Sankhyā : The Indian Journal of Statistics
1985, Volume 47, Series B, Pt. 1, pp. 67-77.

BALANCING INCOMPLETE BLOCK DESICNS

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SUMMARY. Concept of balance of incomplete block designs has been generalised. Several new series of efficiency balanced designs are presented. A generalised efficiency balanced design is illustrated.

1. Introduction

Balanced designs available in literature are either variance balanced or efficiency balanced. The variance balanced designs can have both equal and unequal numbers of replications and block sizes.

With the introduction of efficiency balanced design through the work of Calinski (1971), Puri and Nigam (1975a), Williams (1975) and others, the concept of balance has undergone a change. Once such a change gains ground further modifications are likely to follow.

The present paper thus aims at a more general definition of balance in designs such that all the existing concepts of balance become its special cases. Several new series of balanced designs including efficiency balanced designs have also been presented as reinforced incomplete block designs.

2. On balanced designs

Balanced incomplete block (B.I.B.) designs were first introduced by Yates (1936) for varietal trials. As these designs require a large number of experimental plots, alternative designs like partially balanced incomplete block (PBIB) designs, Lattice designs etc. were subsequently introduced through the work of Yates, Bose, Nair, and others. In these designs the concept of balance consisted of comparisions between two designs, namely an incomplete block design and a randomised block design having the same number of treatments and replications, such that the ratio of the variances of the estimates of any treatment contrast for the two designs is constant. When

AMS (1980) *subject classification :* 62K10.

Key words and phrases : Generalised efficiency balance, Efficiency factor, Balance from comparison of designs.

subsequently the utility of incomplete block designs with unequal numbers of replications was realised, further series of incomplete block designs with unequal numbers of replications of treatments were introduced. Some of these series of designs were balanced but not in the above sense. Because of unequal numbers of replications of the treatments, the designs to be compared, are an incomplete block design and a corresponding completely randomised design with the same numbers of varieties and replications, the former being balanced if the ratio of the variances of the estimates of any treatment contrast obtainable through the two designs is constant. Designs which are balanced according to the above concept have been called efficiency balanced designs. In the present investigation the above concept of balance has been further extended in the following way. When an incomplete block design is compared against any other design i.e., either completely randomised or randomised block, both having the same number of treatments but not necessarily the same number of replications such that the ratio of the variances of the estimates of any treatment contrast for the two designs is constant, then such an incomplete block design has been called generalised efficiency balanced design.

2.1 *Requirement of balanced designs and their analysis.* In two way classified data n_{ij} observations in the (i, j)-th cell $(i = 1, 2, \ldots, t; j = 1, 2, \ldots, b)$, defined by the levels of two factors, say, treatments and blocks, the reduced normal equations for estimating the treatments effects t_i $(i = 1, 2, \ldots, t)$ after eliminating the block effects are,

$$C_{ii}t_i - \sum_{i \neq m} C_{im}t_m = Q_i$$

where
$$C_{im} = \sum_j \frac{n_{ij}n_{mj}}{n_{.j}} \text{ and } Q_i\text{'s}$$

are the adjusted treatment totals. It is known that

$$C_{ii} - \sum_{m \neq i} C_{im} = 0.$$

Thus
$$C_{ii} = \sum_{m \neq i} C_{im}.$$

Let $s_1, s_2, \ldots, s_t$ be a set of t positive numbers. We assume further,

$$C_{im} = cs_is_m$$

where c is a constant. On this assumption the reduced normal equations become

$$C_{ii}t_i - cs_i \sum_{m \neq i} s_m t_m = Q_i \quad (i = 1, 2, \ldots, t).$$

Since

$$C_{ii} = \sum_{m \neq i} C_{im} = cs_i \sum_{m \neq i} s_m,$$

the above normal equations become

$$\Big(\sum_{m \neq i} s_m\Big) t_i - \sum_{m \neq i} s_m t_m = \frac{Q_i}{cs_i}.$$

Taking the restriction $\sum_{m=1}^{t} s_m t_m = 0$, the equation reduces to

$$\Big(\sum_{m=1}^{t} s_m\Big) t_i = Q_i / cs_i.$$

i.e

$$t_i = \frac{1}{c\Sigma s_m} \frac{Q_i}{s_i}$$

The s.s. due to treatments adjusted for blocks is thus

$$\Sigma t_i Q_i = \frac{1}{c\Sigma s_m} \Sigma Q_i^2 / s_i$$

$$\text{variance of } (t_i - t_m) = \frac{\sigma^2}{c\Sigma s_m} \Big(\frac{1}{s_i} + \frac{1}{s_m}\Big).$$

Thus a design in which

$$C_{im} = c_i s_i s_m \quad (i, m = 1, 2, \ldots, t)$$

can be said to be balanced with reference to a design with the same set of treatments for which

$$\text{variance of } (t_i - t_m) = \sigma_1^2 \Big(\frac{1}{s_i} + \frac{1}{s_m}\Big)$$

where, σ_1^2 may be different from σ^2.

The efficiency factor of such a balanced design is thus $c.\Sigma s_m$. These designs have been called general efficiency balanced designs. This definition of efficiency balance has wider scope than the definition of existing efficiency balanced designs. If $s_1, s_2, \ldots, s_v$ are equated to the numbers of replications of the treatments in a design, and the design against which it is compared is taken as completely randomised design with these replications, we get the existing E.B. designs. If $s_1 = s_2 = \ldots = s_t = s$ where s is the number of replication of the design against which the design in question is compared, we get the variance balanced designs, where the block sizes and replications numbers may be either equal or unequal. In all other situations regarding the behaviour of the s 's, the design will be neither the existing efficiency balanced nor the usual balanced designs. Suppose there is an incomplete block design where the different treatments are unequally replicated and it is intended to compare its efficiency in relation to a randomised block design with $\bar{r}$ as the number of replications where $\bar{r}$ is the average number of replications of the treatments, then each of s_i is to be taken as $\bar{r}$ for obtaining the efficiency factor. Such designs belong to the third category mentioned above. Other situations where the set of numbers $s_1, s_2, \ldots, s_t$ need not be equal to the replications of the treatments for obtaining the efficiency of a design are not unlikely.

For such a balanced design if for any given pair of treatments

$$(i \text{ and } m), \quad \sum_j \frac{n_{ij} n_{mj}}{n_{.j}}$$

is denoted by λ_{im}, then

$$c = \frac{\lambda_{im}}{s_i s_m}.$$

Puri and Nigam (1975b) pointed out that by collapsing a group of treatments in a B.I.B. design so as to replace each of them by a new treatment, efficiency balanced designs are obtained. These results are also true when more than one such group is taken such that each group is replaced by a different treatment. These results actually hold for the general efficiency balanced designs defined here.

Let a set of treatments $t_1, t_2, \ldots, t_n$ out of t treatments in a general efficiency balanced design be each replaced by a new treatment, t_N. Now in the thus modified design there are $t-n+1$ treatments. The set of numbers s_i's for the new designs are taken as $s_N = s_1+s_2+\ldots+s_n$, s_i's for the non-collapsed treatments remain the same.

In the modified design

$$\begin{aligned} C_{iN} &= \sum_j \frac{n_{ij}(n_{1j}+n_{2j}+\ldots+n_{nj})}{n_{.j}} \\ &= cs_i(s_1+s_2+\ldots+s_n) \\ &= cs_i s_N. \end{aligned}$$

Thus the modified design remains balanced with same c.

The same results hold when there are several sets of treatments and each set of treatments is replaced by a different treatment. Some particular cases of these designs as obtainable by allotting different sets of values to s_i's are discussed below.

2.2 *Efficiency balanced designs.* In the existing efficiency balanced designs s_i's are taken to be equal to r_i's ($i = 1, 2, \ldots, t$) and so,

$$V(t_l - t_m) = \frac{\sigma^2}{c\Sigma r_i}\left(\frac{1}{r_l}+\frac{1}{r_m}\right)$$

$$= \frac{1}{cn}\left(\frac{1}{r_l}+\frac{1}{r_m}\right)$$

where n is the total number of observations in the design.

In such designs

$$C_{lm} = cr_l r_m.$$

This has been pointed out by Puri and Nigam (1977).

2.3 *Variance balanced designs.* When $s_1 = s_2 = \ldots = s_t = s$

$$\text{var}(t_l - t_m) = \frac{2\sigma^2}{cts^2}.$$

The efficiency factor in relation to a randomised block design with replications r is thus $\frac{cts^2}{r}$. Here $s = r$.

These are actually variance balanced designs which can have both equal and unequal numbers of replications and block sizes.

3. Some new series of efficiency balanced designs

Das (1958) defined a series of design called Reinforced designs which were obtained by including (i) several additional treatments in each block of an existing incomplete block design and also (ii) any number $n \geqslant 0$ of additional blocks each containing all the treatments. It appears that the efficiency balanced block designs available so far are obtained from one or more B.I.B. designs by some type of reinforcement of B.I.B. designs. It is interesting to note incidentally that no efficiency balanced design could be obtained where the number of treatments is greater than $v+1$ where v is the number of treatment in the B.I.B. design which has been reinforced. We present below some new series of designs as Reinforced designs and avoid matrix notation.

3.1. *Efficiency balanced designs with v treatments.* A B.I.B. design with parameters v, b, r, k and λ when reinforced by taking any number $n \geqslant 0$ of extra blocks such that in each of these blocks each treatment occurs once except that any particular treatment, say, the first one occurs r/λ times in each of the n extra blocks, gives an efficiency balanced design with

$$s_i = r_i, \ c = \frac{\lambda}{kr(r+n)}$$

$$r_1 = \frac{r(n+\lambda)}{\lambda}$$

and $r_i = r+n$ for other treatments $(i \neq 1)$. These designs are illustrated in Section 4.

3.2. *Efficiency balanced designs with $v+1$ treatments.* A B.I.B. design when reinforced by taking $n \geqslant 0$ blocks each containing each of the existing treatments once together with one more treatment, say, t_0 occurring $\frac{r-\lambda}{\lambda}$ times, is an efficiency balanced design with $t = v+1$

$$s_i = r_i, \ c = \frac{\lambda}{rk(r+n)}$$

$$s_0 = r_0 = \frac{n(r-\lambda)}{\lambda}, \ r_i = r+n \ (i \neq 0),$$

where s_0 denotes the number for the extra treatment.

3.3. *Another series of efficiency balanced design with $v+1$ treatments.* A B.I.B. design when reinforced by one extra treatment, say t_0, which occurs p times in each of the b blocks and by $n > 0$ extra blocks each containing each of the v treatments in the B.I.B. design q times, but not the extra treatment gives a series of efficiency balanced design when r, p and q are so taken that $npq = r-\lambda$.

For this series of designs $c = \dfrac{r}{b(k+p)(r+nq)}$

$$s_0 = r_0 = bp, \ s_i = r_i = r+nq \ (i \neq 0).$$

where s_0 denotes the number for the extra treatment.

In all the designs in 3.1, 3.2 and 3.3 only two values of C_{lm}'s, say, C_{lm} and $C_{l'm'}$ are possible when all possible pairs of treatments are considered. Given the set of numbers $s_1, s_2, \ldots, s_t$, these can be expressed as $C_{lm} = c_1 s_l s_m$

and $C_{i'm'} = c_2 s_{i'} s_{m'}$. For balance c_1 has to be equal to c_2 and this equality can be ensured by suitably fixing the frequency of occurrence of one of the existing treatments as in 3.1 or of an extra treatment as in 3.2 and 3.3.

4. Some particular cases of efficiency balanced designs

4.1. The efficiency balanced design reported by Calinski (1971) can be obtained as a particular case from the series of designs in 3.1 when the B.I.B. design used has parameters

$$v = b = 3,\ r = k = 2,\ \lambda = 1 \text{ and } n = 2.$$

4.2. The E.B. design reported by Williams (1975) can be obtained as a particular case from the series of designs in 3.2 when the B.I.B. design used has again the same parameters

$$v = b = 3,\ r = k = 2,\ \lambda = 1 \text{ and } n = 2.$$

4.3. The first series of E.B. design reported by Dey and Singh (1980) can be obtained as a particular case of the series of designs in 3.3 by taking $p = q = n = 1$ when the B.I.B. design used has parameters

$$v = b,\ r = k = 1,\ \lambda = 0.$$

4.4. By taking $p = q = 1$ and $n = r = \lambda$ in 3.3 we can always get binary E.B. designs from any B.I.B. design.

4.5. As $r-\lambda$ is the same both for a B.I.B. design and its complementary design, we can get two E.B. designs from the above two B.I.B. designs with the same values of n, p and q.

5. Balanced designs

We have presented below some new series of balanced designs where $s_1 = s_2 = \ldots = s_i = 1$ so that the variance of $(t_1 - t_m)$ is constant for any choice of i and m.

5.1. *Balanced designs with* $v+1$ *treatments.* A B.I.B. design when reinforced with any number $n > 0$ blocks each containing each of the existing v treatments once together with one extra treatment t_0 which occurs in each of the n extra blocks p times, is a balanced design when

$$p = \frac{1+\lambda(v+1)}{nk-\lambda}.$$

Here $c = \dfrac{np}{v+p}$, $r_0 = np$ and $r_i = r+n$ $(i \neq 0)$.

5.2. *Another series of balanced designs with $v+1$ treatments.* A B.I.B design when reinforced with one extra treatment t_0 which occurs p times in each of the b blocks and $n > 0$ extra blocks each of which contains each of the v treatments q times but not the extra treatment, is balanced when p, q and n are related as shown below.

$$nq = \frac{v(pr-\lambda)}{k+p} .$$

Here $c = \frac{pr}{k+p}$, $r_0 = bp$, $r_i = r+nq$ $(i \neq 0)$.

5.3. *Some particular series of balanced designs obtained as particular cases of designs in* 5.1.

5.3.1. Using the B.I.B. designs with parameters $v = b$, $r = k = 1$, $\lambda = 0$ and $n > 0$, we get a series of balanced binary design with $c = \frac{n}{v+1}$, $r_0 = n$ and other $r_i = n+1$.

5.3.2. By using B.I.B. designs with $k = 2$ we get balanced designs where

$$p = 1+\frac{v+1}{2n-1} .$$

By suitably choosing n, we can always get integral values of p.

5.3.3. Using the series of B.I.B. designs with parameters $v = t^2$, $b = t^2+t$, $r = t+1$, $k = t$, $\lambda = 1$ we get balanced designs when

$$p = 1+\frac{t^2+1}{tn-1} .$$

For specific values of t and n, p becomes integral.

5.3.4. From many other series of B.I.B. designs balanced designs can be obtained.

5.4. *Balanced designs obtainable as particular cases of designs in* 5.2.

5.4.1. Using the series of B.I.B. designs with parameters $v = b$, $r = k = 1$, $\lambda = 0$, $n > 0$, we have

$$nq = \frac{vp}{p+1}$$

$$= v/2 \quad \text{when } p = 1.$$

Thus by taking a B.I.B. designs of the above series with even v and $n = v/2$ we get binary balanced designs with $c = \frac{1}{2}$, $r_0 = b$, $r_i = n+1$.

5.4.2. When $p = 1$, if we can get integral values for nq, say $nq = w$, then by taking $n = w$, we get binary designs since $p = q = 1$.

In general when $p = 1$

$$nq = \frac{v(r-\lambda)}{k+1}.$$

Using B.I.B. designs with parameters

$$v = \frac{k(k+1)}{2}, \quad b = \frac{(k+1)(k+2)}{2}, \; r = k+2, \; k, \; \lambda = 2$$

we get $nq = k^2/2$.

Thus when k is even and $n = k^2/2$, we get binary balanced design from the above series of B.I.B. designs.

5.4.3. Using the series of B.I.B. designs with parameters $v, b = 2v$, $r = v-1$, $k = \frac{v-1}{2}$, $\lambda = \frac{v-3}{2}$ we get balanced designs with $p = 1$ and $nq = v$. Here also we can get binary designs by taking v extra vlocks.

Many other binary designs can be obtained by using other series of B.I.B. designs. When $p > 1$ again several series of balanced designs can be obtained.

6. *A generalised efficiency balanced design which is balanced relative to a completely randomised design having other type of replications.*

We have so far discussed efficiency balanced designs which are either variance balanced or efficiency balanced as per the definition of Nigam and Puri (1975). We shall now illustrate designs which are balanced relative to designs which have replications unequal to those in the designs under consideration. The following design (shown by its incidence matrix) is balanced against a completely randomised design having replications 9, 3, 3, 3, 3 when in the design under consideration, the replications of the corresponding treatments are 6, 3, 3, 3, 3 :

$$\text{Incidence matrix of the design} \quad \begin{vmatrix} 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 & 1 \end{vmatrix}.$$

The efficiency of the design is

$$c\Sigma s_i = \frac{1}{27}\times 21 = \frac{7}{9} \text{ as } c = \frac{1}{27}$$

and

$$\Sigma\, s_i = 9+3+3+3+3 = 21.$$

Some work in regard to balanced designs has also been reported by Puri and Nigam (1977), Raghavarao (1962), Hedayat and Federer (1969) and others.

After the paper was submitted for publication the author's' attention was drawn by referee to a paper entitled "Balance and Design : Another Terminological Tangle" by Preece (1982). In the above paper, the author has mentioned variance balance and efficiency balance as two separate concepts of balancing in relation to incomplete block designs. In the present paper, we have given a condition namely,

$$\sum_j n_{ij} n_{mj} = c s_i s_m \quad (m \neq i)$$

where c is a constant, for making a design balanced. When this condition is satisfied, the design can be either variance balanced of efficiency balanced according as $s_i = r$ or $s_i = r_i$ for all i's, r_i being number of replications when s_i's are neither equal to r or r_i but is a separate set of numbers, then also the design can be balanced. As such this is a more general condition for balance.

Acknowledgements. The authors are thankful to Dr. K. R. Parthasarathy, Head, Delhi Centre, Indian Statistical Institute for providing necessary facilities and incentives for conducting the research. They are also thankful to I.C.A.R., New Delhi for providing grant which supported the research.

REFERENCES

CALINSKI (1971) : On some derivable patterns in block designs. *Biometrics*, 27, 275–292.

DAS, M. N. (1958) : Reinforced incomplete block designs. *J. Ind. Soc. Agr. Statist.*, 10, 73–77.

DEY, A. and SINGH, M. (1980) : Some series of efficiency balanced designs. *Austra. J. Statist.*, 22(3), 364–367.

HEDAYAT, A. and FEDERER, W. T. (1974) : Pairwise and variance balanced incomplete block designs. *Ann. Instt. Stat. Math.*, 26, 331–338.

PREECE, D. A. (1982) : Balance and designs : Another terminological tangle. *Utilitas Mathematica*, 21c, 85–186.

PURI, P. D. and NIGAM, A. K. (1975a) : On patterns of efficiency balanced designs. *J. Roy. Stat. Soc.*, B, 37, 357–58.

PURI, P. D. and NIGAM, A. K. (1975b): A note on efficiency balanced designs. *Sankhyā*, B, 37, 457–460.

PURI, P. D. and NIGAM, A. K. (1977): Balanced block designs. *Commun. Statist. Theor-Math.*, A 6(12), 1171–1179.

RAGHAVARAO, D. (1962): On balanced unequal block designs. *Biometrika*, 49, 561–562.

WILLIAMS, E. R. (1975): Efficiency balanced designs. *Biometrika*, 62, 686–688.

YATES, F. (1936): A new method of arranging varietal trials involving a large number of varieties. *Ann. Eugen.*, 7, 319–22.

Paper received : December, 1981.
Revised : January, 1984.