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ROBUSTNESS OF BLOCK DESIGNS AGAINST MISSING DATA

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Abstract: Robustness of incomplete block designs against missing data is investigated. Necessary and sufficient conditions for robustness of an arbitrary incomplete block design are derived under two patterns of missing observations. Simple sufficient conditions are also provided. Some classes of designs that are robust are identified. The efficiency of the residual design is evaluated for certain group-divisible designs when all observations in a block are lost. Finally, a lower bound to the efficiency of the residual design is obtained when a single observation is missing in an arbitrary incomplete block design.

Key words and phrases: Criteria for robustness, conditions for robustness, robust designs, efficiency.

1. Introduction

When one or more observations become nonavailable in a designed experiment, it is of interest to examine the loss of information, defined suitably, that is incurred due to missing data. Designs for which this loss is "small" may be termed *robust.* The robustness of incomplete block designs against missing data has been investigated in the literature from different angles; see for example, Hedayat and John (1974), John (1976), Ghosh (1982), Ghosh, Rao and Singhi (1983), Baksalary and Tabis (1987), Dey and Dhall (1988), Srivastava, Gupta and Dey (1990) and Mukerjee and Kageyama (1990). For an excellent review of the subject up to 1988, refer to Kageyama (1990).

A criterion of robustness of designs (in particular, incomplete block designs), was introduced by Ghosh (1982). According to this criterion (to be called Criterion 1) an incomplete block design is robust against the loss of t (≥ 1) observations if the *residual* design obtained by deleting these t observations remains connected. It was shown by Ghosh (1982) that Balanced Incomplete Block (BIB) designs are robust according to Criterion 1 against the loss of *any* $r-1$ observations, where r is the common replication of the original BIB design. Similar results on certain Partially Balanced Incomplete Block (PBIB) designs were obtained by Ghosh et al. (1983). See also Baksalary and Tabis (1987), who presented sufficient conditions for arbitrary block designs to be robust under Criterion 1.

Another criterion of robustness (to be called Criterion 2) that has received attention, is in terms of the efficiency of the residual design. As per Criterion 2, a design is said to be robust if the efficiency of the residual design relative to the original one is not too small. The papers by John (1976), Dey and Dhall (1988), Whittinghill (1989), Srivastava et al. (1990), Mukerjee and Kageyama (1990) and Ghosh, Kageyama and Mukerjee (1991) are in this spirit.

Perusal of the existing literature reveals that most studies (the work of Baksalary and Tabis (1987) appears to be the only exception) on robustness of block designs, using either Criteria 1 or 2 have been restricted to specific classes of designs with a specific pattern of missing observations. The purpose of this communication is to present some results on robustness of *arbitrary* incomplete block designs with equal block sizes. In Section 2, we obtain necessary and sufficient conditions for robustness as per Criterion 1 when the missing observations appear in the following patterns: (i) $t \geq 1$ observations pertaining to the same treatment are missing, and (ii) all observations in a block are missing. Simple *sufficient* conditions for robustness according to Criterion 1 are obtained in each of the above cases. Some designs that are robust are identified. In Section 3, the efficiency of the residual design, when all observations in a block are lost, is evaluated for group-divisible designs satisfying $k = n$, where k is the block size and n, the number of treatments in a group. In particular, the exact efficiency of the residual design is evaluated when all observations in a block of a Regular group-divisible design satisfying $k = n$ and $\lambda_1 > 0$ are lost.

The efficiency of the residual design when a single observation is missing has been evaluated by Whittinghill (1989) for a balanced block design and by Ghosh et al. (1991) for BIB and some group-divisible designs. In Section 4, we present a lower bound to the efficiency of the residual design when a single observation is missing in an arbitrary incomplete block design. Using this bound, it is found that most PBIB designs with two associate classes are robust against the loss of a single observation as per Criterion 2.

2. Conditions for Robustness

2.1. Two basic results

To begin with, we introduce some notation to be followed throughout. All matrices and vectors are real, vectors being written as column vectors. We denote an n-component vector of all unities by $\mathbf{1}_n$ and by I_n, an identity matrix of order n. For a matrix A, A', $\mathcal{M}(A)$, A^- and A^+ will respectively denote the transpose, column span (range), a generalized inverse (g-inverse) and the Moore-Penrose inverse of A.

We now have the following results.

Theorem 1. *Let A and B be a pair of symmetric, nonnegative definite matrices of order n and let*

$$A = B + GG' \tag{1}$$

where G is an $n \times m$ matrix such that $\mathcal{M}(G) \subset \mathcal{M}(A)$. Then

$$\text{Rank}(A) = \text{Rank}(B) \tag{2}$$

if and only if

$$I_m - G'A^-G \quad \textit{is positive definite.} \tag{3}$$

Proof. Let

$$D = \begin{bmatrix} I_m & G' \\ G & A \end{bmatrix},\ E = \begin{bmatrix} I_m & 0 \\ -G & I_n \end{bmatrix},\ F = \begin{bmatrix} I_m & -G'A^- \\ 0 & I_n \end{bmatrix}. \tag{4}$$

Clearly, E and F are nonsingular. Now,

$$EDE' = \begin{bmatrix} I_m & 0 \\ 0 & A - GG' \end{bmatrix} = \begin{bmatrix} I_m & 0 \\ 0 & B \end{bmatrix}$$

is nonnegative definite, which implies that D is nonnegative definite.

Also, since $\mathcal{M}(G) \subset \mathcal{M}(A)$, $G = AH$ for some matrix H. Hence

$$G'A^-A = H'AA^-A = H'A = G'. \tag{5}$$

Similarly,

$$AA^-G = AA^-AH = AH = G. \tag{6}$$

Therefore,

$$FDF' = \begin{bmatrix} I_m - G'A^-G & 0 \\ 0 & A \end{bmatrix}$$

is nonnegative definite, because D is so and F is nonsingular. This implies that $I_m - G'A^-G$ is nonnegative definite. Finally,

$$\begin{aligned} \text{Rank}(D) &= \text{Rank}(EDE') = \text{Rank}(B) + m & (7)\\ &= \text{Rank}(FDF') = \text{Rank}(A) + \text{Rank}(I_m - G'A^-G). & (8) \end{aligned}$$

The result then follows from (7) and (8) and remembering that $(I_m - G'A^-G)$ is nonnegative definite.

Remark 1. Observe that since $G'A^-G = H'AH$, $G'A^-G$ is invariant with respect to any choice of a g-inverse of A.

Remark 2. The result of Theorem 1 for the special case $m = 1$ was proved by Ghosh et al. (1991).

The following result is known (see, e.g., Pringle and Rayner (1971, p.32)).

Theorem 2. *Let A, B, G be as in Theorem 1 and suppose that $I_m - G'A^-G = I_m - G'A^+G$ is positive definite. Then*

$$B^+ = A^+ + A^+G(I_m - G'A^+G)^{-1}G'A^+. \tag{9}$$

2.2. Conditions for robustness when t (≥ 1) observations pertaining to the same treatment are lost

Consider a connected, binary block design d_0 with v treatments, b blocks and constant block size k. Let t (≥ 1) of the bk observations be missing and let all these t observations pertain to the same treatment. Without loss of generality, we may assume that these t observations pertain to the first treatment in the first t blocks. We further assume that these t "affected" blocks are *not all* identical. Let the residual design, obtained by deleting these t observations from d_0 be called d_t. If N_0 (respectively N_t) is the incidence matrix of d_0 (respectively d_t), then

$$N_0 = \begin{bmatrix} \mathbf{1}_t' & \mathbf{e}' \\ F & M \end{bmatrix}, \quad N_t = \begin{bmatrix} \mathbf{0}' & \mathbf{e}' \\ F & M \end{bmatrix}$$

where $\mathbf{e}$ is a $(b-t)$-component $(0,1)$ vector and F and M are $(0,1)$-matrices of orders $(v-1) \times t$ and $(v-1) \times (b-t)$ respectively. Denote by $C_0(C_t)$, the usual C-matrix of $d_0(d_t)$. Then, it can be shown, after some routine algebra, that

$$C_0 = C_t + UU' \tag{10}$$

where U is a $v \times t$ matrix, given by

$$U = \{k(k-1)\}^{-1/2} \begin{bmatrix} (k-1)\mathbf{1}_t' \\ -F \end{bmatrix}. \tag{11}$$

Clearly,

$$\mathbf{1}_v'U = \mathbf{0}'. \tag{12}$$

Also, since $\text{Rank}(C_0) = v - 1$ (as d_0 is assumed to be connected) and $\mathbf{1}_v'C_0 = \mathbf{0}'$, it follows that $\mathcal{M}(U) \subset \mathcal{M}(C_0)$. Thus, using Theorem 1, we arrive at

Theorem 3. *The design d_0 is robust against the loss of any t (≥ 1) observations pertaining to the same treatment according to Criterion 1 if and only if $I_t - U'C_0^- U$ is positive definite.*

Corollary 1. *The design d_0 is robust against the loss of any single observation according to Criterion 1 if and only if*

$$\mathbf{u}'C_0^-\mathbf{u} < 1 \tag{13}$$

where

$$\mathbf{u}' = \{k(k-1)\}^{-1/2}(k-1, -\mathbf{f}') \tag{14}$$

and $\mathbf{f}$ *is a* $(0,1)$ *vector representing the incidence of the* $(v-1)$ *"unaffected" treatments in the first block containing the missing observation.*

The result of Corollary 1 has been obtained by Ghosh et al. (1991) in terms of the Moore-Penrose inverse C_0^+, using a different approach.

The necessary and sufficient condition for robustness given in Theorem 3 is not very convenient in the sense that its verification depends on the structure (incidence) of the unaffected treatments in the t affected blocks through the matrix U. A simpler sufficient condition in terms of the smallest positive eigenvalue of C_0 is given in

Theorem 4. *The design d_0 is robust as per Criterion 1 against the loss of t (> 1) observations pertaining to the same treatment if t does not exceed the smallest positive eigenvalue of C_0.*

Proof. From Theorem 3, we know that d_0 is robust against the loss of t (> 1) observations pertaining to the same treatment if and only if $I_t - U'C_0^- U$ is positive definite, or, equivalently, if and only if all the eigenvalues of $U'C_0^- U$ are strictly smaller than unity. Let $\lambda_{\max}(A)$ denote the largest eigenvalue of a symmetric nonnegative definite matrix A. Then,

$$\lambda_{\max}(U'C_0^- U) = \lambda_{\max}(U'C_0^+ U) = \lambda_{\max}(C_0^+ UU'). \tag{15}$$

Also, it is known (cf. Marshall and Olkin (1979, p.247)) that for a pair of symmetric, nonnegative definite matrices A and B,

$$\lambda_{\max}(AB) \leq \lambda_{\max}(A)\lambda_{\max}(B). \tag{16}$$

Hence, from (15) and (16), we have

$$\begin{aligned} \lambda_{\max}(U'C_0^+ U) &\leq \lambda_{\max}(C_0^+)\lambda_{\max}(UU') \\ &= \lambda_{\max}(C_0^+)\lambda_{\max}(U'U) \\ &< \lambda_{\max}(C_0^+) tr(U'U), \end{aligned} \tag{17}$$

where $tr(\cdot)$ stands for the trace of a square matrix. Note that $tr(U'U) = \lambda_{\max}(U'U)$ if and only if $U'U$, and hence U, is of rank unity. In such a case, it is easily seen that all the t "affected" blocks of d_0 must be identical. Since we have already excluded such designs from our discussion, we have strict inequality in (17). Now,

$$U'U = \{k(k-1)\}^{-1}\Big[(k-1)^2 J_t + F'F\Big] \tag{18}$$

where J_t is a square matrix of order t with all elements unity. Hence,

$$tr(U'U) = \{k(k-1)\}^{-1}\Big[t(k-1)^2 + t(k-1)\Big] = t. \tag{19}$$

Using (19) in (17) and remembering that $\lambda_{\max}(C_0^+) = \{\lambda_1(C_0)\}^{-1}$, where $\lambda_1(C_0)$ is the smallest positive eigenvalue of C_0, we get the required result.

Corollary 2. *The design d_0 is robust against the loss of a single observation, according to Criterion 1, if the smallest positive eigenvalue of C_0 is strictly larger than unity.*

Proof. When $t = 1$, $\mathbf{u}\mathbf{u}'$ is of rank unity, where $\mathbf{u}$ is given by (14). Hence

$$\lambda_{\max}(\mathbf{u}\mathbf{u}') = tr(\mathbf{u}\mathbf{u}') = tr(\mathbf{u}'\mathbf{u}) = 1 \quad \text{and}$$

$$\lambda_{\max}(\mathbf{u}'C_0^+\mathbf{u}) \le \lambda_{\max}(C_0^+)\lambda_{\max}(\mathbf{u}\mathbf{u}') = \lambda_{\max}(C_0^+) = \{\lambda_1(C_0)\}^{-1}.$$

Suppose d_0 is a connected, equireplicate, binary, block design with common replication number r and constant block size k. Clearly, if the residual design d_t is to be connected, the number of missing observations pertaining to a treatment cannot exceed $r-1$. If $t_{\max}$ is the maximum value of t such that d_t is connected, then from Theorem 4, we have

$$t_{\max} \ge \Big[\lambda_1(C_0)\Big]$$

where $[\cdot]$ is the greatest integer function. We now derive a sufficient condition under which $t_{\max} = r-1$. Let N_0 be the incidence matrix of d_0. If $\phi_1 = rk > \phi_2 \ge \cdots \ge \phi_v$ are the eigenvalues of N_0N_0' then $\lambda_1(C_0) = r - \phi_2/k$. Hence $[\lambda_1(C_0)] = r-1$ if and only if

$$0 < \phi_2 \le k. \tag{20}$$

The sufficient condition (20) can be used to identify designs for which $t_{\max} = r-1$. However, since (20) is only a sufficient condition, designs other than those satisfying (20) may exist for which $t_{\max} = r-1$. Note that the sufficient condition (20) is applicable to *any* equireplicate incomplete block design with constant block size, k.

2.3. Conditions for robustness when all observations in a block are missing

Suppose d_0 is a connected, binary block design with v treatments, b blocks and block size k, and suppose that for some reason, all the observations in a block are missing. Without loss of generality, let the missing observations pertain to the first k treatments in the first block. If $C_0(C_k)$ denotes the C-matrix of d_0 (residual design d_k), then it can be shown that

$$C_0 = C_k + VV' \tag{21}$$

where V is a $v \times k$ matrix given by

$$V' = [I_k - k^{-1}J_k, 0]. \tag{22}$$

It is easily seen that $\mathbf{1}_v'V = \mathbf{0}'$ and hence $\mathcal{M}(V) \subset \mathcal{M}(C_0)$. Thus, using Theorem 1, we have

Theorem 5. *The design d_0 is robust as per Criterion 1 against the loss of all observations in a block if and only if $I_k - V'C_0^- V$ is positive definite.*

Note that

$$VV' = \begin{bmatrix} I_k - k^{-1}J_k & 0 \\ 0 & 0 \end{bmatrix}$$

which is a symmetric, idempotent matrix of rank $(k-1)$, and thus, $\lambda_{\max}(VV') = 1$. Hence, proceeding as in the proof of Theorem 4, we arrive at the following sufficient condition.

Theorem 6. *The design d_0 is robust as per Criterion 1 against the loss of all observations in a block if the smallest positive eigenvalue of C_0 is strictly larger than unity.*

Corollary 3. *The following designs satisfy the sufficient condition of Theorem 6 and are thus robust against the loss of all observations in a block:*

(i) *All* BIB *designs.*

(ii) *All group-divisible designs with the exception of the design with parameters* $v = 4 = b$, $r = 2 = k$, $m = 2 = n$, $\lambda_1 = 0$, $\lambda_2 = 1$.

(iii) *All triangular designs with the exception of the design with parameters* $v = 10$, $b = 15$, $r = 3$, $k = 2$, $\lambda_1 = 0$, $\lambda_2 = 1$.

(iv) *All Latin-square type (L_i type, $i \geq 2$)* PBIB *designs with the exception of L_2 designs with parameters* $v = s^2$, $r = 2$, $k = s$, $b = 2s$, $\lambda_1 = 1$, $\lambda_2 = 0$.

(v) *All* PBIB *designs based on Partial geometries with more than two replicates.*

Remark 3. It should be noted, however, that since Theorem 6 provides only a sufficient condition for robustness, designs excluded in Corollary 3 could well be robust against the loss of all observations in a block.

3. Efficiency of the Residual Design When all Observations in a Block Are Lost

Criterion 1 of robustness is in terms of the connectedness of the residual design. However, even if a design is robust according to Criterion 1, the residual design may have poor efficiency relative to the original design. It is, therefore, of interest to examine the efficiency of the residual design and decide robustness on the basis of Criterion 2. If d_0 is a binary block design with constant block size and d_t is the residual design, we take as a measure of efficiency of the residual design, the quantity E given by

$$\begin{aligned} E &= \frac{\text{sum of reciprocals of non zero eigenvalues of } C_0}{\text{sum of reciprocals of non zero eigenvalues of } C_t} \\ &= \frac{tr(C_0^+)}{tr(C_t^+)} \end{aligned} \qquad (23)$$

where $C_0(C_t)$ is the C-matrix of $d_0(d_t)$.

In this section, we evaluate the efficiency of the residual design when all observations in a block are missing. Srivastava et al. (1990) and Mukerjee and Kageyama (1990) evaluated this efficiency for BIB designs and Singular, Semi-regular and Regular Group Divisible (GD) designs with $\lambda_1 = 0$. For Regular GD designs with $\lambda_1 > 0$, Mukerjee and Kageyama (1990) give lower and upper bounds for the efficiency. In this section, we evaluate the exact efficiency in the case of GD designs with $k = n$, when all observations in a block are lost. For a definition of GD designs see, e.g., Dey (1986).

Let (without loss of generality) the missing observations pertain to the treatments $1, 2, \ldots, k$ in the first block of d_0, where d_0 is a GD design with parameters $v, b, r, k, m, n, \lambda_1, \lambda_2$ satisfying $k = n$. Theorem 6 tells us that such a design is robust as per Criterion 1 against the loss of all observations in a block. If d_k is the residual design and $C_0(C_k)$ is the C-matrix of $d_0(d_k)$, then from Theorem 2, we have

$$\begin{aligned} tr(C_k^+) &= tr(C_0^+) + tr[C_0^+ V(I - V'C_0^+V)^{-1}V'C_0^+] \\ &= tr(C_0^+) + tr[(I - V'C_0^+V)^{-1}V'C_0^+C_0^+V], \end{aligned}$$

where V is as in (22).

For a GD design d_0, it is well known that

$$C_0 = \phi_1 L_1 + \phi_2 L_2 \qquad (24)$$

where $\phi_1 = \{r(k-1)+\lambda_1\}/k$, $\phi_2 = v\lambda_2/k$, $L_1 = I_m \otimes (I_n - n^{-1}J_n)$, $L_2 = (I_m - m^{-1}J_m) \otimes n^{-1}J_n$ and $\otimes$ denotes the Kronecker product of matrices.

Observe that $L_i^2 = L_i$ and $L_iL_j = 0$ for $i \neq j$, $i,j = 1,2$. Hence

$$C_0^+ = \phi_1^{-1}L_1 + \phi_2^{-1}L_2 \tag{25}$$

and

$$C_0^+C_0^+ = \phi_1^{-2}L_1 + \phi_2^{-2}L_2. \tag{26}$$

Using (25), we have, since $k = n$,

$$V'C_0^+V = \phi_1^{-1}(I_n - n^{-1}J_n) \tag{27}$$

and

$$V'C_0^+C_0^+V = \phi_1^{-2}(I_n - n^{-1}J_n). \tag{28}$$

Also,

$$(I_n - V'C_0^+V)^{-1} = (1-\phi_1^{-1})^{-1}(I_n - n^{-1}\phi_1^{-1}J_n). \tag{29}$$

Thus,

$$\begin{aligned}(I_n - V'C_0^+V)^{-1}V'C_0^+C_0^+V &= \phi_1^{-2}(1-\phi_1^{-1})^{-1}(I_n - n^{-1}J_n)\\ &= \{\phi_1(\phi_1 - 1)\}^{-1}(I_n - n^{-1}J_n).\end{aligned} \tag{30}$$

Further, since the positive eigenvalues of C_0 are ϕ_1 and ϕ_2 with respective multiplicities $(v-m)$ and $(m-1)$,

$$\begin{aligned}tr(C_0^+) &= (v-m)\phi_1^{-1} + (m-1)\phi_2^{-1}\\ &= \{(v-m)\phi_2 + (m-1)\phi_1\}/\phi_1\phi_2.\end{aligned} \tag{31}$$

Hence, using (30) and (31) we get, after simplifying,

$$\begin{aligned}E &= \frac{tr(C_0^+)}{tr(C_k^+)}\\ &= 1 - \frac{(n-1)\phi_2}{(n-1)\phi_2 + (\phi_1 - 1)\{(v-m)\phi_2 + (m-1)\phi_1\}}.\end{aligned} \tag{32}$$

As stated earlier, Mukerjee and Kageyama (1990) evaluated the exact efficiency of the residual design in the case of Singular, Semi-regular and Regular GD designs with $\lambda_1 = 0$. The expression (32) gives the exact efficiency for any GD design satisfying $k = n$. In particular, using (32), one can get the exact efficiency for Regular GD designs with $\lambda_1 > 0$ also, a case for which Mukerjee and Kageyama (1990) give bounds for efficiency, provided these satisfy $k = n$. Of course, the bounds of Mukerjee and Kageyama apply to all Regular GD designs with $\lambda_1 > 0$, not necessarily satisfying $k = n$.

In the tables of Clatworthy (1973), there are 66 Regular GD designs satisfying $k = n$ and $\lambda_1 > 0$. For each of these designs, the value of E was computed. For 62 of the 66 designs, we have $E \geq .90$. For the remaining four designs, $.90 > E > .80$; these designs are $R1$ ($E = 0.875$), $R3$ ($E = 0.846$), $R7$ ($E = 0.89$) and $R45$ ($E = 0.89$). Therefore, Regular GD designs with $k = n$ and $\lambda_1 > 0$ appear to be robust according to Criterion 2 as well.

4. A Lower Bound to the Efficiency When a Single Observation Is Missing

In this section, we obtain a lower bound to the efficiency given by (23), when a single observation in an arbitrary, connected incomplete block design is missing.

From Theorem 2, (23) and Corollary 1 of Theorem 3, we have

$$E = \frac{tr(C_0^+)(1 - \mathbf{u}'C_0^+\mathbf{u})}{tr(C_0^+)(1 - \mathbf{u}'C_0^+\mathbf{u}) + \mathbf{u}'C_0^+C_0^+\mathbf{u}}. \tag{33}$$

Whittinghill (1989) derived the efficiency of the residual design when an observation is lost in a Balanced Block Design. The exact efficiency E has been evaluated by Ghosh et al. (1991) when d_0 is either a BIB design or a Singular, Semi-regular or Regular GD design with $\lambda_1 = 0$. For regular GD designs with $\lambda_1 > 0$, these authors give lower and upper bounds for the efficiency. We obtain lower and upper bounds on E when d_0 is an arbitrary incomplete block design with constant block size, k. Thus, our result is more general than those obtained by Whittinghill (1989) and Ghosh et al. (1991) in the sense that it is not design specific.

We now restrict our attention to block designs satisfying Corollary 2 of Theorem 4, that is, block designs for which the smallest positive eigenvalue of the C-matrix is strictly larger than unity. We need the following well known result (see, e.g., Magnus and Neudecker (1988, p.236)).

Lemma 1. *Let A be a positive definite matrix and B, a symmetric matrix, both of order n. Let $\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n$ be the eigenvalues of $A^{-1}B$. Then, for every nonnull vector* $\mathbf{x}$,

$$\lambda_1 \leq \mathbf{x}'B\mathbf{x}/\mathbf{x}'A\mathbf{x} \leq \lambda_n. \tag{34}$$

Now, from Theorem 2, if C_1 is the C-matrix of the residual design, then,

$$\begin{aligned} tr(C_1^+) - tr(C_0^+) &= (1 - \mathbf{u}'C_0^+\mathbf{u})^{-1}\mathbf{u}'C_0^+C_0^+\mathbf{u} \\ &= \frac{\mathbf{u}'C_0^+C_0^+\mathbf{u}}{\mathbf{u}'(I - C_0^+)\mathbf{u}} \end{aligned} \tag{35}$$

since $\mathbf{u}'\mathbf{u}=1$.

Since the C_0 matrix of any design d_0 satisfying Corollary 2 has all its positive eigenvalues strictly larger than unity, C_0^+ for such designs has all its positive eigenvalues strictly smaller than unity and hence $I - C_0^+$ is positive definite. Also, if $1 < \theta_1 \leq \theta_2 \leq \cdots \leq \theta_{v-1}$ are the positive eigenvalues of C_0, then the eigenvalues of $(I - C_0^+)^{-1}C_0^+C_0^+$ are 0 and $\theta_i^{-1}(\theta_i - 1)^{-1}$ for $i = 1, 2, \ldots, v-1$. Therefore, using Lemma 1, we have

$$0 \leq \frac{\mathbf{u}'C_0^+C_0^+\mathbf{u}}{\mathbf{u}'(I - C_0^+)\mathbf{u}} \leq \theta_1^{-1}(\theta_1 - 1)^{-1}.$$

Hence, from (23), we get

$$\frac{tr(C_0^+)\theta_1(\theta_1 - 1)}{1 + tr(C_0^+)\theta_1(\theta_1 - 1)} \leq E \leq 1. \tag{36}$$

The upper bound in (36) is trivial. Further, if we let

$$E_0 = \frac{tr(C_0^+)\theta_1(\theta_1 - 1)}{1 + tr(C_0^+)\theta_1(\theta_1 - 1)} = 1 - \frac{1}{1 + tr(C_0^+)\theta_1(\theta_1 - 1)} \tag{37}$$

we have

$$E_0 \leq E \leq 1. \tag{38}$$

Since $tr(C_0^+) = \sum_1^{v-1} \theta_i^{-1} = (v-1)/H$, where H is the harmonic mean of the positive eigenvalues of C_0, we have

$$E_0 = 1 - \frac{H}{H + (v-1)\theta_1(\theta_1 - 1)} \leq E \leq 1. \tag{39}$$

The value of E_0 was computed for all Group-divisible, Triangular and Latin-square-type PBIB designs given in Clatworthy (1973) and satisfying the conditions of Corollary 2. Recall that designs listed in Corollary 3 also satisfy the conditions of Corollary 2. The results of these computations for Triangular and Latin-square type designs are given below:

	Number of designs with		
Type of design	$E_0 < 0.80$	$0.80 \leq E_0 < 0.90$	$E_0 \geq 0.90$
Triangular	7	3	90
Latin-square type	2	1	143

Ghosh et al. (1991) have already shown that GD designs are quite robust as per Criterion 2 against the loss of a single observation. This fact, along with the present analysis shows that for two-associate PBIB designs of major types, the loss of efficiency is generally small when a single observation is lost, as E_0 is only a lower bound to the actual efficiency. In other words, most PBIB designs with two associate classes are robust according to Criterion 2.

Remark 4. Ghosh et al. (1991) have derived lower and upper bounds for the efficiency when d_0 is a Regular GD design with $\lambda_1 > 0$. A comparison of these bounds with (38) reveals that the bounds of Ghosh et al. are sharper than those given by (38). This however is expected, as Ghosh et al. take a specific class of designs while in obtaining (38), no information regarding the design structure is used. Moreover, the bound (38) is applicable to *any* incomplete block design with equal block sizes.

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