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On Triangular Representations of Aut(F_r)

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If ρ: $F_r \to GL_n(K)$ is a representation of a finitely generated free group F_r, and $\rho(a)^m = I$ for each *basic* element (i.e., element which occurs in some basis) a, then we show that if $\rho(F_r)$ is triangularisable, it is finite. This can be thought of as a generalisation of the Burnside problem for these linear groups.

1. INTRODUCTION

A classical lemma of Burnside shows that finitely generated, torsion groups that are linear over a field of characteristic 0 are actually finite. One might ask whether finiteness of a matrix group can be enforced by assuming only that a certain subset consists of elements of finite order. Clearly, it is not enough to assume that a set of generators have finite orders as there do exist infinite, linear groups like SL(2, $\mathbb{Z}$) which are generated by finitely many torsion elements. It is easily seen that there exist even triangularisable groups with the above property. If ρ: $F_r \to GL_n(K)$ is a representation of a finitely generated free group F_r, and $\rho(a)^m = I$ for each *basic* element (i.e., element which occurs in some basis) a, then a natural question is whether $\rho(F_r)$ is necessarily finite. Now, by Tits's well-known dichotomy, $\rho(F_r)$ must either contain a nonabelian free group or be virtually solvable, and therefore, virtually triangularisable. But, a conjecture of Formanek asserts [F] that under a representation of Aut(F_r) with $r \geq 3$, the image of Inn(F_r) $\cong F_r$ is virtually solvable and hence virtually triangularisable. Under our additional hypothesis that basic elements have finite order, it is therefore even more likely (although we have not been able to prove it yet) that $\rho(F_r)$ is virtually triangularisable. We prove the following finiteness result:

THEOREM. *Let F_r be a free group of rank $r \geq 2$, and let $\rho: F_r \to GL_n(K)$ be a representation over an arbitrary field K. Assume that for some m which is not a multiple of the characteristic of K, and for each basic element a of F_r, $\rho(a)$ has order m.*

Then $\rho(F_r)$ is triangularisable if, and only if, it is abelian and finite of order dividing m^r.

Since the automorphism group $\mathrm{Aut}(F_r)$ acts transitively on the basic elements of F_r, we have:

COROLLARY 1. *Let ρ: $\mathrm{Aut}(F_r) \to GL_n(K)$ be a representation. Assume that for some basic element a of F_r, the matrix $\rho(\mathrm{Inn}(a))$ has finite order m coprime to char(K). Then $\rho(\mathrm{Inn}(F_r))$ is triangularisable if, and only if, it is abelian and finite.*

COROLLARY 2. *With notation as in the theorem, if $m > n$ then $\rho(F_r)$ is finite if, and only if, it is triangularisable.*

Remarks. (i) A theorem of Formanek and Procesi [FP] shows that the automorphism group of a free group of rank at least 3 does not have a faithful linear representation. Their proof actually shows that under any representation of the automorphism group of a free group F_r of rank $r \geq 3$, the image of any free factor of F_r of rank $\leq r-1$ is virtually solvable and, therefore, virtually triangularisable. Formanek conjectures [F] that under a representation of $\mathrm{Aut}(F_r)$ with $r \geq 3$, the image of F_r itself is virtually solvable.

Thus, our theorem has some implication about certain representations of $\mathrm{Aut}(F_r)$.

(ii) Bass and Lubotzky [BL] recently investigated some questions on the groups $\mathrm{Aut}(F)$ and $\mathrm{Out}(F)$. In particular, they made comments on the question as to whether any virtually solvable subgroup of $\mathrm{Out}(F)$ is virtually abelian.

Proof of Corollary 2. From P.121 of [W], a finite subgroup of $GL_n(K)$ with exponent $m > n$ is triangularisable.

The basic ingredient of the proof of the theorem is a description of basic elements given by a result of Osborne and Zieschang:

THEOREM [OZ]. *In the free group $F_2 = F(x_1, x_2)$ of rank 2, all basic elements, upto conjugacy, are parametrized by pairs m, n of coprime integers and are given by words $w(m, n)$ defined as follows.*

Let $m, n > 0$ and $(m, n) = 1$.

Define $f_{m,n}(k) = 1$ *if* $1 \le k \le m$, $f_{m,n}(k) = 2$ *if* $m < k \le m+n$, *and* $f_{m,n}(k) \equiv f_{m,n}(k')$ *if* $k \equiv k' \bmod (m+n)$. *Then*

$$w(m,n) = \prod_{i=0}^{m+n-1} x_{f_{m,n}(1+im)}$$

For $m < 0$, *form the word* $w'(-m,n)$ *in* x_1^{-1} *and* x_2 *and define* $w(m,n) = w'(-m,n)$. *Similarly, for* $n < 0$, $w(m,n)$ *is defined to be the word* $w'(m,-n)$ *in* x_1 *and* x_2^{-1}. *If* $mq - np = 1$, *then* $\langle w(m,n), w(p,q)\rangle = \langle x_1, x_2\rangle$.

If the image is abelian, it is obviously triangularisable and we will prove the converse. Assuming that $G := \rho(F_r)$ is triangularisable, we will show that the images in G of two arbitrary basic elements of F_r have to commute. Let $\{a,b\}$ be any two basic elements of F_r. We write $A = \rho(a)$ and $B = \rho(b)$. Then

$$A = \begin{pmatrix} a_1 & * & \cdots & * \\ 0 & \ddots & \cdots & * \\ \vdots & 0 & \ddots & * \\ 0 & \cdots & 0 & a_n \end{pmatrix}$$

$$B = \begin{pmatrix} b_1 & * & \cdots & * \\ 0 & \ddots & \cdots & * \\ \vdots & 0 & \ddots & * \\ 0 & \cdots & 0 & b_n \end{pmatrix}.$$

LEMMA 1. *For* $i < j$, *the basic word* $w = w(r,s)$ *in* a, b *given by the theorem of Osborne and Zieschang is such that its image* $W = \rho(w)$ *satisfies* $w_{ii} = w_{jj}$.

Proof of Lemma 1. If either $a_i = a_j$ or $b_i = b_j$, there is nothing to prove. We may assume that $a_i \ne a_j$ and $b_i \ne b_j$. Let r, s be coprime integers. We notice that the diagonal entries of $\rho(w)$ are $\{a_i^r b_i^s : 1 \le i \le n\}$, since r, s are respectively the total powers of a and b occurring in w. So we need to show that $\exists (r,s) = 1$ such that $a_i^r b_i^s = a_j^r b_j^s$. Now, a_i, b_i are mth roots of unity in K. Let us write $a_i = \zeta^{t_i}$ and $b_i = \zeta^{s_i}$, where ζ is a generator of the cyclic group $\{x \in K : x^m = 1\}$. So $a_i^r b_i^s = a_j^r b_j^s$ in coprime integers r, s if, and only if,

$$\begin{pmatrix} t_i & s_i \\ t_j & s_j \end{pmatrix}\begin{pmatrix} r \\ s \end{pmatrix} = \begin{pmatrix} D \\ D \end{pmatrix} \qquad \text{for some integer } D.$$

Let us look at the matrix

$$\Omega = \begin{pmatrix} t_i & s_i \\ t_j & s_j \end{pmatrix}.$$

If $\det(\Omega) = D$, we have

$$\Omega \begin{pmatrix} s_j - s_i \\ t_i - t_j \end{pmatrix} = \begin{pmatrix} D \\ D \end{pmatrix}.$$

Note that $t_i \neq t_j$, and $s_i \neq s_j$. Note further that if $d = (s_j - s_i, t_i - t_j)$, then d divides D and therefore, we can divide by d to get coprime r, s such that $\Omega\binom{r}{s} = \binom{D/d}{D/d}$. Hence the lemma is proved.

LEMMA 2. *Let $g \in GL_n(K)$ be an upper triangular matrix of finite order. If $g_{ii} = g_{jj}$ for some $i < j$, then g_{ij} is a linear combination of the products of g_{kl} with $l - k < j - i$ with coefficients depending only on g_{ii}.*

Proof. If $g^k = I$, then a simple computation shows

$$0 = (g^k)_{ij} = k g_{ii}^{k-1} g_{ij} + S$$

where S is a linear combination of the products of g_{kl} with $l - k < j - i$ with coefficients depending only on g_{ii}.

COROLLARY 3. *For $i \geq 1$, let w be basic word (as in Lemma 1) with $w_{ii} = w_{i+1,i+1}$. Then $w_{i,i+1} = 0$.*

LEMMA 3. *Let $s \geq 0$ and assume that $(AB)_{ij} = (BA)_{ij}$ for all $j - i \leq s$. Then, for a word w in A, B and an element $\tilde{w} \in w[G, G]$, we have $w_{ij} = \tilde{w}_{ij}$ for all $j - i \leq s$.*

Proof. We prove this by induction on $j - i$. The start of induction at $j - i = 0$ is trivial. Assume that $i < j$ and that the assertion is true for k, l with $l - k < j - i$. First, it is proved easily by induction on $u + v$ that $(A^u B^v)_{ij} = (B^v A^u)_{ij}$ for all $u, v \geq 0$. As A, B have order m, this assertion is true for all integers u, v. Now, let w be any word in A, B say, $w = A^{r_1} B^{s_1} \cdots A^{r_n} B^{s_n}$. Once again, a routine proof by induction on $\sum |r_i| + |s_i|$ shows that $w_{ij} = (A^u B^v)_{ij} = (B^v A^u)_{ij}$, where u, v are, respectively, the powers of A, B occurring in w. This proves the lemma.

PROPOSITION 1. *Let w be a basic element of $\langle a, b \rangle$. Then there exist $g_1, g_2 \in \langle A, B \rangle$ such that:*

(i) *$W := \rho(w) = g_1 B A g_2$, and*

(ii) *W is conjugate to $g_1 A B g_2$ by an element of $\langle A, B \rangle$.*

We first complete the proof of the theorem using the above proposition and the lemmata.

PROOF OF THE THEOREM

We shall prove by induction on $j-i$ that $(AB)_{ij} = (BA)_{ij}$. This is true for $j-i=0$. Let $i \geq 1$. By Lemma 1, we find a basic element w in $\langle a, b \rangle$ such that $W = \rho(w)$ satisfies $w_{ii} = w_{i+1,\,i+1}$. By Corollary 3, $w_{i,\,i+1} = 0$. Now, the proposition gives elements $g_1, g_2 \in \langle A, B \rangle$ such that $W = g_1 AB g_2$ and W is conjugate to $\tilde{W} := g_1 BA g_2$. Hence $\tilde{W}^m = I$. By applying Lemma 2, we get $\tilde{w}_{i,\,i+1} = 0$. Thus,

$$\begin{aligned} w_{i,\,i+1} &= (g_1 AB g_2)_{i,\,i+1} \\ &= (g_1)_{ii}(g_2)_{i,\,i+1}(AB)_{ii} + (g_1)_{i,\,i+1}(g_2)_{i+1,\,i+1}(AB)_{i+1,\,i+1} \\ &\quad + (g_1)_{ii}(g_2)_{i+1,\,i+1}(AB)_{i,\,i+1} \\ &= \tilde{w}_{i,\,i+1} = (g_1 BA g_2)_{i,\,i+1} \\ &= (g_1)_{ii}(g_2)_{i,\,i+1}(BA)_{ii} + (g_1)_{i,\,i+1}(g_2)_{i+1,\,i+1}(BA)_{i+1,\,i+1} \\ &\quad + (g_1)_{ii}(g_2)_{i+1,\,i+1}(BA)_{i,\,i+1} \end{aligned}$$

evidently gives $(AB)_{i,\,i+1} = (BA)_{i,\,i+1}$. Assume that $i < j$ and that $(AB)_{kl} = (BA)_{kl}$ for all $l-k < j-i$. Once again, we can choose a basic element w in $\langle a, b \rangle$ with $w_{ii} = w_{jj}$ and elements $g_1, g_2 \in \langle A, B \rangle$ so that $W = \rho(w) = g_1 AB g_2$ is conjugate to $\tilde{W} := g_1 BA g_2$. Hence $\tilde{W}^m = I$. By applying Lemma 2, we get $\tilde{w}_{ij}$ is a linear combination of the products of $\tilde{w}_{kl}$ with coefficients depending only on $\tilde{w}_{ii} = w_{ii}$. Hence, $w_{ij} = \tilde{w}_{ij}$. Now, by the induction hypothesis, we have $(AB)_{kl} = (BA)_{kl}$ for all $l-k < j-i$. This implies, by Lemma 3, that for all $h \in \langle A, B \rangle$ and $\tilde{h} \in h[G, G]$, one has $h_{kl} = \tilde{h}_{kl}$ for all $l-k < j-i$. Expanding w_{ij} and $\tilde{w}_{ij}$, all terms match except possibly the term corresponding to $(AB)_{ij}$ and $(BA)_{ij}$. Thus, these terms have to match too. Therefore, $(g_1)_{ii}(AB)_{ij}(g_2)_{jj} = (g_1)_{ii}(BA)_{ij}(g_2)_{jj}$. This proves $(AB)_{ij} = (BA)_{ij}$ and the theorem is proved.

PROOF OF THE PROPOSITION

We look more closely at the description of basic elements given by the theorem of Osborne and Zieschang.

It is easily seen that for coprime $r, s > 0$, the corresponding basic word is $w = ab^{q_1}ab^{q_2}\cdots ab^{q_r}$ with

$$q_1 = [(s-1)/r], \qquad q_i = [(is-1)/r] - [((i-1)s-1)/r] \qquad \text{for } i > 1.$$

Without loss of generality, we may assume that $r < s$ (for the case $r = s = 1$, the proposition is trivial). Let us write $s = lr + k$ with $1 \leq k < r$. Then a simple analysis shows that

$$w = ab^{l+1}(ab^l)^{f_1}ab^{l+1}(ab^l)^{f_2}\cdots ab^{l+1}(ab^l)^{f_k}$$

where

$$f_i = \begin{cases} g_i - 1 & \text{if } ri, r(i-1) \not\equiv -1(k) \\ g_i & \text{if } r(i-1) \equiv -1(k) \\ g_i - 2 & \text{if } ri \equiv -1(k) \end{cases}$$

and where $g_i = [(ir+1)/k] - [((i-1)r+1)/k]$. Thus, $f_1 = [r/k] - 1$, $f_k = [r/k]$, and each f_i is either $[r/k]$ or $[r/k] - 1$.

If $k = 1$, then $w = ab^{l+1}(ab^l)^{r-1}$. Writing $g_1 = I$ and $g_2 = B^l(AB^l)^{r-1}$, we have $W = \rho(w) = g_1 AB g_2$. Then $g_1 BA g_2 = B(AB^l)^r = (AB^l)^{-1} W (AB^l)$, which proves the proposition in the case $s \equiv 1 \bmod r$.

So we may assume that $k > 1$. We write $r = uk + v$ with $1 \le v \le k - 1$. Then the above expression for the f_i's can be further rewritten as

$$f_i = \begin{cases} u - 1 & \text{if } vi \equiv -1(k) \\ u & \text{if } v(i-1) \equiv -1(k) \\ u - 1 & \text{if } vi, v(i-1) \not\equiv -1(k), \\ & \quad [(iv+1)/k] = [((i-1)v+1)/k] \\ u & \text{if } vi, v(i-1) \not\equiv -1(k), \\ & \quad [(iv+1)/k] = [((i-1)v+1)/k] + 1. \end{cases} \qquad (*)$$

When $v = 1$,

$$W = \{AB^{l+1}(AB^l)^{u-1}\}^{k-1} AB^{l+1}(AB^l)^u = g_1 BA g_2$$

where $g_1 = \{AB^{l+1}(AB^l)^{u-1}\}^{k-1} AB^l$ and $g_2 = B^l(AB^l)^{u-1}$. Therefore, we have

$$\tilde{W} := g_1 AB g_2 = \{AB^{l+1}(AB^l)^{u-1}\}^{k-2} AB^{l+1}(AB^l)^u AB^{l+1}(AB^l)^{u-1}.$$

Noticing that $\tilde{W} = gWg^{-1}$ with $g = \{AB^{l+1}(AB^l)^{u-1}\}^{-1}$, the proposition follows in the case $v = 1$.

When $v = k - 1$, one has

$$W = AB^{l+1}(AB^l)^{u-1}\{AB^{l+1}(AB^l)^u\}^{k-1} = g_1 BA g_2$$

where $g_1 = AB^{l+1}(AB^l)^u$ and $g_2 = B^l(AB^l)^{u-1}\{AB^{l+1}(AB^l)^u\}^{k-2}$. As $g_1 AB g_2 = gWg^{-1}$ with $g = AB^{l+1}(AB^l)^u$, the case $v = k - 1$ also follows.

So we may assume that $1 < v < k - 1$. As usual, to prove the general case, we require more information on the v_i's which is contained in the following lemma. We let $1 < v^{-1} < k - 1$ denote the inverse of v modulo k. Then:

LEMMA 4. *$f_{k-v^{-1}} = u - 1$, $f_{k+1-v^{-1}} = u$, and*

$$f_i = \begin{cases} f_{i-v^{-1}} & \text{if } k > i > v^{-1} \\ f_{k+i-v^{-1}} & \text{if } 1 < i \le v^{-1}. \end{cases}$$

In other words, defining for each integer n, f_n to be f_i where $n \equiv i$ mod k and $1 \leq i \leq k$, the lemma asserts $f_i = f_{i-v^{-1}}$ for all $i \neq 1, k$. Write

$$\tilde{i} = \begin{cases} i - v^{-1} & \text{if } i > v^{-1} \\ k + i - v^{-1} & \text{if } i \leq v^{-1}. \end{cases}$$

The hypothesis of the lemma means that $\tilde{i} \neq k - v^{-1}, k + 1 - v^{-1}$. Now, $\tilde{i}v \equiv iv - 1 \not\equiv -1$ and $(\tilde{i} - 1)v \equiv (i-1)v - 1 \not\equiv -1$ modulo k since $i \neq 1, k$. Therefore, $f_{\tilde{i}} = u - 1$ or u accordingly as $[(\tilde{i}v + 1)/k] = [((\tilde{i} - 1)v + 1)/k]$ or $[(\tilde{i}v + 1)/k] = [((\tilde{i} - 1)v + 1)/k] + 1$.

Proof of Lemma 4. The assertions $f_{k-v^{-1}} = u - 1$ and $f_{k+1-v^{-1}} = u$ are clear from the description in the expression $(*)$. Let $i \leq k$.

There are four cases:

Case I. $iv \equiv -1$ mod k i.e., $f_i = u - 1$. Now, $(\tilde{i}v \equiv iv - 1 \equiv -2$ modulo k. So $[(\tilde{i}v + 1)/k] = [(\tilde{i}v + 2)/k - 1/k] = (\tilde{i}v + 2)/k - 1$, and $[((\tilde{i} - 1)v + 1)/k] = [(\tilde{i}v + 2)/k - (v+1)/k] = (\tilde{i}v + 2)/k - 1$ since $v < k - 1$. Thus, we have, by $(*)$, that $f_{\tilde{i}} = u - 1 = f_i$.

Case II. $(i-1)v \equiv -1$ mod k, i.e., $f_i = u$. Then $\tilde{i}v \equiv iv - 1 \equiv v - 2$ modulo k. Since $[((\tilde{i} - 1)v + 1)/k] = [((\tilde{i} - 1)v + 2)/k - 1/k] = (\tilde{i} - 1)v + 2/k - 1$, and $[(\tilde{i}v + 1)/k] = [((\tilde{i} - 1)v + 2)/k + (v-1)/k] = (\tilde{i} - 1)v + 2/k$, we get $f_{\tilde{i}} = u$.

Case III. $iv, (i-1)v \not\equiv -1$ mod k but $f_i = u - 1$. Now, by $(*)$, $[(iv + 1)/k] = [((i-1)v + 1)/k] = d$, say. Then $(iv + 1)/k = d + \theta$ for some $0 < \theta < 1$. From $d = [((i-1)v + 1)/k] = [d + \theta - v/k]$, one concludes that $\theta > v/k$. Then

$$[(\tilde{i}v + 1)/k] = \begin{cases} d + \theta - 1/k & \text{if } i > v^{-1} \\ d + \theta - 1/k + v & \text{if } i \leq v^{-1} \end{cases}$$

$$= \begin{cases} d & \text{if } i > v^{-1} \\ d + v & \text{if } i \leq v^{-1} \end{cases}$$

since $\theta > v/k > 1/k$.

Similarly,

$$[((\tilde{i} - 1)v + 1)/k] = \begin{cases} d + \theta - (v+1)/k & \text{if } i > v^{-1} \\ d + \theta - (v+1)/k + v & \text{if } i \leq v^{-1}. \end{cases}$$

As $k\theta > v$ and is an integer, we have $k\theta > v + 1$, which immediately gives $[(\tilde{i}v + 1)/k] = [((\tilde{i} - 1)v + 1)/k]$, i.e., $f_{\tilde{i}} = u - 1 = f_i$.

Case IV. $iv, (i-1)v \not\equiv -1 \bmod k$ but $f_i = u$. By $(*)$, $[(iv+1)/k] = [((i-1)v+1)/k] + 1 = d$, say. Writing $(iv+1)/k = d + \theta$ with $0 < \theta < 1$, the fact that $d - 1 = [((i-1)v+1)/k] = [d + \theta - v/k]$ gives $\theta < v/k$. Now,

$$[((\tilde{i}-1)v+1)/k] = \begin{cases} d+\theta-(v+1)/k & \text{if } i > v^{-1} \\ d+\theta-(v+1)/k+v & \text{if } i \le v^{-1} \end{cases}$$

$$= \begin{cases} d-1 & \text{if } i > v^{-1} \\ d-1+v & \text{if } i \le v^{-1} \end{cases}$$

since $\theta < v/k < (v+1)/k$.

On the other hand,

$$[(\tilde{i}v+1)/k] = \begin{cases} d+\theta-1/k & \text{if } i > v^{-1} \\ d+\theta-1/k+v & \text{if } i \le v^{-1} \end{cases}$$

$$= \begin{cases} d & \text{if } i > v^{-1} \\ d+v & \text{if } i \le v^{-1} \end{cases}$$

since $k\theta$, being a non-zero positive integer, is at least 1. Hence, again we have $[(\tilde{i}v+1)/k] = [((\tilde{i}-1)v+1)/k] + 1$, i.e., $f_{\tilde{i}} = u = f_i$.

This completes the proof of the lemma in all cases.

The proof of the proposition is now completed in the following way.

$$W = AB^{l+1}(AB^l)^{f_1} AB^{l+1}(AB^l)^{f_2} \cdots AB^{l+1}(AB^l)^{f_k} = g_1 BA g_2$$

where

$$g_1 = AB^{l+1}(AB^l)^{f_1} \cdots AB^{l+1}(AB^l)^{f_{k-v^{-1}}} AB^l$$

and

$$g_2 = B^l(AB^l)^{f_{k-v^{-1}+1}-1} AB^{l+1}(AB^l)^{f_{k-v^{-1}+2}} \cdots AB^{l+1}(AB^l)^{f_k}.$$

Hence

$$\tilde{W} := g_1 AB g_2 = AB^{l+1}(AB^l)^{h_1} AB^{l+1}(AB^l)^{h_2} \cdots AB^{l+1}(AB^l)^{h_k}$$

where $h_i = f_i$ for $i \neq k - v^{-1}, k - v^{-1} + 1$, $h_{k-v^{-1}} = f_{k-v^{-1}} + 1 = u$, and $h_{k-v^{-1}+1} = f_{k-v^{-1}+1} - 1 = u - 1$. Hence, $\tilde{W} = gWg^{-1}$ with $g = AB^{l+1}(AB^l)^{h_1} \cdots AB^{l+1}(AB^l)^{h_{k-v^{-1}}}$. This completes the proof of the proposition.

2. CONCLUDING REMARKS

When one of the basic elements, say a, maps to a diagonal matrix, then the proof of the fact that $A = \rho(a)$ commutes with $B = \rho(b)$, for every basic element b is somewhat easier and we present it below.

Suppose, if possible, that $AB \neq BA$. Let $(AB)_{i,j} \neq (BA)_{i,j}$ with $j - i$ least possible. Now, $(AB)_{i,j} \neq (BA)_{i,j}$ means that $a_i b_{i,j} \neq a_j b_{i,j}$. So, $b_{i,j} \neq 0$. Let $i < k_1 < \cdots < j$ be any chain of positive integers. By the minimality of $j - i$ with the property that $(AB)_{i,j} \neq (BA)_{i,j}$, it follows that if none of $b_{i,k_1}, b_{k_1,k_2}, \ldots, b_{k_r,j}$ is zero, then $a_i = a_{k_1} = \cdots = a_j$, a contradiction. Hence, the product $b_{i,k_1} \cdots b_{k_r,j} = 0$ for any chain $i < k_1 \cdots < j$. Hence, we have, by Lemma 2, that $b_i \neq b_j$. From the proof of Lemma 1, there are r, s coprime integers such that the corresponding basic element $w = w(r, s)$ has the property that $W_{i,i} = W_{j,j}$. Here, as before, W stands for $\rho(w)$. In fact, recall from the proof of Lemma 1 that the r, s respectively divide $s_i - s_j$ and $t_i - t_j$. Since we have $a_i \neq a_j$ and $b_i \neq b_j$, we have that $\zeta^r \neq 1$ and $\zeta^s \neq 1$ (we may assume that $0 \leq t_k < m$). We have the following proposition.

PROPOSITION 2.

$$W_{i,i} = W_{j,j}$$

$$W_{i,j} = \zeta^{rt_j} \zeta^{(s-1)t_{\sigma(i)}} \zeta^{r(s-1)+1} \frac{\zeta^s - 1}{\zeta - 1} b_{i,j}.$$

In particular, $W_{ij} \neq 0$.

Proof. Now, $w = w(r, s) = ab^{\beta_1} ab^{\beta_2} \cdots ab^{\beta_r}$ for some non-negative integers β_i such that $\sum_{i=1}^{r} \beta_i = s$. An elementary calculation shows that for positive integers s, l, we have

$$((AB^l)^s)_{i,j} = \frac{(b_i^l - b_j^l)(a_i^s b_i^{sl} - a_j^s b_j^{sl})}{(b_i - b_j)(a_i b_i^l - a_j b_j^l)} a_i b_{i,j}.$$

Using this, and simplifying, we get

$$W_{i,j} = \frac{\zeta^{rt_j} \zeta^{(s-1)t_{\sigma(i)}}}{\zeta^r - 1} b_{i,j} \theta \qquad (\spadesuit)$$

where

$$\begin{aligned}\theta &= \zeta^{rs+s} - \zeta^{r(\beta_2+\cdots+\beta_r)+s} + \zeta^{r(\beta_2+\cdots+\beta_r)+2s} - \zeta^{r(\beta_3+\cdots+\beta_r)+2s} \\ &\quad + \cdots + \zeta^{r\beta_r+rs} - \zeta^{rs} \\ &= \zeta^{rs}(\zeta^s - 1)(1 + \zeta^{s-r\beta_1} + \cdots + \zeta^{(s-r\beta_1)+\cdots+(s-r\beta_{r-1})}).\end{aligned}$$

Now, from the theorem of Osborne and Zieschang, it is not difficult to show

$$\beta_1 = q_1 + 1, \qquad \text{where } s-1 = q_1 r + l_1 \text{ with } 0 \le l_1 < r,$$

$$\beta_{i+1} = q_{i+1}, \qquad \text{where } s + l_i = q_{i+1} r + l_{i+1} \text{ for } 1 < i < r.$$

Feeding this into the expression for θ, we get

$$\begin{aligned}\theta &= \zeta^{rs}(\zeta^s - 1)(1 + \zeta^{1+l_1-r} + \cdots + \zeta^{1+l_{r-1}-r}) \\ &= \zeta^{r(s-1)+1}(\zeta^s - 1)(\zeta^{r-1} + \zeta^{l_1} + \cdots + \zeta^{l_{r-1}}).\end{aligned}$$

Now, the expressions for l_i show that $l_i \equiv is - 1 \bmod r$. Hence, $l_1, \dots, l_{r-1}$ are distinct and are different from $r-1$ since r, s are coprime. Thus, they are just the integers from 0 to $r-2$. Therefore,

$$\theta = \zeta^{r(s-1)+1}(\zeta^s - 1)(1 + \zeta + \cdots + \zeta^{r-1}) = \zeta^{r(s-1)+1}\frac{(\zeta^s - 1)(\zeta^r - 1)}{\zeta - 1}.$$

Putting this in (♠), we get the expression in the proposition.

The proof is completed by the following observation. For any $k < l$ it is evident that $W_{k,l}$ is a sum of terms of the form $c_I b_{k,k_1} \cdots b_{k_u,l}$, where I is a chain $k \le k_1 \le \cdots \le k_u \le l$. Therefore, the hypothesis implies that $W_{k,l} = 0$ for all $k < l$ with $l - k < j - i$. So, by Lemma 2, $W_{i,j} = 0$. This contradicts the above proposition.

3. ACKNOWLEDGMENTS

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