Since $a_x Z$ is fixed by elements of A, we have

$$\alpha(\exp_q(w_x + L(Z)) = \exp_q(w_x + L(Z))$$

for all $\alpha \in A$ where $\exp_q$ denotes the exponential map of the Lie group N/Z. Since N is a connected nilpotent Lie group, $\exp_q$ is a diffeomorphism of the Lie algebra $L(N)/L(Z)$ onto N/Z (see Theorem 3.6.2 of [V]). This implies that

$$\alpha(w_x + L(Z)) = w_x + L(Z)$$

for all $\alpha \in A$ and hence

$$\alpha(w_x) - w_x \in L(Z)$$

for all $\alpha \in A$. Since $w_x \in W$ which is an A-invariant subspace, we have

$$\alpha(w_x) - w_x \in W \cap L(Z) = (0)$$

for all $\alpha \in A$. This implies that $\alpha(w_x) = w_x$ for all $\alpha \in A$ and hence

$$\alpha(a_x) = a_x$$

for all $\alpha \in A$ and all $x \in X$.

Now for any sequence (z_{x_n}), and for each $\alpha \in A$,

$$x_n \alpha(x_n^{-1}) = b_{x_n} a_{x_n} z_{x_n} \alpha(z_{x_n}^{-1}) \alpha(a_{x_n}^{-1}) \alpha(b_{x_n}^{-1}) = z_{x_n} \alpha(z_{x_n}^{-1}) b_{x_n} \alpha(b_{x_n}^{-1}).$$

This implies that $(\{z_{x_n} \alpha(z_{x_n}^{-1}))$ is relatively compact. Now the result follows from the abelian case. □

Lemma 3.2 *Let U be a unipotent algebraic group and K be a compact group of automorphisms on U. Let X be a subset of U such that for any sequence (x_n) in X, the sequence $(x_n \alpha(x_n^{-1}))$ is relatively compact. Then XU^K is relatively compact in U/U^K where U^K is the group of all K-fixed points of U.*

Proof Since U is a unipotent algebraic group, exponential is a diffeormorphism of the Lie algebra of U onto U. Since K is compact, the induced action of K on the Lie algebra of U is semisimple. Thus, one may prove the lemma by arguing as in Lemma 3.1.

The next result extends Lemma 3.1, to connected solvable groups with a faithful representation and when the group of automorphisms is a compact connected group.

Lemma 3.3 *Let G be a connected solvable Lie group with a faithful representation and K be a compact connected group of automorphisms of G. Suppose X is a subset of G such that for every sequence (x_n) in X, the sequence $(x_n\alpha(x_n)^{-1})$ is relatively compact for all $\alpha \in K$. Then for each $x \in X$, there exist (b_x) and (a_x) in G such that*

$$x = b_x a_x,$$

$\{b_x \mid x \in X\}$ is relatively compact and $\alpha(a_x) = a_x$ for all $\alpha \in K$ and all $x \in X$.

Proof Let G be a connected solvable Lie group with a faithful representation. Then there exists a compact connected abelian subgroup T of G and a simply connected normal subgroup H of G such that $G = TH$ (see [Ho]). Let N be the nilradical of H and $\mathcal{H}$ be the Lie algebra of H. Let $\mathcal{H}_1$ be the Lie subalgebra of H such that

$$\mathcal{H}_1 = \{v \in \mathcal{H} \mid \alpha(v) = v, \text{ for all } k \in K\}.$$

Let H_1 be the connected Lie subgroup of G corresponding to the subalgebra $\mathcal{H}_1$. Then H_1 is a simply connected closed subgroup of H (see Theorem 3.18.12 of [V]) and $\alpha(h) = h$ for all $h \in H_1$ and all $\alpha \in K$. By Leptin's Theorem (see [BJR]), $H = NH_1$. Now the lemma follows from Lemma 3.1. $\square$

We need the following lemmas which are quite useful in establishing the main result of this section.

Lemma 3.4 *Let G be a locally compact second countable group and K be a compact subgroup of G. Let (τ_n) be a sequence of automorphisms of G. Suppose $(\tau_n(\omega_K))$ is a relatively compact sequence in $\mathcal{P}(G)$. Then $(\tau_n(k))$ is a relatively compact sequence in G, for all $k \in K$.*

Proof Suppose $(\tau_n(k_0))$ is not relatively compact for some $k_0 \in K$. Then by passing to a subsequence, if necessary we may assume that $(\tau_n(k_0))$ has no convergent subsequence and $\tau_n(\omega_K) \to \rho \in \mathcal{P}(G)$. Since each of $\tau_n(\omega_K)$ is an idempotent, ρ is an idempotent and hence $\rho = \omega_M$ for some compact subgroup M of G (see Theorem 1.2.10 of [He]). Since G is second countable, $\mathcal{P}(G)$ is metrizable. Let (U_i) be a decreasing sequence of compact neighbourhoods of M. Since $\tau_n(\omega_K) \to \omega_M$, for each $i \geq 1$, there exists an n_i such that

$$\omega_K(\tau_{n_i}^{-1}(U_i)) > 1 - \frac{1}{2^i}$$

(see [P]) and we may assume that $n_i < n_{i+1}$ for all $i \geq 1$. Let

$$B = \cup_{m=1}^{\infty} \cap_{i=m}^{\infty} \tau_{n_i}^{-1}(U_i).$$

Then B is a Borel subset of G and

$$\omega_K(G \setminus B) \leq \sum_{i=m}^{\infty} \frac{1}{2^i}$$

for all $m \geq 1$. This implies that $\omega_K(B) = 1$. Let $b \in B$. Then there exists a $m \geq 1$ such that $\tau_{n_i}(b) \in U_i$ for all $i \geq m$. Since (U_i) is a decreasing sequence, we have $\tau_{n_i}(b) \in U_m$ for all $i \geq m$ and hence $(\tau_{n_i}(b))$ is relatively compact for all $b \in B$. Let H be the set of all k in K such that $(\tau_{n_i}(k))$ is relatively compact. Then H is a co-null subgroup of K. By Proposition B.1 of [Zi], $H = K$. This implies that $(\tau_n(k_0))$ has a convergent subsequence. This is a contradiction. Thus, we prove the lemma. □

We make the following observation which is essentially Lemma 2.1 of [DR].

Lemma 3.5 *Let V be a finite-dimensional algebra over real or p-adic field. Let (τ_n) be a sequence of algebra automorphisms of V. Then there exists a subalgebra W of V such that*

1. $W = \{w \in V \mid (\tau_n(w)) \text{ is bounded } \}$ *and*
2. *if $(\tau_n(\mu))$ is relatively compact for $\mu \in \mathcal{P}(V)$, then μ is supported on W.*

Proof Since V is of finite-dimension, there exists a vector subspace W of V such that $(\tau_n(w))$ is bounded if and only if $w \in W$. Now let $\mu \in \mathcal{P}(V)$ be such that $(\tau_n(\mu))$ is relatively compact. We now claim that the support of μ is contained in W, in other words, for each $v \in S(\mu)$, $(\tau_n(v))$ is a bounded sequence. Suppose for some $v \in V$, the sequence $(\tau_n(v))$ is not bounded. Then there exists a subsequence (τ_{k_n}) of (τ_n) such that

$$\tau_{k_n}(v) \to \infty \quad \text{and} \quad \tau_{k_n}(\mu) \to \nu$$

for some $\nu \in \mathcal{P}(V)$. Then by Lemma 2.1 of [DR], there exists a subspace W_0 of V such that $(\tau_{k_n}(w))$ converges for all $w \in W_0$ and μ is supported on W_0. This implies in particular, that $(\tau_{k_n}(v))$ converges. This is a contradiction. Thus, $v \in W$. This proves the lemma. □

Proposition 3.1 *Let G be an almost connected Lie group and G^0 is a semisimple Lie group. Let K be a compact subgroup of G. Suppose (G, K) is a Gelfand pair and S is the semigroup of all K-biinvariant probability measures on G. Let (μ_n) be a relatively compact sequence in S and (λ_n) be a seqeunce such that for each $n \geq 1$, λ_n is a factor of μ_n. Then there exists a sequence (x_n) from the center of G such that $(x_n\lambda_n)$ is realtively compact.*

Proof Since (μ_n) is relatively compact, by Theorem 1.2.21 of [He], there exists a sequence (g_n) in G such that $(g_n\lambda_n)$ is relatively compact. Let Ad be the adjoint representation of G and let $H = \mathrm{Ad}\,(G)$. Then H is a connected algebraic semisimple group. Thus, the center of H is finite and $H \simeq G/Z$ where Z is the center of G. By Proposition 2.2, we have $(H, KZ/Z)$ is also a Gelfand pair. Let M be a maximal compact subgroup of H containing KZ/Z. Then (H, M) is a Gelfand pair. Let $p\colon G \to H$ be the cannonical quotient map. Then $(p(\mu_n) * \omega_M)$ is relatively compact and $p(\lambda_n) * \omega_M$ is a factor of $p(\mu_n)$ for all $n \geq 1$. Then there exists a sequence (h_n)

in H such that $(h_n\omega_M * p(\lambda_n))$ and $(p(\lambda_n) * \omega_M h_n)$ are relatively compact (see [DM]). This implies by Theorem 1.2.21 of [He], that $(h_n\omega_M h_n^{-1})$ and $(h_n^{-1}\omega_M h_n)$ are relatively compact. By Cartan decomposition (see [W]), $H = MAM$ for an abelian group A and hence for each $n \geq 1$, there exist $a_n \in A$ and $m_n, m_n' \in M$ such that $h_n = m_n a m_n'$. This implies that $(a_n\omega_M a_n^{-1})$ and $(a_n^{-1}\omega_M a_n)$ are relatively compact. By Lemma 3.4, $(a_n k a_n^{-1})$ and $(a_n^{-1} k a_n)$ are relatively compact for all $k \in M$. Let $\rho: H \to GL(V)$ be a faithful rational representation of H and W be the subalgebra of End (V) generated by H. For each $n \geq 1$, define $\tau_n: W \to W$ by $\tau_n(w) = a_n w a_n^{-1}$. Then by Lemma 3.5, there exists a subalgebra W_0 of W such that $(\tau_n(w))$ is relatively compact if and only if $w \in W_0$. Since A is abelian and $(a_n k a_n^{-1})$ is relatively compact for all $k \in M$, we have $KA \subset W_0$. Since $H = MAM$, we have $H \subset W_0$. Since H generates W, $W = W_0$ and hence (τ_n) is relatively compact in End (W). Since $(a_n^{-1} k a_n)$ is relatively compact, we may prove in a similar manner that (τ_n^{-1}) is relatively compact in End (W). Thus, (τ_n) is relatively compact. Since H is semisimple algebraic group, the center of H is finite. This implies that (a_n) is relatively compact in H. This proves that (h_n) is relatively compact and hence $(\omega_M * p(\lambda_n))$ is relatively compact. By Theorem 1.2.21 of [He], $(p(\lambda_n))$ is relatively compact. Since $(g_n\lambda_n)$ is relatively compact, once again by Theorem 1.2.21 of [He], $(p(g_n))$ is relatively compact. Thus, there exists a relatively compact seqeunce (b_n) in G and a sequence (x_n) from Z such that $g_n = b_n x_n$ for all $n \geq 1$. Since $(g_n\lambda_n)$ is relatively compact, $(x_n\lambda_n)$ is relatively compact. □

We now prove the factor compactness for connected Gelfand pairs when K is a maximal compact group.

Proposition 3.2 *Let G be a connected locally compact group and K be a maximal compact subgroup of G. Suppose (G, K) is a Gelfand pair and S is the semigroup of K-biinvariant probability measures on G. Let (μ_n) be a relatively compact sequence in S and (λ_n) be a sequence in S such that for each $n \geq 1$, λ_n is a factor of μ_n. Then there exists a sequence (x_n) in G such that $(x_n\lambda_n)$ is relatively compact and the sequence (x_n) is relatively compact in $G/N(K)$ where $N(K)$ is the normalizer of K in G.*

Proof Since G is a connected group, there exists a compact normal subgroup M of G such that G/M is a connected Lie group. Since K is a maximal compact subgroup, M is contained in K. Thus, by replacing G by G/M, we may assume that G is a connected Lie group.

Let (μ_n) be a sequence of K-biinvariant probability measures on G and (λ_n) be a sequence in S such that for each $n \geq 1$, λ_n is a factor of μ_n. Let R be the solvable radical of G. Then G/R is a connected semisimple Lie group. Then there exists a sequence (x_n) in G such that $(x_n\lambda_n)$ is relatively compact and by Proposition 3.1, (x_n) is relatively compact in $(G/R)/Z(G/R)$. This implies that there exists a sequence (g_n) in G such that $x_n = q_n g_n$ for all $n \geq 1$, $g_n R \in Z(G/R)$ and (q_n) is relatively compact. This implies that $(g_n\lambda_n)$ is relatively compact and $g_n g g_n^{-1} g^{-1} \in R$ for all $g \in G$ and all $n \geq 1$. Since λ_n is K-biinvariant, we have that $(g_n \omega_K g_n^{-1} g_n \lambda_n)$ is relatively compact and hence by Theorem 1.2.21 of [He], $(g_n \omega_K g_n^{-1})$ is relatively compact. By Lemma 3.4, we have $(g_n k g_n^{-1})$ is relatively compact for all $k \in K$. Thus, $(g_n \lambda_n)$ and for $k \in K$, $(g_n k g_n^{-1})$ are relatively compact. Let $H = \text{Ad}\,(G)$. Then H is a connected Lie group with a finite-dimensional faithful representation. Let S be the solvable radical of H. Then S contains $\text{Ad}\,(R)$. Let $h_n = \text{Ad}\,(g_n)$. Then $h_n h h_n^{-1} h^{-1} \in S$ for all $h \in H$. This implies that $(h_n S)$ is contained in the center of H/S which is a semisimple algebraic group and hence $(h_n S)$ is relatively compact. There exists a relatively compact sequence (c_n) in H such that $h_n = c_n y_n$ and for some $y_n \in S$ for all $n \geq 1$. Since (c_n) is relatively compact, we have $y_n k y_n^{-1}$ is relatively compact for all $k \in \text{Ad}\,(K)$. This implies that the sequence $(y_n k y_n^{-1} k^{-1})$ in S is relatively compact in S for all $k \in \text{Ad}\,(K)$. Now by Lemma 3.3, there exists a relatively compact sequence (b_n) in S and a sequence (a_n) in S such that $y_n = b_n a_n$ and $k a_n k^{-1} = a_n$ for all $k \in \text{Ad}\,(K)$ and all $n \geq 1$. Thus, $h_n = c_n b_n a_n$ where $(c_n b_n)$ is relatively compact and $k a_n k^{-1} = a_n$ for all $k \in \text{Ad}\,(K)$ and all $n \geq 1$. This implies that there exists a relatively compact sequence (d_n) in G and a sequence (u_n) in G such that

$$x_n = d_n u_n \quad \text{and} \quad k u_n k^{-1} u_n^{-1} \in Z$$

for all $n \geq 1$ and all $k \in K$ where Z is the center of G. Now for each $n \geq 1$, the map $k \mapsto k u_n k^{-1} u_n^{-1}$ is a continuous homomorphism of K into Z. Since K is a maximal compact subgroup, we get that $k u_n k^{-1} u_n^{-1} \in K$ and hence $u_n \in N(K)$ for all $n \geq 1$. Thus, (x_n) is relatively compact in $G/N(K)$. □

We now prove the factor compactness result for certain almost connected Gelfand pairs. To do this we need the following results on the structure of Gelfand pairs. The following proves that Gelfand pair is invariant under conjugation.

Proposition 3.3 *Let (G, K) be a Gelfand pair and τ be an automorphism of G. Then $(G, \tau(K))$ is also a Gelfand pair.*

Proof Let λ and μ be $\tau(K)$-biinvariant probability measures. Define $\lambda' = \tau^{-1}(\lambda)$ and $\mu' = \tau^{-1}(\mu)$. Then

$$\omega_K \lambda' \omega_K = \omega_K \tau^{-1}(\lambda) \omega_K = \tau^{-1}(\omega_{\tau(K)} \lambda \omega_{\tau(K)}) = \lambda'.$$

This shows that λ' and μ' are K-biinvariant probability measures on G. Since (G, K) is a Gelfand pair, we have

$$\lambda\mu = \tau(\lambda')\tau(\mu') = \tau(\lambda'\mu') = \tau(\mu'\lambda') = \mu\lambda.$$

This proves that $(G, \tau(K))$ is also a Gelfand pair. □

Proposition 3.4 *Let G be a almost connected Lie group and (G, K) be a Gelfand pair. Let G^0 be the connected component of identity in G and Ad be the adjoint* *representation of G on its Lie algebra. Suppose Ad (G^0) is an almost algebraic group. Let S be the solvable radical of Ad (G^0). Then S is type R, that is S is a compact extension of a connected nilpotent normal subgroup of Ad (G).*

Proof Since S is a solvable almost algebraic group. There exists a compact abelian subgroup T and a diagonalizable almost algebraic group D such that $S = TDU$ where U is the unipotent radical of G. To prove the lemma it is enough to prove that D centralizes U. Now let $u \in U$ and (d_n) is a sequence in D such that $d_n u d_n^{-1} \to e$. Let M be a maximal compact subgroup of Ad (G) containing T. Since Ad (K) is contained in a coset of G, by Proposition 3.3, we get that $(Ad\ (G), M)$ is also a Gelfand pair. Now for each $n \geq 1$, $d_n u = c_n u c'_n d_n c_n"$ for some c_n, c'_n and $c_n"$ in M. This implies that

$$d_n u d_n^{-1} d_n c_n" d_n^{-1} = c_n u c'_n$$

for all $n \geq 1$. Since (c_n) and (c'_n) are relatively compact, by passing to a subsequence, we may assume that $c_n \to c$, $c'_n \to c'$ and $d_n u d_n^{-1} \to e$. This implies that $d_n c_n" d_n^{-1} \to cuc'$. Let R be the semisimple Levi subgroup of Ad (G^0) containing M^0. Then it is easy to see that R contains D and hence $u \in RM$. Since RM is a finite extension of a semisimple group, it does not contain any unipotent normal subgroup but $U \cap RM$ is an unipotent normal subgroup of RM and hence $u = e$. Thus, D centralizes U. This proves the proposition. □

We now prove the factor compactness.

Proposition 3.5 *Let G be an almost connected group and K be a maximal compact subgroup of G. Suppose (G, K) be a Gelfand pair and there exists a compact normal subgroup M of G such that G/M is a Lie group and Ad (G/M) is an almost algebraic group. Let S be the semigroup of K-biinvariant probability measures on G. Let (μ_n) be a relatively compact sequence in S and (λ_n) be a sequence such that for each $n \geq 1$, λ_n is a factor of μ_n in S. Then there exists a sequence (x_n) in $N(K)$ such that $(x_n\lambda_n)$ is relatively compact.*

Proof Since K is a maximal compact subgroup of G, M is contained in K. Since considered measures are all K-biinvariant, we may assume that G is a Lie group. Let S be the solvable radical of G^0, the connected component of identity in G. Then S is normal in G and G/S is a finite extension of a connected semisimple Lie group. Let $p: G \to G/R$ be the canonical quotient map. Since $p(\lambda_n)$ is a factor of $p(\mu_n)$ for all $n \geq 1$ and (μ_n) is relatively compact, by Proposition 3.1, there exists a sequence (x_n) in G such that $(x_n\lambda_n)$ is relatively compact and $x_n g x_n^{-1} g^{-1} \in R$ for all $g \in G$. Since G is almost connected we may assume that $x_n \in G^0$ for all $n \geq 1$. Since λ_n is K-biinvariant, we have that $(x_n \omega_K x_n^{-1} x_n \lambda_n)$ is relatively compact and hence by Theorem 1.2.21 of [He], $(x_n \omega_K x_n^{-1})$ is relatively compact. By Lemma 3.4, $(x_n k x_n^{-1})$ is relatively compact. Let $\psi: G \to G/Z \simeq$ Ad (G) be the natural map and S be the solvable radical of Ad (G^0). Then for $g_n = \psi(x_n)$ for all $n \geq 1$, we have $g_n x g_n^{-1} x^{-1} \in$ Ad (S) for all

$n \geq 1$ and $(g_n k g_n^{-1})$ is relatively compact. Since center of H^0/S is finite, there exists a relatively compact sequence (d_n) and a sequence (y_n) in S such that $g_n = d_n y_n$. This implies that $(y_n k y_n^{-1} k^{-1})$ is relatively compact for all $k \in K$. By Proposition 3.4, S is a compact extension of its nilradical. Thus, there exists a sequence u_n in the nilradical, say N of S such that $(x_n u_n^{-1})$ is relatively compact and hence $(u_n k u_n^{-1} k^{-1})$ is realtively compact. By Lemma 3.1, there exists a bounded sequence (b_n) and sequence (z_n) such that $u_n = b_n a_n$ and $k a_n k^{-1} = a_n$ for all $n \geq 1$. Thus, there exists a bounded sequence (c_n) and a sequence (h_n) such that

$$g_n = c_n h_n \quad \text{and} \quad k h_n k^{-1} h_n^{-1} = e$$

for all $n \geq 1$ and all $k \in \mathrm{Ad}\,(K)$. This implies that there exists a sequence (z_n) in G such that $(z_n \lambda_n)$ is relatively compact and $x_n k x_n^{-1} k \in Z$ for all $n \geq 1$ where Z is the center of G. Since K is a maximal compact subgroup of G, we get that $x_n K x_n^{-1} \subset K$ for all $n \geq 1$, since K is a Lie group, $x_n K x_n^{-1} = K$ for all $n \geq 1$. Thus, $(x_n \lambda_n)$ is relatively compact and $x_n \in N(K)$. □

4 Khinchin's first Theorem for connected Gelfand pairs

In this section we prove the Khinchin's factorization theorem for Gelfand pairs when G is a connected locally compact group. We now look at shift compactness in a general Hausdorff commutative topological semigroup. Let S be any commutative Hausdorff topological semigroup and $X \subset S$. Then we say that

(1) X is *weakly shift compact* if for every sequence (x_i) in X, there is a relatively compact sequence (y_i) in S such that x_i is an associate of y_i.

(2) X is *strongly shift compact* if there exists a compact set Y of S such that every element of X is an associate of some element of Y.

We now use a version of an argument in Section 3.6 of [RS] to prove the following lemma; it proves the converse of Statement 21.8 of Chapter 2 of [RS] in the case of the semigroup of probability measures on Gelfand pairs.

Lemma 4.1 *Let G be a locally compact σ-compact group and S be a closed abelian subsemigroup of probability measures on G satisfying (H-2). Suppose the group of units in S is exactly equal to $\{g\eta \mid g \in H\}$ for some subgroup H of G where η is the identity in S. Then weakly shift compact subsets of S are also strongly shift compact.*

Proof Let X be a weakly shift compact subset of S. For any $\mu \in S$ and any compact subset M of G, let

$$C(\mu;\ M) = \sup_{x\in H}\mu(xM).$$

We first claim that for any given $0 < \theta < 1$, there exists a compact subset M of G such that $C(\mu;\ M) > \theta$ for all $\mu \in X$. Suppose for some $0 < \theta < 1$ and for each compact subset M of G, there exists $\mu \in X$ such that $C(\mu;\ M) \leq \theta$. Let $M_1 \subset M_2 \subset \cdots M_n \subset \cdots$ be a sequence of compact sets in G such that M_n is contained in the interior of M_{n+1} and $G = \cup M_n$ (this is possible because G is locally compact σ-compact). Then for each n there exists a $\mu_n \in X$ such that $C(\mu_n;\ M_n) \leq \theta$. Since the sequence (μ_n) is in X, there exists a relatively compact sequence (λ_n) in S such that $\lambda_n \sim \mu_n$. Since S is Hungarian, there exists a sequence (x_n) in H such that $(x_n\mu_n)$ is relatively compact and hence there is a compact subset M of G such that $\mu_n(x_n^{-1}M) > \theta$ for all n. This implies that $C(\mu_n;\ M) > \theta$ for all n. Since M is compact there exists an M_n such that $M \subset M_n$ and hence $C(\mu_n;\ M_n) > \theta$. This is a contradiction. Thus, our claim is proved.

Let M be a compact subset of G such that $C(\mu;\ M) > 1/2$ for all $\mu \in X$. Let $B = \{\lambda \in S \mid \phi(\lambda) \in \phi(X) \text{ and } \lambda(M) > 1/2\}$. We claim that B is relatively compact and $\phi(X) \subset \phi(B)$. Let $\theta > 1/2$. Then there exists a compact set L of G such that $C(\mu;\ L) > \theta$ for all $\mu \in X$. Let $\lambda \in B$. Then $\lambda(M) > 1/2$ and $\lambda(uL) > \theta$ for some $u \in H$. This implies that $u \in ML^{-1}$ and hence $\lambda(ML^{-1}L) > \theta$. This shows that B is relatively compact. Suppose $\mu \in X$, then there exists $u \in H$ such that $u\mu(M) > 1/2$. This implies that $u\mu \in B$. Thus, $\phi(X) \subset \phi(B)$. Now let $Y = \overline{B}$. Then Y is compact and every element of X is an associate of some element of Y. Thus, X is strongly shift compact. □

The following result is an useful lemma to prove (H-3) and strong stability.

Lemma 4.2 *Let G be a locally compact second countable σ-compact group and (G, K) be a Gelfand pair satisfying condition (*). Let S be the semigroup of all K-biinvariant probability measuures on G. Suppose C is a compact subset of S, such that T_C is weakly shift compact. Then $\phi(T_C)$ is compact where $\phi: S \to S^*$ is the natural map.*

Proof Let C be a compact subset of S such that T_C is weakly shift compact. Let $N(H)$ denote the normalizer of H for any subgroup H of G. By the hypothesis S is a commutative subsemigroup of probability measures on G satisfying (H-2) and the group of units in S is $N(K)\omega_K$. Thus, by Lemma 4.1, T_C is strongly shift compact. This implies that $\phi(T_C) \subset \phi(X)$ for some compact subset X of S, in particular, $\phi(T_C)$ is relatively compact.

We now claim that $\phi(T_C)$ is closed. Let (λ_n) be a sequence in T_C and $\phi(\lambda_n) \to s \in S^*$. Since T_C is weakly shift compact and by passing to a subsequence, we may assume that there exists a sequence (u_n) in $N(K)$ such that $u_n\lambda_n \to \lambda \in S$. Then

$$\phi(\lambda_n) = \phi(u_n\lambda_n) \to \phi(\lambda) \qquad (i)$$

and hence $\phi(\lambda) = s$. Since $u_n \in N(K)$, for all $n \geq 1$, $u_n\lambda_n$ is also in T_C for all $n \geq 1$, that is there exists $\mu_n \in C$ such that

$$u_n\lambda_n * \nu_n = \mu_n \qquad (ii)$$

for all $n \geq 1$ and (ν_n) is a sequence in S. By Theorem 1.2.5 of [He], (ν_n) is relatively compact and hence since C is compact, by passing to a subsequence, we may assume that $\mu_n \to \mu \in C$ and $\nu_n \to \nu \in S$. By (i) and (ii), we get that $\lambda * \nu = \mu$. This proves that $\lambda \in T_C$ and hence $s \in \phi(T_C)$. Thus, $\phi(T_C)$ is a closed set. This proves that for any compact set C, such that T_C is weakly shift compact, $\phi(T_C)$ is compact in S^*. □

We now prove the main result of this section.

Theorem 4.1 *Let G be a connected locally compact group and (G, K) be a Gelfand pair. Let S be the semigroup of all K-biinvariant probability measures on G. Suppose the pair (G, K) satisfies condition (*). Then S is a strongly stable Hungarian semigroup and hence Khinchin's factorization theorem holds for the semigroup S.*

Proof It is very routine to verify condition (H-1). By Lemma 2.2 (H-2) is satisfied if and only if the condition (*) is satisfied. We now claim that for any compact set C, $\phi(T_C)$ is compact. Let $N(H)$ be denote the normalizer of H in G, for any closed subgroup H of the group G. Since G is a connected group, there exists a compact normal subgroup M of G such that G/M is a Lie group. Let (μ_n) be a sequence in S and (λ_n) be a sequence such that λ_n is a factor of μ_n in S and (μ_n) is relatively compact. Let L be a maximal compact subgroup of G containing K and M. Then the semigroup S_1 of all L-biinvariant probability measures on G is a commutative subsemigroup of $\mathcal{P}(G)$. Since λ_n is a factor of μ_n, we have that $\lambda_n * \omega_L$ is a factor of $\mu_n * \omega_L$. By Proposition 3.2, there exists a sequence (x_n) such that $(x_n\lambda_n * \omega_L)$ is relatively compact and $x_n \in N(L)$ for all $n \geq 1$. By Theorem 1.2.15 of [He], $(x_n\lambda_n)$ is relatively compact. Now for $x \in N(L)$, we have $xKx^{-1} \subset L$. Since (G, K) satisfies condition (*), we get that $x \in LN(K)$. Thus, there exist sequences (b_n) and (u_n) in G such that

$$x_n = b_n u_n$$

where (b_n) is relatively compact and $u_n \in N(K)$ for all $n \geq 1$. This implies that $(u_n\lambda_n)$ is relatively compact and $u_n \in N(K)$ for all n. By Proposition 2.3, we have λ_n and $a_n\lambda_n$ are associates in S for all n. Thus, for any relatively compact set $\mathcal{N}$ of S, the set of factor $T_{\mathcal{N}}$ of $\mathcal{N}$ is weakly shift compact. Thus, by Lemma 4.2, for any compact subset C of S, $\phi(T_C)$ is compact.

We now verify condition (H-3). Let $x \in S^*$ and $\mu \in S$ be such that $\phi(\mu) = x$. Suppose $s \in S^*$ is a factor of x. Then $st = x$ for some $t \in S$. Let λ and ν in S be such that $\phi(\lambda) = s$ and $\phi(\nu) = t$. Then $\lambda\nu = u\mu$ for some unit u in S. Thus, λ is a factor of μ in S. This implies that $T_x = \phi(T_\mu)$ and hence by Lemma 4.2, T_x is compact. Thus, (H-3) is verified. Hence S is a Hungarian semigroup. Since G is a second countable group, S is metrizable and hence Khinchin's factorization theorem holds for S (see [RS]).

We now prove that S is strongly stable. Let C be any compact set in S. Then by Lemma 4.2, T_C is strongly shift compact and $\phi(T_C)$ is a compact set. Let X be a compact set in S such that

$$\phi(T_C) \subset \phi(X). \qquad (i)$$

Let

$$Y = X \cap \phi^{-1}(\phi(T_C)).$$

Then Y is a compact subset of S and $\phi(Y) \subset \phi(T_C)$. Let $\lambda \in T_C$. Then by (i), there exists a $\nu \in X$ such that

$$\phi(\nu) = \phi(\lambda)$$

and hence $\nu \in Y$. This proves that $\phi(T_C) = \phi(Y)$. Thus, S is a strongly stable Hungarian semigroup. □

We now prove Khinchin's Theorem for certain almost connected Gelfand pairs.

Theorem 4.2 *Let G be an almost connected second countable group and contains a compact normal subgroup M such that G/M is a Lie group and Ad (G/M) is an almost algebraic group. Suppose (G, K) is a Gelfand pair satisfying the condition (*) and S is the semigroup of K-biinvariant probability measures on G. Then S is a strongly stable Hungarian semigroup. Also, S is metrizable and the Khinchin's factorization theorem holds for S.*

Proof The proof of (H-1) is trivial and since (G, K) satisfies condition (*), (H-2) is verified. Now let C be a compact set and T_C be the set of factors of C. We now claim that $\phi(T_C)$ is compact. Let (μ_n) be a relatively compact sequence and for each $n \geq 1$, λ_n is a factor of μ_n. Let L be a maximal compact subgroup of G containing K and M. Then $(\mu_n * \omega_L)$ is relatively compact and $\lambda_n * \omega_L$ is a factor of $\mu_n * \omega_L$. By Proposition 3.2, there exists a sequence (x_n) in $N(L)$ such that $(x_n\lambda_n * \omega_L)$ is relatively compact and hence by Theorem 1.2.15 of [He], $(x_n * \lambda_n)$ is relatively compact. Since (G, K) satisfies the condition (*), we have $N(L) = LN(K)$. Thus, $x_nL = g_nL$ for some $g_n \in N(K)$, for all $n \geq 1$. Thus, $(g_n\lambda_n)$ is relatively compact. This proves that T_C is weakly shift compact. By Lemma 4.2, $\phi(T_C)$ is compact. Now (H-3) and strong stability may be proved by arguing as in Theorem 4.1. □

As an application of strongly stable Hungarian semigroups we obtain the following

Corollary 4.1 *Let (G, K) be a Gelfand pair and S be the semigroup of probability measures on G that are K-biinvariant. Let $T = \mathcal{P}(\mathcal{P}(\cdots(S)\cdots))$. Suppose the pair (G, K) satisfies the condition (*) and*

(a) *G and K are as in Theorem 4.1*

OR

(b) *G and K are as in Theorem 4.2*

OR

(c) *G is a compact metric group.*

Then we have the following:

1. *T is a strongly stable division compact uniformly Hungarian metric semigroup with Prohorov property and consequently Khinchin's factorization theorem holds for T;*

2. *the set of infinitely divisible elements in T is a closed set;*

3. *the set of indecomposable elements in T and the set of anti-indecomposable elements in T are of type G_δ (that is, a countable intersection of open sets).*

Proof By Theorem 4.1 and Theorem 4.2 we get that S is a strongly stable Hungarian semigroup. The divsion compactness and the Prohorov property of S is a consequence of Theorem 2.1 of Chapter 3 and Theorem 6.7 of Chapter 2 of [P].

We now prove that S is uniformly Hungarian. Let A and B be subsets of S and ν_1 and ν_2 be in S such that for every $\lambda \in A$ there exists a $\mu \in B$ such that $\lambda = \nu_1\mu$ and $\mu = \nu_2\lambda$. For any $\lambda \in S$, define

$$M(\lambda) = \{g \in G \mid g\lambda = \lambda = \lambda g\},$$

then by Theorem 1.2.4 of [He], $M(\lambda)$ is a compact group and

$$\omega_{M(\lambda)} * \lambda = \lambda = \lambda * \omega_{M(\lambda)},$$

also

$$\nu * \lambda = \lambda = \lambda * \nu \Leftrightarrow S(\nu) \subset M(\lambda)$$

for any $\nu \in \mathcal{P}(G)$ (see Theorem 1.27 of [He]). Let $M = \cap_{\lambda \in A} M(\lambda)$. Then M contains K and $M(\nu_i)$ for $i = 1, 2$. Now by replacing ν_i by $\nu_i * \omega_M$ for $i = 1, 2$, if necessary we may assume that $M(\nu_i) = M$ for $i = 1, 2$. Now for any $\lambda \in A$, there exists a $\mu \in B$ such that $\lambda = \nu_1\mu$ and $\mu = \nu_2\lambda$ and hence $\lambda = \nu_1\nu_2\mu$. Thus, we have

$$S(\nu_1)S(\nu_2) \subset M(\lambda)$$

for all $\lambda \in A$. This implies that $S(\nu_1)S(\nu_2) \subset M$. Now arguing as in Lemma 2.1, we get that

$$\nu_i = g_i\omega_M$$

for some $g_i \in N(M)$ and for all $i = 1, 2$. Since M contains K and (G, K) satisfies the condition (*), we have $g_i \in N(K)M$. This implies that

$$\nu_i = x_i \omega_M$$

for some $x_i \in N(K)$ and for all $i = 1, 2$. Thus, since ω_M is an idempotent factor of each element of A, for each element $\lambda \in A$, there exists a $\mu \in B$ such that $\lambda = x_1 \mu$ for $x_1 \in N(K)$ and hence $\mu = x_1^{-1}\lambda$. This proves that S is a uniformly Hungarian semigroup. Thus, we have proved that S has all the properties in (1).

Now by applying Theorem 2 of [Ze], we get that the semigroup T also has all these properties and hence the second part of (1) follows from Section 2.23 of [RS]. This proves (1) and the results (2) and (3) follow from (1) and Theorem 26.4 of [RS]. □

5 Khinchin's Theorems for certain Gelfand pairs

We first prove the analogue of Theorem 4.1, for a class of groups which includes discrete groups.

Proposition 5.1 *Let G be a discrete group and (G, K) be a Gelfand pair. Let S be the semigroup of all K-biinvariant probability measures on G. Then the factor set of a relatively compact set is weakly shift compact. In fact, for every seqeunce (λ_n) in T_C, there exists a sequence (x_n) from $Z(K)$ such that $(x_n\lambda_n)$ is relatively compact.*

Proof Let C be a relatively compact set in S and (μ_n) be a sequence in C. Let (λ_n) be a sequence in S such that λ_n is a factor of μ_n for all $n \geq 1$. Then by Theorem 1.2.21 of [He], there exists a sequence (x_n) such that $(x_n\lambda_n)$ is relatively compact. This implies that $(x_n\omega_K x_n^{-1})$ is relatively compact. Since K is finite, we have that $(x_n k x_n^{-1})$ is relatively compact for all $k \in K$. Let $K = \{k_1, k_2, \cdots k_m\}$. Then for $i = 1, 2, \cdots, m$, $(x_n k_i x_n)$ is finite. This implies that there exists a subsequence $(x_{n,i})$ such that $(x_{n,i}k_i x_{n,i}^{-1})$ is a constant sequence for i=1. Now for $i+1$, let $(x_{n,i+1})$ be a subsequence of $(x_{n,i})$ such that $(x_{n,i+1}k_{i+1}x_{n,i+1})$ is a constant sequence. Thus, for $k = |K|$, we have that $(x_{n,k}k_i x_{n,k}^{-1})$ is a constant sequence for all $k_i \in K$. Thus, (x_n), in fact every subsequence of (x_n) has a relatively compact subsequence in $G/Z(K)$. This shows that (x_n) is relatively compact in $G/Z(K)$. Thus the set of factors of a relatively compact set C of S is weakly shift compact in S. □

Corollary 5.1 *Let G be a locally compact second countable group admitting a compact open normal subgroup U and (G, K) be a Gelfand pair. Let S be the semigroup of all K-biinvariant probability measures on G. Suppose (G, K) satisfies condition (*), then the set of factors of a relatively compact set C of S is weakly shift compact in S.*

Proof It is easy to see that G is σ-compact. Let $p: G \to G/U$ be the canonical quotient map. Since U is an open normal subgroup, G/U is discrete. Let $M = UK = KU$. Then $(G/U, M/U)$ is a Gelfand pair. Let (μ_n) be a relatively compact sequence in S and (λ_n) be a sequence in S such that for each $n \geq 1$, λ_n is a factor of μ_n in S. Then $(p(\mu_n))$ is relatively compact and $p(\lambda_n)$ is a factor of $p(\mu_n)$ for all $n \geq 1$. By Proposition 5.1, there exists a sequence (g_n) in G such that $(p(g_n)\lambda_n)$ is relatively compact and for every $n \geq 1$, $p(g_n) \in Z(M/U)$ the centralizer of M in G/U. In particular, for every $n \geq 1$, $g_n \in N(M)$, the normalizer of M in G. Since (G, K) satisfies the condition (*), we have $N(M) = N(K)M$. Thus, there exists a sequence (x_n) in $N(K)$ such that $(x_n\lambda_n)$ is relatively compact. This proves that T_C is weakly shift compact when C is relatively compact. □

The following proves the Khinchin's Theorem for Gelfand pairs considered in the above corollary.

Theorem 5.1 *Let G be a locally compact second countable group admitting a compact open normal subgroup and (G, K) is a Gelfand pair. Suppose Gelfand pair (G, K) satisfies the condition (*). Let S be the semigroup of all K-biinvariant probability measures on G. Then we have the following:*

1. *the semigroup S is a division compact strongly stable uniformly Hungarian semigroup.*

2. *S is first countable and hence Khinchin's factorization theorem holds for S.*

3. *$T = \mathcal{P}(\mathcal{P}(\cdots(\mathcal{P}(S))\cdots))$ satisfies (1), (2) and (3) of Corollary 4.1*

Proof By Corollary 5.1 and by Lemma 4.2, $\phi(T_C)$ is compact for any compact subset C of S. This proves that S satisfies (H-3). The verification (H-1) is trivial and (H-2) follows because (G, K) satisfies the condition (*). Thus, S is a first countable Hungarian semigroup and hence the Khinchin's factorization theorem holds for S. By arguing as in Theorem 4.1 and Corollary 4.1, the remaining parts of the theorem can be proved. □

Let G be a locally compact group and K be a compact subgroup of G. Suppose $X = G/K$ has a G-invariant metric. Then we say that the action of G on X is *doubly transitive* or *G acts doubly transitively on X*, if $d(x, y) = d(x', y')$ implies that there exists a $g \in G$ such that $gx = gx'$ and $gy = gy'$ (it is also known as two-point homogeneous). We now introduce a class of Gelfand pairs that generalizes doubly transitive case. A pair (G, K) consisting of a locally compact group G and a compact subgroup K of G is called *symmetric pair* if $g^{-1} \in KgK$ for all $g \in G$. It is known that if G acts doubly transitively on G/K, then (G, K) is a symmetric pair (see [F]). We now prove that such pairs are Gelfand and semigroup of K-biinvariant probability measures on such pairs form a strongly stable Hungarian semigroup. We first prove factor compactness for such Gelfand pairs.

Proposition 5.2 *Let (G, K) be a symmetric pair. Then (G, K) is Gelfand. Let S be the semigroup of all K-biinvariant probability measures on G. Suppose G is 2-root compact (see [He]). Then for any compact subset C of S, the set of factors is a compact set in S.*

Proof Since all measures in S are symmetric and S is abelian, it is easy to see that (G, K) is a Gelfand pair.

Let (μ_n) be a relatively compact sequence in S and (λ_n) be a seqeunce in S such that for each $n \geq 1$, λ_n is a factor of μ_n in S. Then by Theorem 1.2.21 of [He], there exists a seqeunce (x_n) in G such that $(x_n\lambda_n)$ is relatively compact. Since λ_n is symmetric, we have $(\lambda_n x_n^{-1})$ is relatively compact and hence (λ_n^2) is relatively compact. Since G is 2-root compact, (λ_n) is relatively compact (see [He]). Let C be a compact set in S and (λ_n) be a sequence in T_C such that $\lambda_n \to \lambda \in S$. Then there exists a sequence (μ_n) in C and sequence (ν_n) in S such that $\mu_n = \lambda_n\nu_n = \nu_n\lambda_n$ for all $n \geq 1$. Since C is compact, by Theorem 1.2.21 of [He], (ν_n) is relatively compact. By passing to a subsequence, we may assume that $\mu_n \to \mu \in C$ and $\nu_n\nu \in S$. Then since $\lambda_n \to \lambda \in S$, we have $\mu = \lambda\nu = \nu\lambda$. Thus, $\lambda \in T_C$. This proves that T_C is closed. Thus, the set of factors of a compact set is compact. □

We now prove the Khinchin's theorem for symmetric (Gelfand) pairs .

Theorem 5.2 *Let (G, K) be a symmetric pair. Let S be the semigroup of all K-biinvariant probability measures on G. Suppose G is 2-root compact and the pair (G, K) satisfies the condition (*). Then the conclusions of Theorem 5.1 hold for S.*

Proof Since (H-1) is verified easily and (H-2) follows from the fact that (G, K) satisfies the condition (*). Since G is 2-root compact, by Proposition 5.2, for each compact set C of S, the set of factors T_c is also a compact set. Thus, (H-3) and strong stability are easily verified. The rest may be proved as in Thoerem 5.1. □

Remark M. Voit has proved Khinchin's Theorems for measures for symmetric hyper groups (which includes all symmetric pairs) satisfying a condition $\mathcal{D}$ (see Theorem 5.4.14 and Theorem 5.4.12 of [BH] and 5.4.2 of [BH] for definition of $\mathcal{D}$). Here we prove all three Khinchin's Theoremd for all symmetric Gelfand pairs when G is 2-root compact and our approach is different and quite simple.

We would also like to mention [He1] where H. Heyer proves the Khinchin's Theorems for certain Gelfand pairs.

6 Khinchin's Theorems for p-adic Gelfand pairs

We now prove the analogue of Theorem 4.1 for Zariski-connected p-adic algebraic groups.

We first establish the following result that is useful in proving the factor compactness for Gelfand pairs considered in this section.

Proposition 6.1 *Let G be a Zariski-connected p-adic algebraic group. subgroup of G. Let (μ_n) be a relatively compact sequence of K-biinvariant probability measures on G and (λ_n) be a sequence of K-biinvariant probability measures on G such that λ_n is a factor of μ_n for all $n \geq 1$. Suppose (G, K) is a Gelfand pair. Then there exists a sequence (x_n) in $Z(G)$ such that $(x_n\lambda_n)$ is relatively compact.*

Proof Since G is totally disconnected and K is a compact group, there exists a compact open subgroup M of G containg K. Since for a sequence (x_n) in $Z(G)$, $(x_n\omega_M * \lambda_n)$ is relatively compact implies $(x_n\lambda_n)$ is relatively compact and $\omega_M * \lambda_n$ is a factor of $\omega_M * \mu_n$ for all $n \geq 1$, we may assume that K is a compact open subgroup of G. Since λ_n is a two sided factor of μ_n, there exists a sequence (x_n) in G such that $(x_n\lambda_n)$ and $(\lambda_n x_n)$ are relatively compact.

We now claim that (x_n) is relatively compact in $G/Z(G)$. Since (λ_n) is K-biinvariant, we get that $(x_n\omega_K x_n^{-1} x_n\lambda_n)$ and $(\lambda_n x_n\omega_K x_n^{-1} x_n)$ are relatively compact. By Theorem 1.2.21 of [He], $(x_n\omega_K x_n^{-1})$ and $(x_n^{-1}\omega_K x_n)$ are relatively compact. By Lemma 3.4, for each $k \in K$, the sequence, $(x_n k x_n^{-1})$ and $(x_n^{-1} k x_n)$ are relatively compact. Thus, the group

$$H = \{h \in H \mid (x_n h x_n^{-1}) \text{ is bounded } \}$$

is an open subgroup of G. Since G is a Zariski-connected algebraic group, it has a finite-dimensional faithful rational representation. Let $\rho\colon G \to GL(V)$ be a rational faithful representation of G. Let W be the subalgebra generated by G in End (V). Now for $n \geq 1$, define $\tau_n\colon W \to W$ by $\tau_n(w) = x_n w x_n^{-1}$ for all $w \in W$. Then $\tau_n \in GL(W)$ and $(\tau_n(k))$ is relatively compact for all $k \in K$. Then by Lemma 3.5, there exists a subalgebra of W_0 of W containing H and $(\tau_n(w))$ is relatively compact in W for all $w \in W_0$. Since $G \cap W_0$ is an algebraic subgroup, we get that $W_0 = W$. Thus, (τ_n) is relatively compact in End (V). Similarly we may prove that (τ_n^{-1}) is also relatively compact in End (V). This proves that (τ_n) is relatively compact in $GL(W)$. Since G is an algebraic group, we get that (x_n) is relatively compact in $G/Z(G)$. □

We now prove the Khinchin's Theorem.

Theorem 6.1 *Let G be a Zariski-connected p-adic algebraic group. Suppose (G, K) is a Gelfand pair satisfiying condition (*). Let S be the semigroup of K-biinvariant probability measures on G. Suppose the pair (G, K) satisfies condition (*). Then (1), (2) and (3) of Theorem 5.1 hold for S.*

Proof The verification of (H-1) is quite easy. The verification of (H-2) follows from the assumption that (G, K) satisfies the condition (*). The rest of the proof is quite similar to the proof of Theorem 4.1 and the proof of Corollary 4.1, so we omit the details. □

We now consider compact extension of unipotent algebraic groups. We first, as usual establish the factor compactness.

Proposition 6.2 *Let G be a compact extension of a normal unipotent algebraic group. Let K be a compact subgroup of G. Let (μ_n) be a relatively compact sequence of K-biinvariant probability measures and (λ_n) be a sequence of K-biinvariant probability measures such that for each $n \geq 1$, λ_n is a factor of μ_n. Then there exists a sequence (x_n) in $Z(K)$ such that $(x_n \lambda_n)$ is relatively compact.*

Proof Since λ_n is a factor of μ_n for all $n \geq 1$ and (λ_n) is relatively compact, there exists a sequence (g_n) in G such that $(g_n \lambda_n)$ is relatively compact (see [P]). Let U be a unipotent normal subgroup of G, such that G/U is compact. This implies that $g_n = u_n c_n$ and $u_n \in U$ for all $n \geq 1$ and (c_n) is relatively compact. Since $(g_n \lambda_n)$ is relatively compact, $(u_n \lambda_n)$ is relatively compact. Since λ_n is K-biinvariant,

$$u_n \omega_K u_n^{-1} u_n \lambda_n = u_n \lambda_n$$

for all $n \geq 1$. Thus, $(u_n \omega_K u_n^{-1})$ is relatively compact (see Theorem of [He]). Now from Lemma 3.4, for each $k \in K$, $(u_n k u_n^{-1})$ is relatively compact. This implies that for each $k \in K$, $(u_n k u_n^{-1} k^{-1})$ is relatively compact. Now applying Lemma 3.2, there exists a bounded sequence (b_n) in U such that

$$u_n = b_n x_n \quad \text{and} \quad x_n \in Z(K)$$

for all $n \geq 1$. This proves the proposition. □

We now prove the Khinchin's factorization theorem when G is a compact extension of a unipotent group.

Theorem 6.2 *Let G be a compact extension of an unipotent algebraic group and (G, K) be a Gelfand pair. Let S be the semigroup of all K-biinvariant probability measures on G. Suppose (G, K) satisfies the condition (*). Then conclusions (1), (2) and (3) of Theorem 5.1 hold for S.*

Proof Using Proposition 6.2, one may prove the result by arguing as in Theorem 6.1. □

Remark In Theorem 6.2 if G is replaced by a compact extension of a connected nilpotent (real) Lie group, the conclusions are still valid.

7 Limit theorems and embedding

Let $\mathcal{N}$ be any set in a topological semigroup S. Then $\mathcal{R}(\mathcal{N}) = \{\nu^k \mid \nu^n \in \mathcal{N}, k \leq n\}$. In proving limit theorems and embedding of probability measures we need the compactness of the root set $\mathcal{R}(\mathcal{N})$: see [S1] and [Te] for results on limit theorems for probability measures on general locally compact groups. An element s of a Hungarian semigroup S is said to be *weakly infinitely divisible* if for each $n \geq 1$, there exists an s_n in S such that s_n^n is an associate of s and s is said to be *infinitely divisible* if for each $n \geq 1$, there exists an s_n in S such that $s_n^n = s$. An element s of a Hungarian semigroup S is said to be *embeddable* in a continuous convolution semigroup in S if there exists a continuous homomorphism $t \mapsto s_t$ from $[0, \infty)$ into S such that $s_1 = s$.

Lemma 7.1 *Let G be a locally compact second countable group and K be a compact subgroup of G such that (G, K) is a Gelfand pair. Let S be the semigroup of K-biinvariant probability measures on G. Suppose $N(K)$ is a strongly root compact group where $N(K)$ is the normalizer of K in G and $\mathcal{R}(\mathcal{N})$ is strongly shift compact. Then $\mathcal{R}(\mathcal{N})$ is relatively compact in S*

Proof Since $\mathcal{R}(\mathcal{N})$ is strongly shift compact, we get that for each $\nu \in \mathcal{R}(\mathcal{N})$, there exists a unit and hence by Proposition 2.3, an element $x(\nu) \in N(K)$ such that $\{x(\nu)\nu \mid \nu \in \mathcal{R}(\mathcal{N})\}$ is relatively compact. Let $\epsilon \in (0, \frac{1}{3})$. Let $\nu \in \mathcal{R}(\mathcal{N})$ be such that $\nu^n \in \mathcal{N}$. Then for each $1 \leq k \leq n$ there exists a $x_k \in N(K)$ and a compact set $C(\epsilon)$ such that $\nu^k(Cx_k) > 1 - \epsilon$. Replacing C by KCK which is again a compact set containing C, if necessary we may assume that $xC = Cx$, for all $x \in N(K)$. Now for any $1 \leq k, l, k + l \leq n$,

$$\nu^{k+l}(Cx_kCx_l) \geq \nu^k(Cx_k)\nu^l(Cx_l) \geq (1 - \epsilon)^2$$

and hence

$$\nu^{k+l}(Cx_kCx_l \cap Cx_{k+l}) \geq \nu^{k+l}(Cx_kCx_l) - \nu^{k+l}(G \setminus Cx_{k+l}) \geq (1 - \epsilon)^2 - \epsilon \geq 1 - 3\epsilon$$

Since $0 < \epsilon < \frac{1}{3}$,

$$Cx_kCx_l \cap Cx_{k+l} \neq \emptyset.$$

This implies that

$$CCx_kx_l \cap Cx_{k+l} \neq \emptyset.$$

Thus, we may choose $\{x_1, x_2, \cdots, x_n\}$ a set of n points in $N(K)$ such that for any $1 \leq k, l, k + l \leq n$,

$$x_kx_lx_{k+l}^{-1} \in CCC^{-1}.$$

Let $D = CCC^{-1} \cap N(K)$. Then since $D \subset DDD^{-1}$,

$$x_kx_lx_{k+l}^{-1} \in DDD^{-1} \neq \emptyset$$

and hence since $xD = Dx$ for all $x \in N(K)$, we have

$$Dx_kDx_l \cap Dx_{k+l} \neq \emptyset$$

for all $1 \leq k, l, k+l \leq n$. Since $N(K)$ is strongly root compact, there exists a compact set B such that $x_i \in B$, for all $1 \leq i \leq n$. This implies that $\nu(CB) \geq \nu(Cx_1) \geq 1 - \epsilon$. By Prohorov's Theorem we deduce that $\mathcal{R}(\mathcal{N})$ is relatively compact. □

Remark Suppose G is a Lie group. Then $N(K)$ is a strongly root compact group which may be seen as follows: Since (G, K) is a Gelfand pair, $N(K)/K$ is a commutative Lie group and hence by Theorem 3.12 and Remark 2 of [He], $N(K)$ is a strongly root compact group.

We now prove the functional limit theorem.

Theorem 7.1 *Let (G, K) be a Gelfand pair and G is a locally compact σ-compact second countable group. Suppose the semigroup S of all K-biinvariant probability measures on G is strongly stable and the normalizer $N(K)$ of K in G is strongly root compact. Let (ν_n) be a sequence in S such that $\nu_n^{k_n} \to \mu$ as $k_n \to \infty$. Then*

(a) μ has an associate λ that is infinitely divisible in S and

(b) there exists a one-parameter continuous convolution semigroup (λ_t) in S and a compact connected subgroup C of $N(K)/M$ such that $\lambda_1 = c\lambda$ and $cM \in C$ for some compact subgroup M containing K.

Proof Let $I(\mu) = \{g \in G \mid g\mu = \mu g = \mu\}$. Then by replacing ν_n by $\nu_n \omega_{I(\mu)}$, we may assume that

$$\nu_n \omega_{I(\mu)} = \nu_n$$

for all $n \geq 1$. Let $A = \{\nu_n^k \mid k \leq k_n, n \geq 1\}$. Then since S is strongly stable, A is weakly shift compact and hence by Lemma 7.1, A is relatively compact. Since $\nu_n \in A$ for all $n \geq 1$, by passing to a subsequence we may assume that (ν_n) converges. Let ν be the limit point of (ν_n). Then $\nu^j \in \overline{A}$ and hence (ν^j) is relatively compact. We now claim that ν^j is a factor of μ for all $j \geq 1$. For any $j \geq 1$, we have for large n,

$$\nu_n^{k_n} = \nu_n^{k_n - j} \nu_n^j.$$

By letting $n \to \infty$, over a subsequence of (n), we get that

$$\mu = \lambda_j \nu^j$$

for all $j \geq 1$. Thus, ν^j is a factor of μ for all $j \geq 1$. Then by 22.12 of [RS], $\nu\mu$ is an associate of μ and hence by 22.13 of [RS], ν is an associate of $\omega_{I(\mu)}$. Let u be a unit in S such that $\nu = u\omega_{I(\mu)}$. Then it is easy to see that $\eta_n = u^{-1}\nu_n \to \omega_{I(\mu)}$. Since $\nu^j \in \overline{A}$ for all j, we get that (u^j) is relatively compact, in particular (u^{k_n}) is relatively compact. Then again by passing to a subsequence (k_n), if necessary, we may assume that $u^{k_n} \to u'$, a unit in S. This implies that $\eta_n^{k_n} \to u'^{-1}\mu = \lambda$, say.

Let $\rho = \omega_{I(\mu)}$. Then $\rho\lambda = \lambda$. It is clear that η_n is $I(\mu)$-invariant. Now using diagonal process and by passing to a subsequence we may assume that

$$\eta_n^{[\frac{k_n}{m}]} \to \lambda_{\frac{1}{m}}$$

as $n \to \infty$ for all $m \in \mathbb{N}$. Also since each η_n is $I(\mu)$-invariant, $\lambda_{\frac{1}{m}}$ is also $I(\mu)$-invariant.

We now claim that $\lambda_{\frac{1}{m}}^m = \lambda$ for all $m \geq 1$. Now for $m \geq 1$,

$$\eta_n^{[\frac{k_n}{m}]m} \eta_n^{r_n} = \eta_n^{k_n}$$

for some $0 \leq r_n < m$. Then by taking limit when $n \to \infty$ over a subsequence of (n), we get that $\lambda_{\frac{1}{m}}^m \rho = \lambda$ and hence $\lambda_{\frac{1}{m}}^m = \lambda$. This proves that an associate of μ is infinitely divisible.

We now claim that $\lambda_{\frac{1}{m!}} = \lambda_{\frac{1}{(m+1)!}}^{(m+1)}$ for all $m \geq 1$. For large n,

$$\eta_n^{[\frac{k_n}{m!}]} = \eta_n^{[\frac{k_n}{(m+1)!}](m+1)} \eta_n^{r_n}$$

for some $0 \leq r_n < m+1$ and for all $m \geq 1$. It follows from $\eta_n \to \rho$ that

$$\lambda_{\frac{1}{m!}} = \lambda_{\frac{1}{(m+1)!}}^{(m+1)}$$

for all $m \geq 1$.

Now By Lemma 3.1.30 of [He] there exists a semigroup homomorphism $f \colon \mathbb{Q}^+ \to S$ such that

$$f(\frac{1}{m!}) = s(\frac{1}{m!})$$

for all $m \in \mathbb{N}$. Since $f((0,\ 1] \cap \mathbb{Q}_+^*) \subset \overline{A}$ is relatively compact by Theorem 3.5.1 of [He], there exists a compact connected group C in S and a continuous convolution semigroup (λ_t) such that $\lambda_1 = c\lambda$ for some $c \in C$. It is easy to see that, for $r < 1$, $\overline{f((0,r) \cap \mathbb{Q}^+)}$ is contained in the set of factors of λ. Since $C = \cap_{r<1} \overline{f((0,r) \cap \mathbb{Q}^+)}$, we get that C is contained in the set of factors of λ. This implies that identity of C is an idempotent factor of λ. Thus, there exists a compact subgroup L of G such that C is a compact connected subgroup of $N(L)/L$ and $\lambda * \omega_L = \lambda$. Since $I(\mu) = I(\lambda)$, we get that $L \subset I(\mu)$. Since $N(L) = N(K)L$, we get that $\lambda_1 = u\lambda$ and u is contained in a compact connected subgroup of $N(K)/M$ where $M = N(K) \cap I(\mu)$. □

As a consequence of the above theorem and the results in the previous sections, we have the following functional limit theorem for Gelfand pairs.

Theorem 7.2 *Let G be a locally compact second countable group and (G, K) be a Gelfand pair satisfying the condition (*). Suppose G is either (a) connected or (b) almost connected and G has a compact normal subgroup M such that G/M is a Lie group and Ad (G/M) is an almost algebraic group or (c) G has a compact open normal subgroup or (d) p-adic algebraic group or a compact extension of a unipotent algebraic group (connected nilpotent Lie group). Let S be the semigroup of all K-binvariant probability measures on G. Suppose $N(K)$ is strongly root compact. Then the conclusions of Theorem 7.1 hold for S.*

Remark It should be remarked that by verifying conditions in Theorem 2.3 of [Te] one may try to prove the functional limit Theorem but our proof is simpler (may be because it is a particular case) and arguments are independent of proof of Theorem 2.3 of [Te].

Theorem 7.3 *Let (G, K) be a symmetric pair. Suppose G is 2-root compact and $N(K)$ is strongly root compact and (G, K) satisfies condition (*). Let S be the semi-group of all K-biinvariant probability measures on G and (ν_n) be a sequence in G such that $\nu_n^n \to \mu \in S$. Then μ is embeddable in a continuous convolution semigroup in S.*

Proof We first claim that $N(K)$ is compact. For any $x \in N(K)$, since (G, K) is a symmetric pair, $x^{-1} \in KxK = xK$ which implies $x^2 \in K$. Since G is 2-root compact, $N(K)$ is compact. By Theorem 7.1, μ has an associate λ that is infinitely divisible and $u\lambda$ is embeddable in a continuous convolution semigroup in S for some u contained in a compact connected subgroup $N(K)/M$ where $M = N(K) \cap I(\mu)$. Since all K-biinvariant measures are symmetric, $N(K)/M$ is a abelian group and all its elements are of order 2. This implies that $N(K)/M$ has no connected subgroups and hence λ is embeddable. It is clear from the proof of Theorem 7.1 that $\lambda = u\mu$ and u is a limit of (u^{k_n}). Choosing $k_n = 2n$, we may prove that $\lambda = \mu$. Thus, μ itself is embeddable in a continuous convolution semigroup. $\square$

Remark In [He2], it is proved that infinitely divisible measures are embeddable for discrete Gelfand pairs.

8 Normability and inifinitesimal limits

In this section we study the second and third theorems of Khinchin. Let S be a any Hausdorff topological semigroup and I be a directed set. An I-*array* is a system $((t_{ij})_{j=1}^{n(i)})_{i \in I}$, $t_{i,j} \in S$. In particular, if I is the set of positive integers, we say that (t_{ij}) is a *triangular system*. An I-array (t_{ij}) is *infinitesimal* if for every neighbourhood U of identity in S, there is an $i_0 \in I$ such that $t_{ij} \in U$ for all $i > i_0$ and all $1 \leq j \leq n(i)$. We say that $s \in S$ is an *infinitesimal limit* if there exists an infinitesimal I-array (t_{ij}) such that $s = \lim(\prod_j t_{ij})$. Khinchin's third theorem says that any infinitesimal limit is infinitely divisible in $S = \mathcal{P}(\mathbb{R})$. Khinchin's third theorem was extended to general abelian groups by Ruzsa and Szekely (see [RS]). We prove an analogue of this for Gelfand pairs. We first establish the normability of the semigroup S of K-biinvariant probability measures on G for Gelfand pairs (G, K). For any $\lambda \in S$ define $\check{\lambda}$ by

$$\check{\lambda}(E) = \lambda(E^{-1})$$

Where $E^{-1} = \{x^{-1} \mid x \in E\}$ for all Borel subsets of G. It is easy to

$$\lambda * \check{\lambda} = \check{\lambda} * \lambda$$

for all $\lambda \in S$, that is measures in S are all normal.

We denote by S^s the semigroup of symmetrization of measures in S. Let $\Phi: S \to S^s$ be the map defined by

$$\Phi(\lambda) = \lambda * \check{\lambda}$$

for all $\lambda \in S$. Since S is a commutative semigroup, we get that Φ is a homomorphism of S into S^s. For any $\lambda \in S$, by T_λ we denote the set of all factors of λ in S and for any $\mu \in S^s$, by T^s_μ we denote the set of all factors of μ in S^s. We now prove the following:

Lemma 8.1 *For any $\lambda \in S$, $\Phi(\lambda)$ is a shift of an idempotent if and only if λ is an idempotent. For $\lambda \in S$, let $\mu = \Phi(\lambda)$. If there is a continuous partial homomorphism f_λ from T^s_μ into $[0, \infty)$ such that $f_\mu(\mu) > 0$, then there exists a continuous partial homomorphism f_λ from T_λ into $[0, \infty)$.*

Proof Suppose $\lambda * \check{\lambda}$ is a shift of an idempotent, say ω_M. Then $S(\lambda)S(\lambda)^{-1} \subset uM$ for some $u \in N(K)$. This implies that uM contains the identity of G and hence $u \in M$. Thus, $\lambda\check{\lambda}$ is an idempotent. Since λ is normal, by Proposition of [E], we get that λ is a shift of an idempotent. The converse part and the second part of the lemma are obvious. □

We now prove the existence of a partial homomorphism for any commutative hypergroup (see [BH] for details on hyper groups): the proof is quite similar to the case of locally compact abelian groups (see [RS]).

Proposition 8.1 *Let K be a commutative second countable hypergroup. Let $\lambda \in \mathcal{P}(K)$ be a measure such that $\lambda * \check{\lambda}$ is not an idempotent. Then there exists a continuous partial homomorphism $f_\lambda: T_\lambda \to [0, \infty)$ such that $f_\lambda(\lambda) > 0$.*

Proof Since $\lambda * \check{\lambda}$ is not an idempotent, by Theorem 2.2.4 of [BH], there exists a continuous bounded multiplicative function χ on K such that

$$0 < |\lambda(\chi)|^2 < 1.$$

Let $f_\lambda: T_\lambda \to [0, \infty)$ be the map defined by

$$f_\lambda(\nu) = -\log(|\nu(\chi)|^2)$$

for all $\nu \in T_\lambda$. Then by Theorem 2.2.4 of [BH], f_λ is a continuous partial homomorphism with the required condition.

We now deduce the normability for Gelfand pairs.

Corollary 8.1 *Let G be locally compact second countable group and (G, K) be a Gelfand pair. Suppose the semigroup S of all K-biinvariant probability measures on G is a Hungarian semigroup. Then S is a normable Hungarian semigroup.*

Proof By Theorem 1.1.9 of [BH], the double coset space $G//K$ is a hyper group. By Theorem 1.5.20 of [BH], S is isomorphic to probability measures on $G//K$. Since (G,K) is a Gelfand pair, $G//K$ is a commutative semigroup and hence the corollary follows from Proposition 8.1. □

We now prove the Khinchin's Theorems for certain Gelfand pairs.

Theorem 8.1 *Let G be a locally compact second countable group and (G,K) is a Gelfand pair. Suppose G is as in Theorem 4.1 or Theorem 4.2 or Theorem 5.1 or Theorem 5.2 or Theorem 6.1 or Theorem 6.2. Suppose $N(K)$ is strongly root compact. Then the semigroup S of all K-biinvariant probability measures on G is a stable normed Hungarian semigroup. Consequently, Khinchin's first, second and third Theorems hold for S. Moreover, if $N(K)$ is compact or $N(K)/K$ is divisible and strongly root compact, then any anti-indecomposable element in S or any infinitesimal limit in S has an associate that is embeddable in S.*

Proof It is already proved that the semigroup S of all K-biinvariant probability measures on G is a stable Hungarian semigroup. Now by Corollary 8.1, S is a stable normable Hungarian semigroup. By Theorem 24.17 and Theorem 26.9 of [RS], anti-indecomposable elements of S and inifinitesimal limits in S are weakly infinitely divisible.

Let μ in S be either anti-indecomposable in S or μ is a limit of an infinitesimal I-array. Then for each $n \geq 1$, there exists a μ_n and a unit u_n such that $\mu = u_n\mu_n^n$. If $N(K)$ is compact, then

$$\mu_{k_n}^{k_n} \to \lambda \in S$$

for some subsequence (k_n) of (n) and λ is an associate of μ. Now from Theorem 7.2, we deduce that λ has an associate that is embeddable. Thus, an associate of μ is embeddable.

Suppose $N(K)/K$ is divisible and strongly root compact. Since $u_n \in N(K)$, there exists a $x_n \in N(K)$ such that $x_n^n\omega_K = u_n\omega_K$ for all $n \geq 1$. This implies that

$$\mu = (x_n\nu_n)^n$$

for all $n \geq 1$. By Theorem 7.2, an associate $u\mu$ of μ is embeddable for some unit u contained in a compact subgroup of $N(K)$. □

Remark Let (G,K) be a Gelfand pair such that (a)G is connected or (b) G is a almost connected group considered in the article or (c) G is discrete or (d) G is a Zariski-connected p-adic algebraic group or (e) G is a compact extension of an unipotent algebraic group or a connected nilpotent Lie group (not necessarily satisfying the condition (*)) and S be the semigroup of all K-biinvariant probability measures on G. Suppose $\lambda \in S$ is a bald element, that is a measure having no idempotent factors. Then using the factor compactness results (modulo the group of units, which follows for connected or almost connected case from the fact that for an almost connected Lie group G and a compact subgroup M containing K, $N(M)/MN(K)$ is finite (see [HHSWZ])) and applying the arguments of K. R. Parthasarathy (Chapter IV, Theorem 11.3 of [P]) as in Theorem 5.4.14 of [BH], one may prove the Khinchin's factorization theorem for any bald λ. Thus, it shows that

(a) any measure in S is a product of a idempotent in S and a measure in S that has a Khinchin type decomposition in S and

(b) any measure $\mu \in S$ has Khinchin's decomposition in a subsemigroup of S.

As a consequence of Theorem 8.1 we obtain Khinchin's Theorems for compact symmetric spaces.

Corollary 8.2 *Let G be a compact connected semisimple Lie group and K be a compact connected subgroup of G such that G/K is a compact Riemannian symmetric space. Then the semigroup S of K-biinvariant probability measures on G is a normable strongly stable Hungarian semigroup and Khinchin first, second and third Theorems hold for S. Moreover, any antiindecomposable or infinitesimal limit has an associate that is embeddable.*

Proof Let S be the semigroup of K-biinvariant probability measures on G. Since G/K is a Riemannian symmetric space, S is a commutative semigroup. By lemma 2.3, S satisfies condition (*). Thus, since G is second countable, the corollary follows from Theorem 8.1 □

Remark We note that I. R. Truhina [Tr] and J. Lamperti [L] have earlier inverstigated and proved the Khinchin's factorization theorem for probability measures on irreducible compact symmetric spaces of Rank 1. In [L], this was achieved by covering by delphic semigroups. Thus, our treatment is completely different and we consider the general cases.

9 Gaussian measures on compact Gelfand pairs

Gaussian measures on $\mathbb{R}$ have two well-known properties:

1. Gaussian measures on $\mathbb{R}$ have only Gaussian factors which is known as Cramer's theorem;

2. Gaussian measures have no indecomposable factors.

Since Gaussian measures on $\mathbb{R}$ are infinitely divisible, second property follows from the first property (see [Fe]).

On the other hand Marcinkiewicz [Ma] showed that these properties are not verified by Gaussian measures on the circle. Trukhina [Tr] proved that $SO(n)$- invariant Gaussian measures on the spheres $S^n \simeq SO(n+1)/SO(n)$ are not anti-indecomposable. In this section we extend these results to compact Gelfand pairs. As in the previous sections for any compact subgroup L, we denote the normalized Haar measure on L by ω_L.

Theorem 9.1 *Let G be a compact connected Lie group and K be a compact subgroup of G. Suppose (G, K) is a Gelfand pair. Let μ be a K-biinvariant probability measure on G such that the inequality*

$$\mu(E) \geq a\omega_G(E), \qquad 0 < a < 1 \tag{i}$$

holds for all Borel subsets E of G. Then μ has an indecomposable factor.

In [Fe], G. M. Feldman proved a similar theorem for measures on compact abelian groups. We apply some of the techniques of Theorem 4.17 of [Fe]. We first prove the following lemma.

Lemma 9.1 *Let G, K and m_G be as in Theorem 9.1. Suppose μ is a K-biinvariant probability measure on G verifying the inequality (i). Then μ decomposes in the following way*

$$\mu = \frac{\mu - a\omega_G}{1-a} \ * \ [(1-a)\omega_K + a\omega_G] \tag{ii}$$

where $0 < a < 1$ is as in the inequality (i).

Proof We now verify the result using the obvious properties:

$$\mu * \omega_K = \mu, \quad \omega_G * \omega_K = \omega_G, \quad \text{and} \quad \mu * \omega_G = \omega_G.$$

$$\begin{aligned} \tfrac{\mu - a\omega_G}{1-a} * [(1-a)\omega_K + a\omega_G] &= (\mu - a\omega_G)\omega_K + \tfrac{a}{1-a}(\mu - a\omega_G)\omega_G \\ &= \mu - a\omega_G + \tfrac{a}{1-a}\omega_G - \tfrac{a^2}{1-a}\omega_G \\ &= \mu - \omega_G \tfrac{[a(1-a) - a + a^2]}{1-a} \\ &= \mu. \end{aligned}$$

□

We now define K-biinvariant compound Poisson measures on Gelfand pairs. Let G be a locally compact group and K be a compact subgroup of G. Let γ be a K-biinvariant signed bounded measure on a locally compact group G. We define the *compound Poisson measure* $e(\gamma)$ by

$$e(\gamma) = e^{-\gamma(G)} \sum_{n=0}^{\infty} \frac{\gamma^n}{n!}$$

where, by definition, $\nu^0 = \omega_K$. It is easy to see that if γ is a positive measure, $e(\gamma)$ is a K-biinvariant probability measure on G. We now prove Theorem 9.1

Proof (of Theorem 9.1) Observe that if $k > 0$, then

$$e(k\omega_G) = e^{-k}\omega_K + (1 - e^{-k})\omega_G. \qquad (iii)$$

Choosing $k > 0$ such that $a = 1 - e^{-k}$, we see by Lemma 9.1 that $e(k\omega_G)$ is a factor of μ. It is will be sufficient to show that $e(k\omega_G) \notin I_0$, the set of all anti-indecomposable measures. By hypothesis, there exist two points $x, y \in G$ but $x, y \notin K$ such that $y \notin KxK$. Set $z = yx^{-1} \in G$, so that $y = zx$. By Urysohn's Theorem, there exist two open sets U_1 and U_2 such that

$$U_i = KU_iK$$

for all $i = 1, 2$ and

$$U_1 \cap U_2 = \emptyset,$$

$$x \in U_1, \quad y \in U_2,$$

$$e, z \notin U_2.$$

Define $U_3 = K(zU_1) \cap U_2$ and $V = G \setminus \overline{U}_3$. Then U_3 and V are K-biinvariant open sets such that $\overline{U_1} \subset V$ and $z \in V$. It follows that

$$\overline{U_3} \subset Kz\overline{U_1} \subset VV$$

and hence

$$VV = G.$$

Let γ be the restriction of ω_G to the open set V. Then since ω_G is K-biinvariant and V is K-biinvariant, it is easy to see that the measure γ is K-biinvariant. Also, it can be seen that since $\omega_G - \gamma$ is a positive measure, $e(k\gamma)$ is a factor of $e(k\omega_G)$. We now claim that $\gamma * \gamma$ has a strictly positive density $\chi_V * \chi_V$. It is well-known that since V is an open set, $\chi_V * \chi_V$ is a continuous function. Let $x \in VV$. Then there exist $v_1, v_2 \in V$ such that $x = v_1v_2$. Since V is a open set and the map $g \mapsto xg^{-1}$ is continuous, there exists a neighbourhood W of v_2 in G such that

$$xW^{-1} \subset V.$$

This implies that

$$xw^{-1} \in V$$

for all $w \in W$. Thus,

$$\begin{aligned} \chi_V * \chi_V(x) &= \int_V \chi_V(xg^{-1}) d\omega_G(g) \\ &\geq \omega_G(V \cap W) \end{aligned}$$

since $v_2 \in V \cap W$, $V \cap W$ is a non-empty open set in G and hence

$$\chi_V * \chi_V(x) > 0$$

for all $x \in VV$. Since $G = VV$, $\chi_V * \chi_V$ is strictly positive. It is easy to see that $\chi_V * \chi_V$ is the density of $\gamma * \gamma$. Since $\chi_V * \chi_V$ is continuous and G is compact, there exists a constant $c > 0$, such that

$$\chi_V * \chi_V(x) > c$$

for all $x \in G$. It follows that for any Borel subset E of G,

$$\gamma(E) \geq c\omega_G(E)$$

and hence for some constant $0 < b < 1$,

$$e(k\gamma)(E) \geq b\omega_G(E)$$

for all Borel subsets E of G. Using again Lemma 9.1, and (iii) we get that

$$e(k\gamma) = \mu_1 * e(k_1\omega_G) \qquad (iv)$$

for a measure $\mu_1 \in \mathcal{P}_K(G)$. We may suppose that $0 < k_1 < k$. Note that the constants a, b, k, k_1 can be made arbitrarily small. Since $e(k\gamma)$ is a factor of $e(k\omega_G)$, in order to prove that $e(k\omega_G)$ has an indecomposable factor, it is sufficient to show that $e(k\gamma)$ has an indecomposable factor.

In the last part of the proof we will show that for $0 < k < k_1$ small enough, the measure $\mu_1 \notin I_0$. Suppose on the contrary that for any $0 < k_1 < k$, $\nu_1 \in I_0$. By Theorem 6.2, the measure μ is infinitely divisible and a power of it is embeddable. Now it is easy to see that any power of μ_1 also satisfies the equation (iv). Thus, by replacing μ_1 by a suitable power of μ_1, we may assume that μ_1 is embeddable in a ω_L-continuous convolution semigroup $(\mu_t)_{t>0}$, where L is a compact subgroup of G containing K. So (G, L) is also a compact Gelfand pair. Compact Gelfand pairs are strong hypergroups (see 4.3.23 of [BH]) and by 5.2.15 and 5.2.29 of [BH], we have the following Lévy Khinchine formula for the semigroup $(\mu_t)_{t>0}$:

$$\hat{\mu}_t(\chi) = \exp\{-t(a + q(\chi) + \int (1 - \mathrm{Re}(\chi(x))d\eta(x))\}$$

where $a \in \mathbb{R}$, and q is a non-negative K-biinvariant quadratic form on G, η is the Lévy measure (a positive K-biinvariant measure on G) and χ varies over all of the dual of (G, L).

On the other hand (iv) implies that

$$\mu_1 = e(k\gamma - k_1\omega_G).$$

In a similar way as Feldman (see 4.13 of [Fe]) one shows that $a = 0$ and $q = 0$ and that if k and k_1 are sufficiently small, then the Lévy measure η is finite and $\eta(G)$ is arbitrarily small. This yield that $\mu_1 = e(\eta)$ and it follows from (iv) that

$$k\gamma - k_1\omega_G = \eta$$

which is impossible because

$$\eta \geq 0$$

and

$$(k\gamma - k_1\omega_G)(U_3) < 0.$$

Thus, $\mu_1 \notin I_0$, that is μ_1 is not anti-indecomposable and hence μ is not anti-indecomposable. □

Remark In the proof of Theorem 9.1, we use the hypothesis that G is a connected Lie group to get the following: (a) the double coset space $K\backslash G/K$ has more than two points and (b) a finite power of a unit in $\mathcal{P}_K(G)$ is embeddable in $\mathcal{P}_K(G)$. The first statement (a) is always true for a connected group as the double coset space $K\backslash G/K$ is a connected space and the second statement (b) is true if (G, K) is a symmetric pair. Thus, Theorem 9.1 may also be proved for more general compact Gelfand pairs which satisfy (a) and (b).

Corollary 9.1 *Let (G, K) be as in Theorem 9.1 and μ be a K-biinvariant probability measure on G. Suppose μ is an absolutely continuous measure with everywhere positive continuous density. Then μ is not anti-indecomposable.*

Proof Suppose μ has a everywhere positive density, say f, then it is easy to see that for some $a > 0$,

$$f(x) > a$$

for all $x \in G$. This implies that

$$\mu(E) > a\omega_G(E)$$

for all Borel subsets E of G. Now the corrollary is immediate from Theorem 9.1. □

Corollary 9.2 *Let (G, K) be a Gelfand pair with G a compact connected Lie group and K a compact subgroup of G. Let $(\gamma_t)_{t>0}$ be the heat semigroup generated by the Laplace-Beltrami operator Δ for the Riemannian homogeneous manifold G/K. Then $\gamma_t \notin I_0$ for all $t > 0$. In particular the measures $(\gamma_t)_{t>0}$ have non-Gaussian factors, that is, Cramer theorem does not hold for $(\gamma_t)_{t>0}$.*

Proof We use the fact that the measures $(\gamma_t)_t > 0$ are K-invariant and that they have everywhere positive smooth densities (see Theorem 5.2.1 of [Da]). □

The following is easily deduced from Corollary 9.2.

Corollary 9.3 *Let G/K be a Riemannian symmetric space of compact type. Then the Gaussian measures on G/K (defined as belonging to the heat semigroup) have indecomposable factors and do not satisfy the Cramer Theorem.*

10 Central limit theorems of Lindeberg-Feller type

Central limit theorems for a triangular arrays of measures converging to a Gaussian measure are known on Euclidean spaces, locally compact abelian groups and symmetric spaces. Gaussian meaures on a Gelfand pair were introduced by Heyer (see [He1]) by using the generalized Laplacians, introduced by Duflo [Du].

In order to discuss the central limit problem on Gelfand pairs, we must introduce a notion of *dispersion* D of a mesure $\mu \in \mathcal{P}_K(G)$, having the properties:

$$D: \mathcal{P}_K(G) \to [0, \infty]$$

$$D(\mu_1 * \mu_2) = D\mu_1 + D\mu_2 \qquad (i)$$

and

$$D\mu = \int Q d\mu$$

for a continuous K-biinvariant function Q.

A natural candidate for the function Q is a *positive-quadratic* form on (G, K) defined acccording to Faraut-Harzallah [FH] as a real continuous symmetric K-biinvariant function verifying

$$\int_K Q(xky)dk + \int_K Q(xky^{-1})dk = 2(Q(x) + Q(y)) \qquad (ii)$$

for all $x, y \in G$.

The property (ii) is equivalent to (i) when at least one of the measures μ_1, μ_2 is symmetric. It is then natural to seek for a Gaussian central limit theorem for a triangular array of symmetric measures.

In [G1] such a theorem was proved on non-compact Riemannian symmetric spaces with dispersion defined using functions Q verifying the condition

$$\int_K Q(xky)dk = Q(x) + Q(y)$$

for all $x, y \in G$.

It is easy to see that

$$Q^s = Q + \check{Q}$$

defines a quadratic form and that a similar central limit theorem with dispersion

$$D^s(\mu) = \int Q^s d\mu$$

holds for symmetric K-invariant measures on non-compact symmetric spaces G/K.

Moreover, we can show that the sufficient technical condition in [G1] is also necessary.

Acknowledgement The second author (C. R. E. Raja) would like to thank Dr. Piotr Graczyk for inviting him to work at Universite d'Angers under the fellowship of the Regional commission CCRRDT des Pays de Loire and Departement de Mathematiques of Universite d'Angers for providing all the facilities. We gladly thank Prof. S. G. Dani for his fruitful suggestions and his help in proving Lemma 3.1. Our thanks are also to J. Faraut and G. M. Feldman for various discussions on Gelfand pairs and Khinchin and Cramer theorems.

References

[BH] W. R. Bloom and H. Heyer, Harmonic Analysis on Probability Measures on Hypergroups, Walter de Gruyter, Berlin-New York, 1995.

[BJR] C. Benson, J. Jenkins, G. Ratcliff, On Gelfand pairs associated with solvable Lie groups, Trans. Amer. Math. Soc., 321 (1990), 85-116.

[Da] E. B. Davies, Heat kernels and spetral theory, Cambridge Tracts in Mathematics, 92, Cambridge University Press, Cambridge (1990).

[DM] S. G. Dani and M. McCrudden, Factors, roots and embeddability of measures on Lie groups, Mathematische Zeitschrift 199 (1988), 369-385.

[DR] S. G. Dani and C. R. E. Raja, Asymptotics of measures under group automorphisms and an application to factor set, Proceedings of (1996) International Colloquium on Lie Groups and Ergodic Theory , Tata Institute of Fundamental Research, Bombay, Narosa Publishers, New Delhi, India, (1998), 59-73.

[Du] M. Duflo, Représentations de semi-groupes de mesures sur un groupe localement compact, Ann. Inst. Fourier 28 (1978), 225-249.

[E] P. Eisele, On shifted convolution powers of probability measure, Math. Z., 211 (1992), 557-574.

[F] J. Faraut, Analyse Harmonique sur les Paires de Guelfand et les Espaces Hyperboliques, Analyse Harmonique, les Cors du C.I.M.P.A., Nice, 315-446.

[Fe] G. M. Feldman, Arithematic of probabiity distributions and characterization problems on abelian groups, Translations of Mathematical Monographs, 116, American Mathematical Society, Providence RI, (1993).

[FH] J. Faraut, Kh. Harzalah, distances Hilbertiennes invariantes sur un espeace homogène, Ann. Inst. Fourier 24 (1974), 171-217.

[GV] R. Gangolli and V. S. Varadarajan, Harmonic Analysis on Spherical Functions on Real Reductive Groups, A Series of Modern Surveys in Mathematics, Springer-Verlag, Berlin-Heidelberg, 101 (1988).

[G1] P. Graczyk, Dispersions and a central limit theorem on symmetric spaces, Bull. Sci. Math. 118 (1994), 105-116.

[G2] P. Graczyk, Cramér theorem on symmetric spaces of noncompact type, J. Theoret. Probab. 7 (1994), 609-613.

[G3] P. Graczyk, Factorization theorem for probability measures on symmetric spaces of non-compact type, Journal of Theoretical Probability 12 (1999), 53-61.

[He] H. Heyer, *Probability Measures on Locally Compact Groups*, A Series of Modern Surveys in Mathematics, Springer-Verlag, Berlin-Heidelberg, 17 (1987).

[He1] H. Heyer, Convolution semigroups of probability measures on Gelfand pairs, Exposition. Math. 1 (1983), 3-45.

[He2] H. Heyer, Infinitely divisible probability measures on a discrete Gelfand pair, Nagoya Math. J. 116 (1989), 43-62.

[HHSWZ] W. Hazod, K. H. Hofmann, H.-P. Scheffler, M. Wüstner and H. Zeuner, Normalizers of compact subgroups, the existence of commuting automorphisms, and application to operator semistable measures, Journal of Lie Theory 8 (1998), 189-209.

[Ho] G. Hochschild, *The Structure of Lie Groups*, Holden-Day, Inc., London, (1965).

[L] J. Lamperti, The arithmetic of certain semi-groups of positive operators, Proceedings of Cambridge Philosophical Society 64 (1968), 161-166.

[Le] G. Letac, Problèmes classiques de probabilité sur un couple de Gelfand, Analytical methods in probability theory (Oberwolfach, 1980), Lecture Notes in Math. 861, Springer, Berlin-New York, (1981) pp. 93-120.

[Ma] J. Marcinkiewicz, Sur les variables alétoires enroulées, C. R. Séances et Conf. SMF 1938 (1939), 34-36.

[Mc] M. McCrudden, An introduction to the embedding problem for probabilities on locally compact groups, Positivity in Lie theory: open problems, de Gruyter Exp. Math. 26 (1998), 147-164.

[MV] K. Mokni and G. Van Dijk, Harmonic analysis on a class of generalized Gelfand pairs associated with hyperbolic spaces, Russian J. Math. Phys. 5 (1997), 167-178.

[P] K.R. Parthasarathy, *Probability Measures on Metric Spaces*, Academic Press, New York-London, (1967).

[R] C. R. E. Raja, On a class of Hungarian semigroups and the factorization theorem of Khinchin, Journal of Theoretical Probability 12 (1999), 71-79.

[RS] I. Z. Ruzsa and G. J. Szekely, *Algebraic Probability Theory*, Wiley Series in Probability and Mathematical Statistics: Probability and Mathematical Statistics, John Wiley and Sons, Ltd., Chichester, (1988).

[S1] R. Shah, Semistable Measures and Limit Theorems on Real and p-adic groups, Mon. Math. 115 (1993), 191-213.

[S2] R. Shah, The central limit problem on locally compact groups, to appear in Israel Journal of Mathematics.

[Te] K. Teloken, Limit Theorems for Probability Measures on Totally Disconnected Groups, Semigroup Forum Vol. 58 (1999), 69-84.

[Tr] I. R. Truhina, A problem related to the arithmetic of probability measures on the sphere, J. Soviet.Math. 17 (1981), 2321-2333.

[V] V. S. Varadarajan, Lie groups, Lie algebras, and their representations, Graduate Texts in Mathematics 102, Springer-Verlag, New York - Berlin, 1984.

[W] N. Wallach, Real Reducive groups, Academic Press Inc., New York - London, 1988.

[Ze] A. Zempléni, On the Heredity of Hun and Hungarian Property, Journal of Theoretical Probability 3 (1990), 599-610.

[Zi] R. J. Zimmer, Ergodic Theory and SemiSimple Lie Groups, Birkäuser, 1984.

P. Graczyk and C. Robinson Edward Raja
Universite d'Angers
Departement de Mathematiques
2, boulevard Lavoisier
49045 Angers Cedex 01.
France.