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Cone complementarity problems with finite solution sets

Madhur Malik*, S.R. Mohan

Indian Statistical Institute, New Delhi-110016, India

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Abstract

We introduce the notion of a complementary cone and a nondegenerate linear transformation and characterize the finiteness of the solution set of a linear complementarity problem over a closed convex cone in a finite dimensional real inner product space. In addition to the above, other geometrical properties of complementary cones have been explored.

Keywords: Linear complementarity problem; Face; Complementary cone; Nondegenerate linear transformation

1. Introduction

Let V be a finite dimensional real inner product space and K be a closed convex cone in V. Given a linear transformation $L : V \to V$ and $q \in V$ the *cone linear complementarity problem* or *linear complementarity problem over K*, denoted LCP(L, q), is to find an $x \in K$ such that $L(x)+q \in K^*$ and $\langle x, L(x)+q\rangle=0$, where $\langle ., .\rangle$ denotes the inner product on V and K^* is the dual of the cone K in V defined as

$$K^* := \{z \in V : \langle x, z\rangle \geqslant 0 \ \forall x \in K\}.$$

The cone LCP is a special case of a more general variational inequality problem [3]. Some important cone LCPs are the LCPs over R^n_+ [2], semidefinite linear complementarity problems (SDLCPs) over the cone of positive semidefinite matrices (S^n_+) in the space of real symmetric matrices (S^n) [4,5,9,11], LCPs over the Lorentz cone [11,14] or in general LCPs over the cone of squares (symmetric cone) in a Euclidean Jordan algebra [6,8]. Though a cone LCP might be considered as a generalization of the well known linear complementarity problem over R^n_+ [2], the solution properties of a LCP over R^n_+ do not carry over to a cone LCP as the cone K need not be isomorphic to R^n_+.

In this article we introduce and study the notion of a *complementary cone*, *nondegenerate* complementary cone and *nondegenerate linear transformation* in connection with the cone LCP, generalizing the notions of a complementary cone and a nondegenerate matrix (a real square matrix whose every principal minor is nonzero) studied in linear complementarity theory, see [2,10]. We study the closedness and the boundary structure of a complementary cone in a cone LCP. We show that closedness of all complementary cones is a necessary condition for the compactness of the solution set of a cone LCP(L, q) for all $q \in V$. Finally, we generalize the earlier results on the finiteness of the solution set of a LCP over specialized cones, see [10,5,8,14], to LCP over a closed convex cone in V.

The set SOL(L, q) denotes the solution set of the LCP(L, q). Orthogonal projection onto the subspace S is denoted by Proj_S and span E represents the linear span of a subset E of a linear space V. A nonempty subset F of a closed convex cone K in V is a *face*, denoted by $F \trianglelefteq K$, if F is a convex cone and

$$x \in K, \quad y - x \in K \quad \text{and} \quad y \in F \Rightarrow x \in F.$$

The *complementary face* of F is defined as

$$F^{\triangle} := \{y \in K^* : \langle x, y\rangle = 0 \ \forall x \in F\}.$$

The *smallest face* of K containing $x \in K$ is defined as the intersection of all the faces of K containing x. It is known that $F \trianglelefteq K$ is the smallest face of K containing $x \in K$ if and only if x lies in the relative interior (ri) of F, see [1]. It is easy to see that for any $x \in \mathrm{ri}\, F$, $F^{\triangle}$ can equivalently be represented as $F^{\triangle} := \{y \in K^* : \langle x, y\rangle = 0\}$. Also for any face F of K, $F \subseteq (F^{\triangle})^{\triangle}$.

Definition 1. A linear transformation $L : V \to V$ has the $\mathbf{R_0}$-property if LCP$(L, 0)$ over K has a unique (zero) solution.

Proposition 1. *L has the $\mathbf{R_0}$-property if and only if the set SOL(L, q) is compact (may be empty) for all $q \in V$.*

Proof. Note that SOL(L, q) is always closed. Let $\{x_n\} \subset$ SOL(L, q) be an unbounded sequence of nonzero terms. Consider the subsequence $\{x_m\}$ of $\{x_n\}$ such that $x_m/\|x_m\|$ converges to some $x \in K$. Then the sequence $L(x_m/\|x_m\|) + q/\|x_m\|$ converges to $L(x) \in K^*$ with $\langle x, L(x)\rangle = 0$, contradicting the $\mathbf{R_0}$-property. The converse is obvious. □

2. Complementary cones and nondegenerate linear transformations

The notion of a complementary cone has been introduced by Murty [10] in relation to a LCP over R^n_+. This notion is well studied in the literature on the LCP theory, see [2]. It has been found useful in studying the existence and multiplicity of solutions to LCP over R^n_+ and in studying a geometric interpretation of Lemke's complementary pivoting algorithm to solve the LCP [2]. The notion of a complementary cone has been extended to the semidefinite linear complementarity problems in [9]. It is further studied in the context of a LCP over a Lorentz cone in [14] and LCP over a symmetric cone in a Euclidean Jordan algebra [8].

Motivated by the above we present the following generalization of the concept of a complementary cone. Subsequently, we show how complementary cones explain the geometry and the solution properties of a cone LCP.

Definition 2. Given a linear transformation $L : V \to V$ a *complementary cone* of L corresponding to the face F of K is defined as

$$\mathscr{C}_F := \{y - L(x) : x \in F, y \in F^{\triangle}\}.$$

Remark 1. The faces of R^n_+ are $\{0\}$, R^n_+ and any set of the form

$$F := P\{(x_1, x_2, \ldots, x_k, 0, \ldots, 0)^{\mathrm{T}} : x_i \geqslant 0, \ 1 \leqslant i \leqslant k\},$$

where P is a permutation matrix and $k \in \{1, \ldots, n\}$. The complementary face of F is given by

$$F^{\triangle} = P\{(0, \ldots, 0, x_{k+1}, \ldots, x_n)^{\mathrm{T}} : x_i \geqslant 0, \ k+1 \leqslant i \leqslant n\}.$$

The complementary face of $\{0\}$ is R^n_+ and R^n_+ is $\{0\}$. Thus, in case of $K = R^n_+$, Definition 2 reduces to Murty's definition of a complementary cone, see [10].

Observation 1. *The linear complementarity problem LCP(L, q) has a solution if and only if there exists a face F of K such that $q \in \mathscr{C}_F$.*

Proof. Suppose $x \in K$ solves the LCP(L, q). Then $y := L(x) + q \in K^*$ and $\langle x, y\rangle = 0$. Let F be the smallest face of K containing x. Then $x \in \operatorname{ri} F$ and $y = L(x) + q \in F^{\triangle}$. Hence $q \in \mathscr{C}_F$. The converse is obvious. □

By the above observation, the union of all complementary cones is the set of all vectors q, for which the LCP(L, q) has a solution. The following example shows that complementary cones are not closed in general. However, it is easy to see that complementary cones are closed when K is a polyhedral cone.

Example 1. Let Λ^3_+, a Lorentz cone in R^3, be defined as $\Lambda^3_+ := \{(x_0, x_1, x_2)^{\mathrm{T}} \in R^3 : (x_1^2 + x_2^2)^{\frac{1}{2}} \leqslant x_0\}$. Let $M : R^3 \to R^3$ be a matrix defined as

$$M(x) = \begin{pmatrix} x_0 + x_1 \\ 0 \\ x_2 \end{pmatrix}.$$

Then $M \begin{pmatrix} \frac{1}{2}(\varepsilon + \frac{1}{\varepsilon}) \\ \frac{1}{2}(\varepsilon - \frac{1}{\varepsilon}) \\ -1 \end{pmatrix} \to \begin{pmatrix} 0 \\ 0 \\ -1 \end{pmatrix}$ as $\varepsilon \to 0$. However, there exists no $x \in \Lambda^3_+$ such that $M(x) = \begin{pmatrix} 0 \\ 0 \\ -1 \end{pmatrix}$. Thus the complementary cone of M corresponding to the face Λ^3_+ is not closed.

In our next proposition we give a sufficient condition for the closedness of a complementary cone of a given linear transformation L and corresponding to a given face F. For this we shall specialize and restate Theorem 9.1, [12], in the context of a closed convex cone.

Lemma 1. *Let K be a closed convex cone in R^n and $A : R^n \to R^m$ be a $m \times n$ real matrix. If $Az = 0$, $z \in K$ implies $z = 0$, then $A(K)$ is closed.*

Proposition 2. *Given a linear transformation $L : V \to V$ and $F \trianglelefteq K$, the complementary cone $\mathscr{C}_F$ is closed if*

$$x \in F, \ L(x) \in F^{\triangle} \text{ implies } x = 0.$$

Proof. By Lemma 1 and the condition described above, it is apparent that $L(F)$ is closed. Let $\widetilde{L} : V \times V \to V$ be defined as $\widetilde{L}(x, y) = x + y$. Let $\mathscr{C}_F = \{y - L(x) : x \in F, y \in F^{\triangle}\}$ be a complementary cone corresponding to the face F. Let $K_1 := \{y : y \in F^{\triangle}\}$ and $K_2 := \{-L(x) : x \in F\}$. Then $\widetilde{L}(K_1 \times K_2) = K_1 + K_2 = \mathscr{C}_F$. Now, $\widetilde{L}(y, -L(x)) = 0$ for some $x \in F$ and $y \in F^{\triangle}$ implies that $y - L(x) = 0 \Rightarrow L(x) \in F^{\triangle}$, which by the given condition gives $x = 0$. Thus we have $y = L(x) = 0$. Appealing to Lemma 1 again, we get $\mathscr{C}_F$ is closed. □

Definition 3. (a) A complementary cone $\mathscr{C}_F$ corresponding to the face F is called *nondegenerate* if

$$x \in \operatorname{span} F, \ L(x) \in \operatorname{span} F^{\triangle} \Rightarrow x = 0.$$

A complementary cone which is not nondegenerate is called *degenerate*.

(b) A linear transformation L is *nondegenerate* if $\mathscr{C}_F$ is nondegenerate for every $F \trianglelefteq K$.

Remark 2. (i) Note that L is $\mathbf{R_0}$ if and only if for every $F \trianglelefteq K$ the following relation holds:

$$x \in F, \ L(x) \in F^{\triangle} \Rightarrow x = 0.$$

Thus, by Proposition 1 and 2, closedness of all complementary cones is a necessary condition for the compactness of the solution set of an LCP(L, q) for all $q \in V$. Also, by Proposition 2, every nondegenerate complementary cone is closed.

(ii) For any $F \trianglelefteq K$, $\det L_{FF} \neq 0$ implies that $\mathscr{C}_F$ is nondegenerate, where $L_{FF} : \operatorname{span} F \to \operatorname{span} F$ is defined as $L_{FF}(x) = Proj_{\operatorname{span} F} L(x)$. Moreover, if $L(\operatorname{span} F) \subseteq \operatorname{span} F + \operatorname{span} F^{\triangle}\ \forall\ F \trianglelefteq K$, then L is nondegenerate if and only if $\det L_{FF} \neq 0\ \forall\ \{0\} \neq F \trianglelefteq K$. In particular, when $K = R^n_+$ and M is a real square matrix, $\det M_{FF}$ for $\{0\} \neq F \trianglelefteq R^n_+$ corresponds to one and only one principal minor of M. Hence, we obtain that a matrix M is nondegenerate if and only if all the principal minors of M are nonzero.

(iii) In the semidefinite setting, Gowda and Song [5] define a nondegenerate linear transformation $L : S^n \to S^n$ as follows:

$$X \in S^n, \ XL(X) = 0 \Rightarrow X = 0. \tag{1}$$

Equivalence of (1) with our Definition 3(b) is an easy consequence of Theorem 3.6 [7], on the characterization of faces of the positive semidefinite cone S^n_+.

Definition 3(b) of a nondegenerate linear transformation is motivated by the uniqueness of a solution to a LCP on a given face and has been explained in the following proposition.

Proposition 3. *Given a linear transformation $L: V \to V$ and $F \trianglelefteq K$, $\mathscr{C}_F$ is a nondegenerate complementary cone if and only if for each $q \in \mathscr{C}_F$ there exist a unique $x \in F$ and $y \in F^{\triangle}$ such that $q = y - L(x)$.*

Proof. Suppose that there exist $x_1,\ x_2 \in F$ and $y_1,\ y_2 \in F^{\triangle}$ such that $q = y_1 - L(x_1) = y_2 - L(x_2)$, which implies that $y_1 - y_2 = L(x_1 - x_2)$, where $x_1 - x_2 \in \operatorname{span} F$ and $y_1 - y_2 \in \operatorname{span} F^{\triangle}$. By the nondegeneracy of $\mathscr{C}_F$, $x_1 = x_2$ and $y_1 = y_2$. Conversely, suppose that there exists an $x \in \operatorname{span} F$ such that $L(x) \in \operatorname{span} F^{\triangle}$. Writing $x = x_1 - x_2$ with $x_1,\ x_2 \in F$ and $L(x) = y_1 - y_2$ with $y_1,\ y_2 \in F^{\triangle}$ we get

$$\overline{q} := y_1 - L(x_1) = y_2 - L(x_2).$$

Since each $q \in \mathscr{C}_F$ has a unique representation in $\mathscr{C}_F$, we get $x_1 = x_2$ and hence $x = 0$. □

Corollary 1. *Given $L: V \to V$ and $q \in V$, LCP(L, q) has infinitely many solutions only if either q is contained in a degenerate complementary cone or q lies in infinitely many complementary cones.*

Proof. Suppose q does not belong to a degenerate complementary cone and lies only in a finite number of nondegenerate complementary cones. Then LCP(L, q) can have only finitely many solutions, contradicting our hypothesis. □

Proposition 4. *Let $\mathscr{C}_F$ be a nondegenerate complementary cone. Then*

(i) *$q \in \operatorname{ri} \mathscr{C}_F$ if and only if there exist $x \in \operatorname{ri} F$ and $y \in \operatorname{ri} F^{\triangle}$ such that $q = y - L(x)$.*
(ii) *Any face $\mathscr{G}$ of $\mathscr{C}_F$ can be represented as*

$$\mathscr{G} = \{y - L(x) : x \in H,\ y \in H'\},$$

where $H \trianglelefteq F$ and $H' \trianglelefteq F^{\triangle}$. Also, any set of the above form is a face of $\mathscr{C}_F$.

Proof. The proof of (i) is easy and is left to the reader. For the proof of (ii) let $\mathscr{G}$ be a face of $\mathscr{C}_F$ for some face F. Then $\mathscr{G}$ can be represented as $\mathscr{G} = \{y - L(x) : x \in H,\ y \in H'\}$, where $H \subseteq F$ and $H' \subseteq F^{\triangle}$. We shall show that $H \trianglelefteq F$ and $H' \trianglelefteq F^{\triangle}$. Since $0 \in \mathscr{G} \trianglelefteq \mathscr{C}_F$ and $\mathscr{C}_F$ is nondegenerate, $0 \in H \cap H'$, and H and H' are convex cones. Let $x \in F$, $z - x \in F$ and $z \in H$. Then $-L(x) \in \mathscr{C}_F$, $-L(z - x) \in \mathscr{C}_F$ and $-L(z) \in \mathscr{G}$. Since $\mathscr{G} \trianglelefteq \mathscr{C}_F$ we get $-L(x) \in \mathscr{G}$, which by the nondegeneracy of $\mathscr{C}_F$ gives $x \in H$. Similarly, we can show that H' is a face of $F^{\triangle}$. Conversely, let $\mathscr{N}$ be defined as $\mathscr{N} := \{y - L(x) : x \in H,\ y \in H'\}$, where $H \trianglelefteq F$ and $H' \trianglelefteq F^{\triangle}$. Then $\mathscr{N}$ is a nonempty convex cone. Let $y - L(x) \in \mathscr{C}_F$, $(y_0 - y) - L(x_0 - x) \in \mathscr{C}_F$, and $y_0 - L(x_0) \in \mathscr{N}$, where $x_0 \in H$, $x \in F$, $y_0 \in H'$ and $y \in F^{\triangle}$. Since $\mathscr{C}_F$ is nondegenerate, $x_0 - x \in F$ and $y_0 - y \in F^{\triangle}$. Thus,

$$x \in F,\ x_0 - x \in F, \text{ and } x_0 \in H,$$

$$y \in F^{\triangle},\ y_0 - y \in F^{\triangle}, \text{ and } y_0 \in H'.$$

Since $H \trianglelefteq F$ and $H' \trianglelefteq F^{\triangle}$, we get $x \in H$ and $y \in H'$. Hence $y - L(x) \in \mathscr{N}$ and $\mathscr{N}$ is a face of $\mathscr{C}_F$. □

Remark 3. In a private communication [13], Dr. Richard E. Stone has pointed out that any face $\mathscr{G}$ of $\mathscr{C}_F$ can be represented as $\mathscr{G} = \{y - L(x) : x \in H,\ y \in H'\}$, where $H \trianglelefteq F$ and $H' \trianglelefteq F^{\triangle}$, without assuming that $\mathscr{C}_F$ is nondegenerate.

Corollary 2. *Given a linear transformation $L: V \to V$ and $q \in V$, the LCP(L, q) has infinitely many solutions if q lies in the relative interior of infinitely many nondegenerate complementary cones.*

Proof. Let $q \in \cap \operatorname{ri} \mathscr{C}_{F_\alpha}$, where F_α is a family of distinct faces of K indexed by α and $\mathscr{C}_{F_\alpha}$ is nondegenerate for each α. Then $q = y_\alpha - L(x_\alpha)$ for $x_\alpha \in \operatorname{ri} F_\alpha$ and $y_\alpha \in \operatorname{ri} F_\alpha^{\triangle}$. Since each $\mathscr{C}_{F_\alpha}$ is a nondegenerate complementary cone, x_α, for every α, are infinitely many distinct solutions to LCP(L, q). □

3. Finiteness of the solution set of a cone LCP

In the context of a LCP over R^n_+, nondegenerate matrices characterize the finiteness of the solution set of a LCP(M, q) for all $q \in R^n$, see [10]. A similar study is made by Gowda and Song [5] where they introduce and study the notion of a nondegenerate linear transformation in the context of a SDLCP. They have shown that when $K = S^n_+$, nondegeneracy of a linear transformation L need not be a sufficient condition for the finiteness of the solution set of SDLCP(L, Q) for all $Q \in S^n_+$. The example below throws more light on the preceding discussion.

Example 2. Let $M : R^3 \to R^3$ be defined as $M(x) = -x$, $K = \{(x_0, x_1, x_2)^{\mathrm{T}} : x_0 \geqslant 0,\ \frac{x_0^2}{4} \geqslant x_1^2 + x_2^2\}$ and $q := (\frac{5}{8}, 0, 0)^{\mathrm{T}}$. It is easy to check that M is nondegenerate, K is closed and convex (but not self-dual) and any point $x = (x_0, x_1, x_2)^{\mathrm{T}}$ lies on the boundary of K if and only if $x_0 \geqslant 0$ and $\frac{x_0^2}{4} = x_1^2 + x_2^2$. Any complementary cone corresponding to a face F is of the form $\mathscr{C}_F = \{y + x : x \in F,\ y \in F^{\triangle}\}$. Except two 3-dimensional complementary cones, namely K^* and K, every other complementary cone is of dimension 2. The infinite set of solutions to LCP(L, q) is given by $\{(\frac{1}{2}, \frac{x_1}{2}, \frac{x_2}{2})^{\mathrm{T}} : x_1^2 + x_2^2 = \frac{1}{4}\}$.

Definition 4. (a) A solution x_0 of LCP(L, q) is locally unique if it is the only solution in a neighborhood of x_0.

(b) A solution x_0 is *locally-star-like* if there exists a sphere $\mathscr{S}(x_0, r)$ such that

$$x \in \mathscr{S}(x_0, r) \cap \mathrm{SOL}(L, q) \Rightarrow [x_0, x] \subseteq \mathrm{SOL}(L, q).$$

The following theorem generalizes the earlier results on the finiteness of the solution set of a LCP over specialized cones, see [10,5,8,14], to LCP over a closed convex cone in V.

Theorem 1. *Given a linear transformation $L : V \to V$, the following statements are equivalent.*

(i) *SOL(L, q) is finite for all $q \in V$.*
(ii) *Every solution of LCP(L, q) over K is locally unique for all $q \in V$.*
(iii) *L is nondegenerate, and for all $q \in V$, each solution of LCP(L, q) is locally-star-like.*

Proof. The assertion (i) $\Rightarrow$ (ii) is obvious. For the reverse implication, note that (ii) implies that L has the $\mathbf{R_0}$- property. Thus SOL(L, q) is compact for all q and hence (in view of (ii)) is finite for all q.

(ii) $\Rightarrow$ (iii): First we shall show that L is nondegenerate. Let $x \in V$ be nonzero such that $x \in \operatorname{span} F$, $L(x) \in \operatorname{span} F^{\triangle}$ for some face F of K. Since $x \in \operatorname{span} F$, we can write $x = x_1 - x_2$ with $x_1,\ x_2 \in F$. Similarly, $L(x) = L(x)_1 - L(x)_2$ with $L(x)_1,\ L(x)_2 \in F^{\triangle}$. Defining $q := L(x)_1 - L(x_1) = L(x)_2 - L(x_2)$ it is observed that LCP(L, q) has two distinct solutions x_1 and x_2 with

$$\langle t x_1 + (1 - t)x_2, t L(x)_1 + (1 - t)L(x)_2\rangle = 0 \quad \forall\, t \in [0, 1],$$

i.e., $[x_1, x_2] \subseteq \mathrm{SOL}(L, q)$ which contradicts (ii).

Also, for any $q \in V$, since the solution $x_0 \in \mathrm{SOL}(L, q)$ is locally unique, it is locally-star-like. (iii) $\Rightarrow$ (ii): Let for some fixed $q \in V$, the solution x_0 of LCP(L, q) be not locally unique. Then there exist a sequence $\{x_k\} \subseteq \mathrm{SOL}(L, q)$ converging to x_0 with $x_k \neq x_0$ for all k. By the locally-star-like property we have $[x_0, x_k] \subseteq \mathrm{SOL}(L, q)$ for all large k. Let F_i be the smallest face of K containing x_i ($x_i \in \operatorname{ri} F_i$) where $i = 0,\ 1,\ 2,\ \ldots$ From the complementarity of solutions we have for all large k

$$x_0 \in \operatorname{ri} F_0 \quad \text{and} \quad L(x_0) + q \in F_0^{\triangle},$$

$$x_k \in \operatorname{ri} F_k \quad \text{and} \quad L(x_k) + q \in F_k^{\triangle}.$$

Also from the fact that $[x_0, x_k] \subseteq \mathrm{SOL}(L, q)$ for large k we get

$$\langle x_0, L(x_k) + q\rangle = 0 \quad \text{and} \quad \langle x_k, L(x_0) + q\rangle = 0.$$

Since $x_0 \in \operatorname{ri} F_0$ and $x_k \in \operatorname{ri} F_k$ we get $L(x_k)+q \in F_0^{\triangle}$ and $L(x_0) + q \in F_k^{\triangle}$. Defining a face $G := F_0^{\triangle} \cap F_k^{\triangle}$ of K^* we get $x_0,\ x_k \in G^{\triangle}$ and $L(x_0) + q,\ L(x_k) + q \in G$. Thus there exists a face $F = G^{\triangle}$ of K such that a nonzero $x := x_0 - x_k \in \operatorname{span} F$ with $L(x) \in \operatorname{span} F^{\triangle}$, which contradicts our assumption that L is nondegenerate. □

Corollary 3. *When K is polyhedral LCP(L, q) has a finite number of solutions for all $q \in V$ if and only if $\det L_{FF} \neq 0$ for all nonzero $F \trianglelefteq K$, or equivalently L is nondegenerate.*

In our next proposition we extend the result, recently observed for a LCP over the Lorentz cone by Tao [14], to any closed convex cone in V.

Definition 5. A linear transformation $L: V \to V$ is said to be monotone (copositive on K) if $\langle x, L(x)\rangle \geqslant 0$ $\forall x \in V$ $(x \in K)$.

Proposition 5. *If L is a monotone linear transformation on V, then L is nondegenerate if and only if LCP(L, q) has a unique solution for all $q \in V$.*

Proof. Suppose L is nondegenerate. Then by Theorem 2.5.10 in [3], LCP(L, q) has a solution for all $q \in V$. Let x_1 and x_2 with $x_1 \neq x_2$ be the two solutions of LCP(L, q) for some $q \in V$. Let $x_1 \in \operatorname{ri} F_1$ and $x_2 \in \operatorname{ri} F_2$ where $F_1,\ F_2$ are the two faces of K. By the monotonicity of L we have

$$0 \leqslant \langle x_1 - x_2, L(x_1 - x_2)\rangle \\ = \langle x_1 - x_2, y_1 - y_2\rangle = -\langle x_1, y_2\rangle - \langle x_2, y_1\rangle \leqslant 0,$$

where $y_i = L(x_i) + q$ for $i = \{1, 2\}$. Thus $\langle x_1, y_2\rangle = 0$ and $\langle x_2, y_1\rangle = 0$. Since $x_1 \in \operatorname{ri} F_1$ and $x_2 \in \operatorname{ri} F_2$, $y_1 \in F_2^{\triangle}$ and $y_2 \in F_1^{\triangle}$. Defining a face $G := F_1^{\triangle} \cap F_2^{\triangle}$ of K^* we get $x_1, x_2 \in G^{\triangle}$ and $L(x_1)+q, L(x_2)+q \in G$. Thus for a face $F = G^{\triangle}$ of K we have a nonzero $x := x_1 - x_2 \in \operatorname{span} F$, such that $L(x) \in \operatorname{span} F^{\triangle}$, which contradicts that L is nondegenerate. The converse is obvious. □

Proposition 6. *Let L be copositive on K. Then L is nondegenerate only if LCP(L, q) has a unique solution for all $q \in K^*$.*

The proof is similar to that of Proposition 5 above and is omitted.

An open problem

We have shown that if LCP(L, q) has a compact solution set for all $q \in V$ then all the complementary cones are closed. However, we do not know whether closedness of all complementary cones is a necessary condition for SOL(L, q) to be nonempty for all $q \in V$.

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