Gravity without metric, torsion, and topological gravity

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The role of θ term in gravity without the metric formulation of Capovilla, Jacobson, and Dell has been investigated when the gauge group is taken to be SL(2,C) and it is shown that this corresponds to the introduction of torsion and as such represents the Einstein–Cartan action. Moreover, as this term is related to the chiral anomaly, this helps us to formulate the Einstein–Hilbert gravitational action as a symmetry-breaking effect in quantum field theory. In view of this one can consider this topological θ term as the fundamental entity. When the chiral nature of matter field is not manifested explicitly, the torsion term effectively gives rise to the cosmological constant. In case there is only the θ term in the action, one can have topological gravity which helps to realize physically Donaldson's theory of four-dimensional space–time geometry.

I. INTRODUCTION

Recently Capovilla, Jacobson, and Dell[1] have formulated a theory of gravitation without the metric which constitutes a Lagrangian formulation of Ashtekar's theory in which the metric or the triad has been completely eliminated in favor of the connection. It is equivalent to Ashtekar's formulation though this equivalence breaks down unless a certain matrix constructed out of the curvature is invertible. The connection carries an SO(3) index. Ashtekar[2] has rewritten Einstein's equations using the SO(3) vector potential and its canonically conjugate momentum as fundamental variables. The latter appears as a standard triad though there is a caveat that these triads are complex-valued and defined on a real Lorentzian manifold. That the Lagrangian formulation of Capovilla, Jacobson, and Dell (CJD) avoids the metric and its equivalence with Ashtekar's Hamiltonian formulation under a certain condition suggests that the metric is not a fundamental entity.

This metric-free action for general relativity implies a link between the covariant quantization of gravity and canonical quantization formulated by Ashtekar. Besides, as emphasized by Capovilla, Jacobson, and Dell, the fact that the Einstein equations can be rewritten in terms of the spin connection may imply a twistor theoretic construction of the theory. It may be recalled that Newman and Penrose[3] formulated the gravitation theory involving a spinorial variable. Carmeli and Malin[4] have formulated the SL(2,C) gauge theory of gravitation which is closely related to the Newman–Penrose formalism. It has been shown in a recent article[5] that SL(2,C) gauge theory of gravitation may be taken to lead to the Einstein–Cartan theory incorporating torsion which appears as a quantum effect. In fact, the torsion term may be taken to be originated from the geometry of microlocal space–time which is associated with the quantization of a fermion and as such may be taken to be an effect of quantum gravity. The torsion may be introduced in the formulation of Capovilla, Jacobson, and Dell by introducing a θ term in the action. Indeed, if we take into account the SL(2,C) gauge theory, this θ term gives rise to the Pontryagin index which contributes to the action as the component of the torsion.

It is our motivation here to study the effect of this θ term in the CJD action when the group structure is taken to be SL(2,C) and we shall show that this may lead to torsion which again can be associated with the cosmological constant. Moreover, we shall show that this topological term can be taken to be the fundamental entity and the standard CJD action which corresponds to the Einstein–Hilbert action for pure gravity can be introduced through the incorporation of the chiral anomaly in the matter field Lagrangian, where the coupling of gravity with the matter field is neglected. In this sense, this is analogous to the contention of Adler[6] that the Einstein–Hilbert gravitational action is obtained as a symmetry-breaking effect in quantum field theory. This helps us to consider that when there is only the θ term in the Lagrangian we have topological gravity which implies the existence of a nilpotent operator Q behaving as a supercharge [Becchi–Rouet–Stora–Tyutin (BRST) charge]. This leads to the Hamiltonian version of the Donaldson theory of the four-dimensional space–time manifold. In this way, the CJD action along with the θ term can be taken to imply not only a formulation of gravity without the metric but also helps us to realize topological gravity.

II. SL(2,C) GAUGE THEORY, θ TERM, AND TORSION

In a recent article,[5] it has been shown that the quantum space–time leads to the realization of SL(2,C) gauge theory for the Einstein–Cartan action which includes the torsion term. Indeed, the quantization of a fermion can be achieved when we introduce Brownian motion processes both in the external space as well as in the internal space and an anisotropy is introduced in the internal space such that the space–time coordinate in the complexified space–time can be written as $z_\mu = x_\mu + i\xi_\mu$ where ξ_μ appears as a "direction vector" attached to the space–time point x_μ.[7] The two opposite orientations of the "direction vector" give rise to two opposite internal helicities corresponding to fermion and antifermion which can easily be formulated in terms of the extended space–time metric $g_{\mu\nu}(x,\theta,\bar{\theta})$, where $\theta(\bar{\theta})$ are two-component spinorial variables. In fact, for a massive spinor, we can choose the chiral coordinates in this extended space as[8]

$$z^\mu = x^\mu + (i/2)\lambda^\mu{}_\alpha \theta^\alpha, \quad (\alpha = 1,2), \tag{1}$$

where we identify the coordinate in the complex manifold as

$$z^\mu = x^\mu + i\xi^\mu, \quad \text{with} \quad \xi^\mu = \tfrac{1}{2}\lambda^\mu{}_\alpha \theta^\alpha.$$

We can now replace the chiral coordinates by their matrix representations

$$z^{AA'} = x^{AA'} + (i/2)\lambda^{AA'}{}_\alpha \theta^\alpha, \tag{2}$$

where

$$x^{AA'} = \frac{1}{\sqrt{2}} \begin{bmatrix} x^0 - x^1 & x^2 + ix^3 \\ x^2 - ix^3 & x^0 + x^1 \end{bmatrix}$$

and

$$\lambda^{AA'}{}_\alpha \in \mathrm{SL}(2,C).$$

This helps us to associate the internal helicity with the spinorial variable θ^α as we can now construct the helicity operator[8]

$$S = -\lambda^{AA'}{}_\alpha \theta^\alpha \bar{\pi}_A \pi_{A'}, \tag{3}$$

where $\bar{\pi}_A(\pi_{A'})$ denotes the spinorial variable corresponding to the four-momentum p_μ (the canonical conjugate of x_μ) and is given by the matrix representation

$$p^{AA'} = \bar{\pi}^A \pi^{A'}. \tag{4}$$

This internal helicity can now be identified with the fermion number. It may be noted that since we have taken the matrix representation of p_μ as $p^{AA'} = \bar{\pi}^A \pi^{A'}$ necessarily implying $p_\mu^2 = 0$, the particle will have its mass due to the nonvanishing character of the quantity ξ_μ^2. It is observed that the complex conjugate of the chiral coordinate given by Eq. (1) will give rise to a massive particle with opposite internal helicity corresponding to an antifermion.

In this complexified space–time exhibiting the internal helicity states, we can write the metric

$$g_{\mu\nu}(x,\theta,\bar{\theta})=g_{\mu\nu}^{AA'}(x)\bar{\theta}_A\theta_{A'}. \tag{5}$$

It has been shown that this metric structure gives rise to the SL(2,C) gauge theory of gravitation and generates the field strength tensor $F_{\mu\nu}$ given in terms of the gauge field A_μ which is matrix-valued, has an SL(2,C) group structure, and is given by[5]

$$F_{\mu\nu}=\partial_\nu A_\mu-\partial_\mu A_\nu+[A_\mu, A_\nu]. \tag{6}$$

Now to study the effect of this quantum geometry in gravitation, following Carmeli and Malin,[4] we choose the simplest SL(2,C) invariant Lagrangian density in spinor affine space

$$L=\mathrm{Tr}(\epsilon^{\alpha\beta\gamma\delta}F_{\alpha\beta}F_{\gamma\delta}). \tag{7}$$

It is noted that this actually corresponds to the θ term in non-Abelian gauge theory when a coupling parameter θ is introduced.

Now writing

$$A_\mu=\mathbf{a}_\mu\cdot\mathbf{g},\quad F_{\mu\nu}=f_{\mu\nu}\cdot\mathbf{g}, \tag{8}$$

where $g=(g_1,g_2,g_3)$ are the generators of the SL(2,C) group in tangent space given by

$$g_1=\begin{bmatrix}0&0\\1&0\end{bmatrix},\quad g_2=\begin{bmatrix}1&0\\0&-1\end{bmatrix},\quad g_3=\begin{bmatrix}0&1\\0&0\end{bmatrix},$$

we can define the current density[4]

$$\mathbf{j}_\theta^\mu=\epsilon^{\mu\nu\alpha\beta}\mathbf{a}_\nu\times\mathbf{f}_{\alpha\beta}=\epsilon^{\mu\nu\alpha\beta}\partial_\nu\mathbf{f}_{\alpha\beta} \tag{9}$$

satisfying the relation

$$\partial_\mu\mathbf{j}_\theta^\mu=\epsilon^{\mu\nu\alpha\beta}\partial_\mu\partial_\nu\mathbf{f}_{\alpha\beta}=0. \tag{10}$$

When the superspace is Riemannian with metric structure, the conserved current may be written as

$$\mathbf{J}^\mu=\tilde{\mathbf{J}}^\mu+(1/\chi)\epsilon^{\mu\nu\alpha\beta}\mathbf{a}_\nu\times\mathbf{f}_{\alpha\beta}, \tag{11}$$

where $\chi=-8\pi G/c^4$, $\tilde{\mathbf{J}}^\mu$ is the contribution to the conserved current due to the energy momentum tensor, $\tilde{\mathbf{J}}^\mu=\tilde{J}^\mu\mathbf{n}$, $\mathbf{n}$ being a unit vector, and the second part of the right hand side of Eq. (11) is the contribution of the spinorial variable θ. The action is now given by

$$S=S_1+S_2=A\int \mathbf{J}_\mu\cdot\mathbf{J}_\mu\, d^4x, \tag{12}$$

where $A=1/k^2=1/16\pi G$; k=Planck length.

Using the relations[4]

$$\tilde{J}^\mu=\tfrac{1}{2}\epsilon^{\mu\alpha\beta\gamma}[R_{\alpha\beta,\gamma}],\quad R_{\alpha\beta,\gamma}=R_{\alpha\beta\gamma\delta}v^\delta, \tag{13}$$

where v^δ is an arbitrary vector and taking

$$R_{\alpha\beta\gamma\delta}=R^{(0)}_{\alpha\beta\gamma\delta}-e_{\gamma\alpha}e_{\delta\beta}, \tag{14}$$

with

$$R^{(0)}_{\alpha\beta\gamma\delta}=-\partial_\alpha\omega_{\beta\gamma\delta}+\partial_\beta\omega_{\alpha\gamma\delta}+\omega^a_{\alpha\gamma}\omega_{\beta a\delta}-\omega^a_{\beta\gamma}\omega_{\alpha a\delta}$$

denoting the Riemannian curvature and related to rotation whereas the second term $e_{\gamma\alpha}e_{\delta\beta}$ in Eq. (14) corresponds to translation, we have

$$S_1=\frac{1}{k^2}\int \tilde{\mathbf{J}}_\mu\cdot\tilde{\mathbf{J}}_\mu\, d^4x=-\frac{1}{k^2}\int Re\, d^4x, \tag{15}$$

where R is the scalar curvature and e is given by the relation

$$\epsilon^{\alpha\beta\gamma\delta}\epsilon_{\alpha\beta\gamma\delta}=-e. \tag{16}$$

Again writing

$$\mathbf{a}_\nu\times\mathbf{f}_{\alpha\beta}=k^2S_{\nu\alpha\beta}\mathbf{n}, \tag{17}$$

with $\mathbf{n}$ being an unit vector, the second part of the action becomes

$$S_2=-\frac{4}{k^2}\int S_{\nu\alpha\beta}S^{\nu\alpha\beta}\, d^4x \tag{18}$$

giving rise to the torsion term. Thus we find that the θ term in the Lagrangian gives rise to the current $\mathbf{j}^\mu_\theta$ associated with the spinorial variable attached to the space–time point and this in turn is responsible for torsion. Indeed, the effect of the internal helicity of a massive fermion associated with the generation of fermion number is manifested through the torsion term in gravitational action in ordinary Minkowski space–time.

III. TORSION, θ TERM, AND GRAVITY WITHOUT THE METRIC

Bengtsson and Peldań[9] have studied the effect of the θ term when added to the CJD action and its possible implications. As it is well known, in its 3+1 form, the action in terms of Ashtekar's variable can be written as

$$S=\int \dot{A}_{ai}E^a{}_i-\mathcal{N}\mathcal{H}-\mathcal{N}^a\mathcal{H}_a-\Lambda_i\mathcal{G}_i,$$

$$\mathcal{H}=\tfrac{1}{2}if_{ijk}E^a{}_iE^b{}_jF_{abk}=\tfrac{1}{2}i\epsilon_{abc}f_{ijk}E^a{}_iE^b{}_jB^c{}_k,$$

$$\mathcal{H}_a=E^b{}_iF_{abi}=\epsilon_{abc}E^b{}_iB^c{}_i,\qquad \mathcal{G}_i=D_aE^a{}_i=\partial_aE^a{}_i+if_{ijk}A_{aj}E^a{}_k. \tag{19}$$

Here a,b,c are spatial indices, i,j,k are SO(3) indices, F_{abi} is an SO(3) curvature, $B^a{}_i=\frac{1}{2}\epsilon^{abc}F_{bci}$ is the corresponding "magnetic" field. The action yields three kinds of constraints viz., the Hamiltonian constraint $\mathcal{H}$, the vector constraint $\mathcal{H}_a$, and SO(3) vector's worth G_i of "internal" constraints which here takes the form of Gauss's law for the "electric" field $E^a{}_i$. The tensor $q^{ab}=gg^{ab}$, where g^{ab} is the metric tensor on the foliating hypersurface and g is its determinant, is given by

$$q^{ab}=E^a{}_iE^b{}_i. \tag{20}$$

Capovilla, Jacobson, and Dell[1] found a very elegant Lagrangian formulation of the above. The CJD action is

$$S^{\mathrm{CJD}}=\frac{1}{8}\int \eta(\Omega_{ij}\Omega_{ij}+a\Omega_{ii}\Omega_{jj}),\quad \Omega_{ij}=\epsilon^{\alpha\beta\gamma\delta}F_{\alpha\beta i}F_{\gamma\delta j}. \tag{21}$$

Here $\alpha,\beta,...$ are space–time indices.

The Lagrange multiplier η is a scalar density of weight 1, and $F_{\alpha\beta i}$ is an SO(3) field strength. They showed that a 3+1 decomposition of this action yields Ashtekar's action directly provided that the parameter $a=-1/2$ and provided that the determinant of the "magnetic" field $B^a{}_i$ is nonzero.[10] In this way the equivalence to Einstein's theory is established. Indeed, with $a=-1/2$, the equivalence to Einstein's theory may also be shown directly in a manifestly covariant way. This demonstration hinges on the tetrad formalism and the space–time metric may be given directly in terms of the curvature when we can write[1]

$$(-g)^{1/2}g^{\alpha\beta}=-(2i/3\eta)K^{\alpha\beta}=-(2i/3\eta)f_{ijk}\epsilon^{\alpha\gamma\delta\rho}\epsilon^{\beta\mu\nu\sigma}F_{\gamma\delta i}F_{\rho\sigma j}F_{\mu\nu k}. \tag{22}$$

The constraint that is obtained when the CJD action is varied with respect to the Lagrange multiplier η is actually the Hamiltonian constraint in disguise.

$$\psi=\Omega_{ij}\Omega_{ij}-\tfrac{1}{2}\Omega_{ii}\Omega_{jj}=i(2\eta^2\det B)^{-1}\mathscr{H}. \tag{23}$$

We can now introduce the matrix ψ_{ij} defined by

$$E^a{}_i=\psi_{ij}B^a{}_j, \tag{24}$$

where we note that such a matrix always exists provided that the magnetic field is nondegenerate. If we insert this expression in the vector constraint $\mathscr{H}_a\approx 0$, we find that the vanishing of the vector constraint is equivalent to the statement that the matrix is symmetric. We also note that as long as $\det B\neq 0$, $\psi\approx 0$ and $\mathscr{H}\approx 0$ are equivalent statements. Gauss' law $\mathscr{G}_i\approx 0$ follows when the action is varied with respect to A_{0i}. Bengtsson and Peldań[9] have shown that if we perform the canonical transformation

$$A_{ai}\to A_{ai},\quad E^a{}_i\to E^a{}_i-\theta B^a{}_i \tag{25}$$

the expression for the Hamiltonian constraint changes though the remaining constraints are unaffected. This corresponds precisely to the addition of a "CP violating" θ term to the CJD Lagrangian when the new action is given by

$$S=\tfrac{1}{8}\int \theta\Omega_{ii}+\eta(\Omega_{ij}\Omega_{ij}-\tfrac{1}{2}\Omega_{ii}\Omega_{jj}). \tag{26}$$

Here we want to show that when the gauge group is taken to be SL(2,C) when i,j corresponds to the SL(2,C) indices, this θ term effectively corresponds to torsion. In fact, the θ term now corresponds to the Lagrangian

$$L=-\tfrac{1}{4}\theta\epsilon^{\mu\nu\alpha\beta}F_{\mu\nu}F_{\alpha\beta},$$

with

$$F_{\mu\nu}=\partial_\mu A_\nu-\partial_\nu A_\mu+[A_\mu,A_\nu],$$

where $A_\mu=\mathbf{a}_\mu\cdot\mathbf{g}$, g^1,g^2,g^3 being the SL(2,C) generators. As discussed in the previous section, this gives rise to the current

$$\mathbf{j}_\theta^\mu = \epsilon^{\mu\nu\alpha\beta}\mathbf{a}_\nu \times \mathbf{f}_{\alpha\beta} = \epsilon^{\mu\nu\alpha\beta}\partial_\nu \mathbf{f}_{\alpha\beta},$$

which in turn is responsible for torsion.

We shall now show that the θ term in the action (26) effectively corresponds to the Pontryagin index when the group structure is taken to be SL(2,C) and this is the case also for the torsion term given by the $\mathbf{j}_\theta^\mu . \mathbf{j}_\theta^\mu$ coupling. To study this aspect, we consider that the θ term is essentially associated with the chiral anomaly which arises when chiral currents interact with a gauge field. Indeed, to describe a matter field in the geometry associated with the Lagrangian (7), we note that in the background of the SL(2,C) gauge fields, the Lagrangian for a Dirac spinor field may be written as (neglecting the mass term)

$$L = -\bar{\psi}\gamma^\mu D_\mu \psi - \tfrac{1}{4}\,\mathrm{Tr}\,\epsilon^{\alpha\beta\gamma\delta}F_{\alpha\beta}F_{\gamma\delta}, \tag{27}$$

where D_μ is the SL(2,C) gauge covariant derivative defined by $D_\mu = \partial_\mu - igA_\mu$ where g is some coupling strength and $A_\mu \in \mathrm{SL}(2,C)$. However, in the Lagrangian (27), if we split the Dirac massless spinor into chiral forms and identify the internal helicity $+1/2$ $(-1/2)$ with left (right) chirality corresponding to θ and $\bar{\theta}$, we can write

$$\bar{\psi}\gamma^\mu D_\mu \psi = \bar{\psi}\gamma^\mu \partial_\mu \psi - ig\bar{\psi}\gamma_\mu A^a_\mu g_a \psi = \bar{\psi}\gamma^\mu \partial_\mu \psi - (ig/2)\{\bar{\psi}_R \gamma^\mu A^1_\mu \psi_R - \bar{\psi}_R \gamma^\mu A^2_\mu \psi_R + \bar{\psi}_L \gamma^\mu A^2_\mu \psi_L + \bar{\psi}_L \gamma^\mu A^3_\mu \psi_L\}. \tag{28}$$

Then the three SL(2,C) gauge field equations give rise to the following three conservation laws:[11]

$$\begin{aligned} &\partial_\mu[\tfrac{1}{2}(-ig\bar{\psi}_R\gamma_\mu\psi_R) + j^1_\mu] = 0, \\ &\partial_\mu[\tfrac{1}{2}(-ig\bar{\psi}_L\gamma_\mu\psi_L + ig\bar{\psi}_R\gamma_\mu\psi_R) + j^2_\mu] = 0, \\ &\partial_\mu[\tfrac{1}{2}(-ig\bar{\psi}_L\gamma_\mu\psi_L) + j^3_\mu] = 0. \end{aligned} \tag{29}$$

These three equations represent a consistent set of equations, if we choose

$$j^1_\mu = -\tfrac{1}{2}j^2_\mu, \quad j^3_\mu = +\tfrac{1}{2}j^2_\mu, \tag{30}$$

which evidently guarantees the vector current conservation. Then we can write

$$\partial_\mu(\bar{\psi}_R\gamma_\mu\psi_R + j^2_\mu) = 0, \quad \partial_\mu(\bar{\psi}_L\gamma_\mu\psi_L - j^2_\mu) = 0. \tag{31}$$

From these we find

$$\partial_\mu(\bar{\psi}\gamma_\mu\gamma_5\psi) = \partial_\mu J^5_\mu = -2\partial_\mu j^2_\mu. \tag{32}$$

Thus the anomaly is expressed here in terms of the second SL(2,C) component of the gauge field current j^2_μ.

The term $\epsilon^{\alpha\beta\gamma\delta}\,\mathrm{Tr}\,F_{\alpha\beta}F_{\gamma\delta}$ in the Lagrangian can actually be expressed as a four divergence of the form $\partial_\mu\Omega^\mu$. In fact Ω^μ is the Chern–Simons secondary characteristic class given by

$$\Omega^\mu = -\frac{1}{16\pi^2}\,\epsilon^{\mu\alpha\beta\gamma}\,\mathrm{Tr}[A_\alpha F_{\beta\gamma} - \tfrac{2}{3}(A_\alpha A_\beta A_\gamma)] \tag{33}$$

and we recognize that the gauge field Lagrangian is related to the Pontryagin density

$$P=-\frac{1}{16\pi^2}\,\mathrm{Tr}^*\,F^{\mu\nu}F_{\mu\nu}=\partial_\mu\Omega^\mu. \tag{34}$$

The Pontryagin index

$$q=\int P\,d^4x \tag{35}$$

is then a topological invariant.

The axial vector current is now modified as[12]

$$\tilde{J}^5_\mu=J^5_\mu+2\hbar\Omega_\mu \tag{36}$$

and though $\partial_\mu J^5_\mu \neq 0$, we have $\partial_\mu \tilde{J}^5_\mu=0$. From the relation (32), it is noted that Ω_μ is associated with j^2_μ and the Pontryagin index can be expressed as[11]

$$q=\int j^2_0\,d^3x=\int \partial_\mu j^2_\mu\,d^4x. \tag{37}$$

Since the $j^2_\mu\cdot j^2_\mu$ term gives rise to the torsion, we note that the Pontryagin index is associated with the torsion term in the Einstein–Cartan action as observed by Drechsler[13] and is related to the gravitational anomaly when we deal only with the Einstein part of the gravitational action. Indeed from the relation

$$\mathrm{Tr}(^*F^{\mu\nu}F_{\mu\nu})\sim\partial_\mu\Omega^\mu$$

we note that the θ term in the action (26) corresponds to the Pontryagin index. This is so for the torsion term $\int j^2_\mu\cdot j^2_\mu\,d^4x$ also. In fact, from the relation (32), we can write

$$j^2_\mu=-\tfrac{1}{2}(J^5_\mu+J^V_\mu), \tag{38}$$

where J^V_μ is any arbitrary vector current which is conserved. Taking the particular solution, we can write

$$j^2_\mu=-\tfrac{1}{2}J^5_\mu=-\tfrac{1}{2}\bar{\psi}\gamma_\mu\gamma_5\psi \tag{39}$$

so that

$$\partial_\mu j^2_\mu=-\tfrac{1}{2}\partial_\mu(\bar{\psi}\gamma_\mu\gamma_5\psi)=-\tfrac{1}{2}[(\partial_\mu\bar{\psi})\gamma_\mu\gamma_5\psi+\bar{\psi}\gamma_\mu\gamma_5(\partial_\mu\psi)]=-im\bar{\psi}\gamma_5\psi. \tag{40}$$

This suggests that we can write

$$\partial_\mu j^2_\mu I=im\gamma_\mu\bar{\psi}\gamma_\mu\gamma_5\psi=im\gamma_\mu j^2_\mu, \tag{41}$$

where I is the identity matrix.

Thus we find

$$-m^2\int j^2_\mu\cdot j^2_\mu\,d^4x=\int(\partial_\mu j^2_\mu)(\partial_\mu j^2_\mu)d^4x=qc, \tag{42}$$

where q is the Pontryagin index given by $q=\int\partial_\mu j^2_\mu\,d^4x$ as discussed earlier and c is any arbitrary constant. Thus we note that the torsion term given by $\int j^2_\mu\cdot j^2_\mu\,d^4x$ actually corresponds to the

Pontryagin index when m is normalized to be 1. In view of this, we find that the net effect of the θ term in the action is just to introduce a torsion term and thus corresponds to the Einstein–Cartan action.

IV. TORSION, COSMOLOGICAL CONSTANT, AND TOPOLOGICAL GRAVITY

It is noted that the θ term which corresponds to torsion is a topological term. If we take the Hermitian representation of the SL(2,C) group structure, we can take the compact group SU(2) as the group manifold. Now we consider a compact region within which $\partial_\mu j^2_\mu \neq 0$ but outside this $\partial_\mu j^2_\mu = 0$. This implies that outside the compact space, we have only the θ term in the action as the Einstein–Hilbert action for pure gravity will be vanishing here. This follows from the fact that there cannot be any matter field in this region as any spinorial matter when written in chiral form demands $\partial_\mu j^2_\mu \neq 0$ as discussed in the previous section. Thus the boundary of the compact space may be taken to be the nucleation point. It may be recalled here that in a recent article,[14] we have shown that the chiral anomaly ($\partial_\mu j^2_\mu \neq 0$) may be taken to be responsible for the origin of mass and the region where $\partial_\mu j^2_\mu = 0$ there is no nucleation.

On the nucleation boundary, the torsion term effectively corresponds to the cosmological constant. Indeed, from the relation (9)

$$j^{\mu(2)} = \epsilon^{\mu\nu\lambda\sigma} \partial_\nu f^{(2)}_{\lambda\sigma}$$

and noting the antisymmetric nature of $f_{\lambda\sigma}$, we can write

$$j^{\mu(2)} = \epsilon^{\mu\nu\lambda\sigma} \epsilon_{\nu\lambda\sigma} c(x), \tag{43}$$

where $c(x)$ is a scalar function. Now from the relation $\partial_\mu j^2_\mu = 0$ we find that $c(x)$ is a constant. So the torsion $j^2_\mu \cdot j^2_\mu$ gives rise to the constant c^2 which now appears as a cosmological constant. This implies that in gravitational theory at the microscopic level where the chiral nature of matter field is not manifested explicitly, the torsion term effectively gives rise to the cosmological constant.

The relationship between the torsion and the cosmological constant has been pointed out by Baekler, Mielke, Hecht, and Hehl[15] also in the Poincaré gauge model of gravity coupled to a massless scalar field where the asymptotic constancy of the torsion compensates the bare cosmological constant and thus helps us to have a solution of the cosmological constant problem. This is also implicit in the canonical quantization procedure as has been shown by Bengtsson and Peldan.[10] Indeed, as the canonical transformation

$$A_{ai} \to A_{ai}, \quad E^a{}_i \to E^a{}_i - \theta B^a{}_i \tag{44}$$

gives rise to the θ term in the action, the canonical transformation

$$E^a{}_i \to E^a{}_i, \quad B^a{}_i \to B^a{}_i - \alpha E^a{}_i \tag{45}$$

gives rise to the cosmological constant. In this case also, the vector constraint is not affected, but the Hamiltonian constraint changes into

$$\mathscr{H} = \tfrac{1}{2} i \epsilon_{abc} f_{ijk} E^a{}_i E^b{}_j (B^c{}_k - \alpha E^c{}_k) = \tfrac{1}{2} i \epsilon_{abc} f_{ijk} E^a{}_i E^b{}_j B^c{}_k - 3 i \alpha \det E. \tag{46}$$

This is precisely the Hamiltonian constraint in the presence of a cosmological constant $\lambda = -6i\alpha$. Since we are adding a vector to an axial vector in both the transformations (44) and (45), the analogy between the θ term and the λ term becomes self-evident. However, the introduction of the cosmological constant leads to a nonpolynomial action.

Now we consider the region where $\partial_\mu j^2_\mu = 0$ everywhere outside a compact space. In this region, we will have only the topological Lagrangian

$$L=\theta\ \mathrm{Tr}^*\ F^{\mu\nu}F_{\mu\nu}.$$

Indeed, there is no massive matter in this region characterized by the absence of nucleation and hence the Einstein part corresponding to the curvature vanishes. That is the CJD Lagrangian ψ vanishes. Thus the Lagrangian here just corresponds to the cosmological term

$$L=\Lambda\sqrt{g}, \tag{47}$$

which may be renormalized to the Lagrangian

$$L=0, \tag{48}$$

where we take $F_{\mu\nu}=0$ corresponding to the pure gauge condition $A_\mu=U^{-1}\partial_\mu U$. This Lagrangian has more symmetries than the usual diffeomorphism invariances. Evidently, general covariance is unbroken here and we have broken symmetry as the nucleation starts at the boundary.

It may be noted here that the torsion term may be incorporated through the stochastic fluctuation of the metric. Indeed, from the geometrical consideration as discussed in Sec. II, we can write the stochastic extension of a relativistic particle by denoting the coordinate and momentum as

$$Q_\mu=q_\mu+\hat{Q}_\mu,\quad P_\mu=p_\mu+\hat{P}_\mu, \tag{49}$$

where $q_\mu(p_\mu)$ is the mean value and $\hat{Q}_\mu(\hat{P}_\mu)$ are stochastic extensions which can be expressed as gauge theoretic extensions given by[16]

$$\hat{Q}_\mu=-i\omega_0\left\{\frac{\partial}{\partial p^\mu}+A_\mu\right\},\quad \hat{P}_\mu=i\omega_0\left\{\frac{\partial}{\partial q^\mu}+B_\mu\right\}, \tag{50}$$

with $\omega_0=\hbar/l_0m_0c$, a dimensionless constant and $A_\mu(B_\mu)\in\mathrm{SL}(2,C)$. Defining the one-form

$$\lambda=A_\mu\ dp^\mu+B_\mu\ dq^\mu \tag{51}$$

the covariant derivative is given by

$$D\lambda=d\lambda+\tfrac{1}{2}[\lambda,\ \lambda] \tag{52}$$

and represents the field strength two-form F corresponding to the symplectic structure

$$F=(i/\omega_0)\ dp_\mu\wedge dq^\mu. \tag{53}$$

Now if we introduce an anisotropic feature in the internal space we can write the commutation relations

$$[\hat{Q}_\mu,\ \hat{P}_\nu]=i\omega_0 g_{\mu\nu}\oplus i\omega_0 g'_{\mu\nu},$$

$$[\hat{Q}_\mu,\ \hat{Q}_\nu]=(l/l_0)^2h'_{\mu\nu},\quad [\hat{P}_\mu,\ \hat{P}_\nu]=(m/m_0)^2h''_{\mu\nu} \tag{54}$$

and since the relations for $h'_{\mu\nu}$ and $h''_{\mu\nu}$ are reciprocally invariant, we can write

$$\left(\frac{l}{l_0}\right)^2 h'_{\mu\nu}=\left(\frac{m}{m_0}\right)^2 h''_{\mu\nu}=\tilde{h}_{\mu\nu}. \tag{55}$$

The field strength two-form F can now be written as

$$\bar{F}=\frac{i}{\omega_0} g^{\mu\nu}\, dp_\mu\wedge dq_\nu \oplus \frac{i}{\omega_0} g'^{\mu\nu}\, dp_\mu\wedge dq_\nu \oplus \frac{\tilde{h}^{\mu\nu}}{\omega_0^2}\, dp_\mu\wedge dp_\nu \oplus \frac{\tilde{h}^{\mu\nu}}{\omega_0^2}\, dq_\mu\wedge dq_\nu. \quad (56)$$

It is evident from our previous discussions, that the field strength tensor associated with the factor $\tilde{h}_{\mu\nu}$ gives rise to the torsion in the gravitational action.

However, on the nucleation boundary where we have $\partial_\mu j_\mu^2=0$ implying that the anisotropic feature associated with chirality is not manifested there, we can incorporate the limiting effect of torsion through the stochastic fluctuation of the metric when it is written as

$$g_{\mu\nu}=\bar{g}_{\mu\nu}+h_{\mu\nu}, \quad (57)$$

where $\bar{g}_{\mu\nu}$ is the mean metric structure and $h_{\mu\nu}$ is the fluctuation. This can be represented through the introduction of a stochastic scalar field $\phi(x)$ where we can write[17]

$$g_{\mu\nu}=(1+\phi)^2\bar{g}_{\mu\nu}, \quad (58)$$

where we can choose $\langle\phi\rangle=0$ and $\langle\phi^2\rangle\neq 0$. Evidently, the quantum effect of torsion can now be represented through the quantity $\chi=\langle\phi^2\rangle$. Following Joshi,[18] we take the general classical background by the space–time metric

$$d\bar{s}^2=dt^2-\bar{g}_{ij}\, dx^i\, dx^j, \quad (59)$$

where $i,j=1,2,3$ denote the space coordinates, and $\bar{g}_{ij}=\bar{g}_{ij}(x^\mu)$ is a function of both the time and space coordinates. Also we take $G=c=\hbar=1$. Neglecting the matter part, the Einstein action is given by

$$\bar{S}=\int \bar{R}\sqrt{-\bar{g}}\, d^4x. \quad (60)$$

Now for the metric $g_{\mu\nu}=(1+\phi)^2\bar{g}_{\mu\nu}$ and taking ϕ as a function of time only, the total action can be written as

$$S=\frac{1}{16\pi}\int [(1+\phi^2)h_1(t)-\dot{\phi}^2h_2(t)]dt, \quad (61)$$

where

$$h_1(t)=\int \bar{R}\sqrt{-\bar{g}}\, d^3x,$$

$$h_2(t)=6\int \sqrt{-\bar{g}}d^3x.$$

Defining the associated conjugate momentum p as $\partial L/\partial\dot{\phi}$, the Hamiltonian for the system is given by

$$H=\dot{\phi}p-L=\frac{4\pi p^2}{h_2(t)}-\frac{1}{16\pi}(1+\phi^2)h_1(t). \quad (62)$$

Now as mentioned above, when there is only the θ term in the action we can take $h_1(t)$ as vanishing. Besides in the region where $\partial_\mu j_\mu^2=0$ or $F_{\mu\nu}\to 0$, the Lagrangian is either a constant [Eq. (47)] or zero [Eq. (48)], and thus we will have $p=0$, i.e., $\partial L/\partial\dot{\phi}=0$. This suggests that the Hamiltonian in this region vanishes.

The vanishing of the Hamiltonian now gives rise to a fermionic operator Q such that $H=\frac{1}{2}\{Q,\bar{Q}\}=0$, $\bar{Q}$ being the adjoint of Q with a minus sign. Indeed, as we know, in supersymmetric quantum mechanics, the ground state energy is zero and this involves the existence of supercharge Q_S so that the Hamiltonian can be written as $H_s=Q_s^2=\frac{1}{2}\{Q,\bar{Q}\}$. The fact that in the domain where the Hamiltonian vanishes and the Lagrangian is given only by the θ term, the action becomes a constant function on the vector potentials. Baulieu and Singer[19] have pointed out that the topological invariance of the path integral

$$\int DAe^{-S_{\rm top}}, \quad \text{where} \quad S_{\rm top}=\frac{1}{8\pi^2}\int_{M_4} F\wedge F$$

follows from (BRST) symmetry. This symmetry and the ghost fields introduced by gauge fixing have a geometrical interpretation on $M_4\times(\mathscr{A}/\mathscr{G})$ where M_4 is the four manifold and $\mathscr{A}/\mathscr{G}$ is the orbit space of vector potentials equivalent under gauge transformations. This is associated with the supersymmetrization of the theory.

To have a geometric interpretation of the supercharge Q_S as well as the BRST (anti-BRST) operator $Q(\bar{Q})$ in our present formalism, we recall that the Pontryagin term (θ term) in the Lagrangian finds its relevance in the "direction vector" ξ_μ attached to the space–time point x_μ as discussed in Sec. II.

We now take into consideration the operators[20]

$$\partial=\frac{1}{2}\left(\frac{\partial}{\partial x}-i\,\frac{\partial}{\partial \xi}\right),\quad \bar{\partial}=\frac{1}{2}\left(\frac{\partial}{\partial x}+i\,\frac{\partial}{\partial \xi}\right) \tag{63}$$

and define the field $\phi^{\pm}(z)=\phi(x)\pm i\phi(\xi)$. For a free field, the Hamiltonian is

$$H=-2\partial\bar{\partial}+m^2\phi^-\phi^+. \tag{64}$$

Now if we identify $\phi^{\pm}(z)=\mp i\sqrt{2}\partial V$, we can construct two operators Q_+ and Q_- such that

$$Q_-=\begin{pmatrix} \partial V & i\partial \\ i\bar{\partial} & -(\partial V)^* \end{pmatrix},\quad Q_+=\begin{pmatrix} (\partial V)^* & i\partial \\ i\bar{\partial} & -\partial V \end{pmatrix} \tag{65}$$

and the Hamiltonian (64) can be expressed as

$$H=\text{Tr}\, Q_+Q_-, \tag{66}$$

where

$$Q_+Q_-=(-\partial\bar{\partial}+|\partial V|^2)\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Since it is the sum of two positive operators it has no zero mode. Besides it maintains the reflection invariance. However, supersymmetry is obtained when reflection invariance is broken and the ground state energy is zero. This can be achieved when we consider the operator

$$Q_-Q_+=Q_+Q_-+\begin{pmatrix} 0 & -i\partial^2 V \\ i(\partial^2 V)^* & 0 \end{pmatrix}. \tag{67}$$

The presence of the nondiagonal elements breaks down the reflection invariance. Indeed, we can now define an operator Q_S such that

$$Q_S = \begin{pmatrix} 0 & Q_- \\ Q_+ & 0 \end{pmatrix} \tag{68}$$

and we can construct the Hamiltonian

$$H_S = Q_S^2 = \begin{pmatrix} Q_-Q_+ & 0 \\ 0 & Q_+Q_- \end{pmatrix}. \tag{69}$$

Due to the presence of the operator Q_-Q_+ in H_S it possesses zero modes. Thus Q_S appears here as the supercharge. Now we note that $-Q_-^* = Q_+$ where Q_-^* is the adjoint of Q_-. This suggests that if we define

$$Q = Q_- \quad \text{and} \quad \bar{Q} = Q_+$$

we can write the supersymmetry Hamiltonian as

$$H_S = \tfrac{1}{2}\{Q, \bar{Q}\}. \tag{70}$$

This helps us to realize Donaldson's theory[21] of four-dimensional geometry in which general covariance is realized when we begin with a gauge field theoretical Lagrangian given by the θ term and thus helps us to realize topological gravity. As mentioned earlier, this general covariance is broken in this formalism by the nonvanishing value of $\partial_\mu j^2_\mu$ corresponding to the chiral anomaly which is responsible for the generation of mass. Since the chiral anomaly is associated with the quantum mechanical symmetry breaking and in the classical level it vanishes, we can consider that general covariance is broken by quantum fluctuation. Since the introduction of the θ term in the CJD action gives rise to the Einstein–Cartan action involving torsion and in the domain where the CJD Lagrangian ψ vanishes, we achieve topological gravity honoring general covariance.

V. DISCUSSION

We have shown above that when the θ term is introduced in the CJD action, this effectively corresponds to the introduction of torsion and as such corresponds to the Einstein–Cartan action. This topological θ term, in the SL(2,C) gauge theoretical representation, is related to the chiral anomaly and Pontryagin density. This topological term can be taken to be the fundamental entity and the CJD action which corresponds to the Einstein–Hilbert action is then induced from the nonvanishing chiral anomaly. This helps us to formulate the Einstein–Hilbert action as a symmetry-breaking effect in quantum field theory. In the region where we have only the θ term in the Lagrangian, we have topological gravity which implies the existence of a nilpotent operator Q behaving as a supercharge (BRST charge). This finds its relevance in the four-dimensional space–time geometry of Donaldson and as such the role of topological field theory in gravitational action can be taken to be of fundamental significance.

The association of the Donaldson theory of four-dimensional space–time manifold with the θ term in CJD action when the gauge group is taken to be SL(2,C) is also of significance from the viewpoint that (2+1)-dimensional gravity can be treated as a Chern–Simons theory and the Chern–Simons term follows from the Pontryagin term when we consider that the three-dimensional space is the boundary of the four-dimensional manifold. Indeed, as shown in a recent article,[21] the topological gauge theories in four, three, and two dimension through BRST invariance is associated with the quantization procedure and the topological gauge theories which incorporate the topological invariance of the partition function then emerge when the superspace generalization of this topological action is taken into account; this suggests a common physical origin of the space–time geometry in four, three, and two dimensions formulated by Donaldson,[21] Floer,[22] and Gromov,[23] respectively. Our above analysis of gravitational action in four-dimensional space–time in terms of topological field theory can then be generalized to three and two space–time dimensional manifolds in a natural way.

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