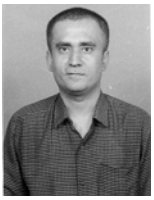
How to Stay Away from Each Other in a Spherical Universe

1. Tammes' Problem

Bhaskar Bagchi



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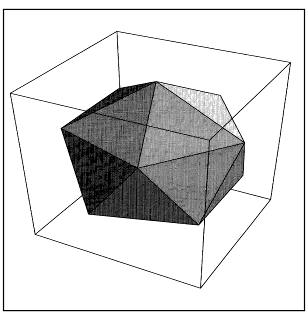
Mathematics is full of innocent looking problems which, when pursued, soon grow to majestic proportions and begin to impinge upon the frontiers of research. One such problem is the subject of this two-part article – the problem of packing spherical caps on the surface of a sphere.

Twelve Men on a Sphere

For most practical purposes, we humans are inhabitants of a (two dimensional) spherical universe: we are constrained to eke out our lives on the surface of the earth. Suppose twelve denizens of this spherical universe hate each other so much that they decide to build their houses as far away from each other as possible. We assume that, except for this morbid hatred, these guys have no other constraints: they may build their houses anywhere on the earth's surface (must be stinkingly rich to have such freedom!). The question is: how should these twelve houses be positioned ? The answer depends, of course, on what we mean by the phrase 'as far away from each other as possible'. Let us take it to mean that the minimum of the pair-wise distances between the twelve houses is to be maximised subjected to the only constraint that they be situated on the spherical surface. With this understanding, it turns out that the problem has a unique solution: the houses must be built at the vertices of an icosahedron inscribed in the sphere. (This is proved in Part 2 of this article.) Now, the icosahedron is a highly structured and symmetrical figure (see *Figure 1*). Its group of rotational symmetries has order 60 - it is the smallest non-abelian 'simple' group. Isn't it amazing that so much structure and symmetry result as the solution of such a simple minded maximisation problem ? Before we finish we

Figure 1 Icosahedron.



Box 1. Symmetry Group

Given a solid body in Euclidean space (of any dimension), its rotational symmetries are those rotations of the ambient space which send the body (as a whole) onto itself. Clearly, applying two such rotations successively one obtains a third rotational symmetry of the body. It follows that all such rotations form a group with composition as the group law. This is called the isometry group of the body. Other names: symmetry group, automorphism group.

Box 2. The Smallest Non-Abelian Simple Group

Recall that a group is called non-abelian if its law is not commutative: i.e., if the product of two elements depends, in general, on the order in which the multiplication is performed. The order of a finite group means the number of elements in the group. A group is called simple if it has no (proper, non-trivial) normal subgroup. Simple groups are to general groups what prime numbers are to arbitrary integers: they are supposed to be the building blocks of general groups. This is why they are so important. Trivialish examples - in fact the only examples - of abelian simple groups are the symmetry groups of regular p-gons for various primes p. The symmetry group of the icosahedron (as we have defined it here - excluding reflections) is the non-abelian simple group of the smallest possible order. Abstractly, this group may be described as the group of all even permutations of a set of five objects. All finite simple groups are known. With 26 exceptions (the so called sporadic simple groups) they fall in neat infinite families.

shall see many more examples of this sort of phenomenon. If, however, we wished to build thirteen houses with the above constraint, then nobody knows what is the optimal solution, or whether the solution is unique or not!

Pollen Grain

This problem (with an arbitrary number n in place of twelve) was first raised by the botanist P M L Tammes in 1930. He wanted to explain the observed distribution of the pores on a grain of pollen. He proposed that, for the sake of maximum biological efficiency, the n pores on the surface of the grain (which may be taken to be a sphere) are placed as far away from each other as possible. Of course, to test this theory,

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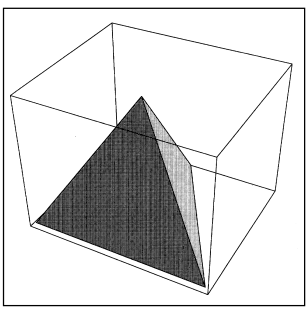
Box 3. Convex Polyhedra

Recall that a set of points in space is called a convex set if the straight line segment joining any two points in the set is entirely contained in the set. Clearly the intersection of any family of convex sets is again convex. It follows that among all convex sets containing a given point set A, there is a smallest one (namely the intersection of all the convex sets containing A); it is called the convex hull of A. Among the simplest examples of convex sets are the half spaces: the removal of any fixed plane from space breaks up space into two parts; each of these two parts is called a half space determined by the plane. The union of a half space with the plane bounding it is called a closed half space. If a bounded set of points can be expressed as the intersection of finitely many closed half spaces then it is called a polyhedron (plural: polyhedra). A point in a polyhedron X is called a vertex (plural: vertices) or extreme point (intuitively, a corner) of X if it is not an interior point of any line segment contained in X. It is not difficult to see that any polyhedron has only finitely many vertices and the polyhedron is the convex hull of the set of its vertices. The straight line segment joining two vertices of a polyhedron X is called an edge of X if it is the intersection of X with a plane. A subset of X is a face of X if it is not a vertex or edge and yet is obtainable as the intersection of X with a plane.

Figure 2 Tetrahedron.



one needs to know the theoretical optimum arrangement(s). This is the 'Problem of Tammes.' The case $n = 13$ of this problem can be traced back to a controversy between Newton and Gregory. Many mathematicians have worked on Tammes' problem, prominent among the early workers being Fejes Toth and B L van der Waerden. And yet, only the cases $n \leq 12$ and (surprise!) $n = 24$ have been solved so far. In all these cases, excepting the case $n = 5$, the optimal solution turns out to be unique. Let us briefly examine atleast some of these solutions. Note that the convex hull of any n points on the sphere is a convex polyhedron whose vertices are precisely the n points that we began with. So it is often convenient to describe a solution to an instance of Tammes' problem as (the set of vertices of) a convex polyhedron. All the polygonal faces which occur in the next two paragraphs are regular (i.e., have equal sides and equal vertical angles). In particular, the triangles are equilateral.

Box 4. Platonic Solids

A flag of a convex polyhedron X is a triplet (x, e, F) where x is a vertex of X, e is an edge of X through x and F is a face of X containing e. X is called a platonic solid if, given any two flags $\phi = (x, e, F)$ and $\phi' = (x', e', F')$ of X, there is an isometry of X (rotation or reflection) which takes ϕ to ϕ' (i.e., maps x to x', e to e' and F to F'.) Since no non-identity isometry of space can fix a flag, it follows that the order of the extended symmetry group (including reflections as well as rotations fixing the body) of a platonic solid equals the number of flags of the solid. For instance, the icosahedron has 12 vertices, 5 edges through each vertex and 2 faces through each edge, so that it has $12 \times 5 \times 2 = 120$ flags, and 120 is the order of its extended symmetry group.

The Platonic Solids

These are objects that have fascinated man since the time of the Greeks. Informally, the platonic solids are the convex polyhedra which look the same from the point of view of each vertex, edge and face. Given one such solid, the centers of its faces form the vertices of a second platonic solid, called the dual of the first. There are only five platonic solids. These are : (i) the (regular) tetrahedron (*Figure 2*) with 4 equidistant vertices, 6 edges and 4 triangular faces, (ii) the cube with 8 vertices, 12 edges and 6 square faces, (iii) the octahedron (*Figure 3*) with 6 vertices, 12 edges and 8 triangular faces, (iv) the icosahedron with 12 vertices, 30 edges and 20 triangular faces, and, finally, (v) the dodecahedron (*Figure 4*) with 20 vertices, 30 edges and 12 pentagonal faces. Of these, the tetrahedron (or regular simplex) is its own dual, the cube and the octahedron are duals of each other, and so are the icosa- and the dodecahedron. Everybody knows the regular tetrahedron and the cube: these are the obvious three dimensional analogues of the equilateral triangle and the square. The octahedron is another - and less obvious - analogue of the square: it may be obtained by putting together two right pyramids on a common square base. There is no such easy description of the remaining two platonic solids. In coordinates, the vertices of the icosahedron may be taken to be the even coordinate permutations of the four

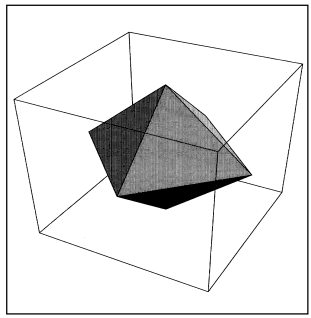
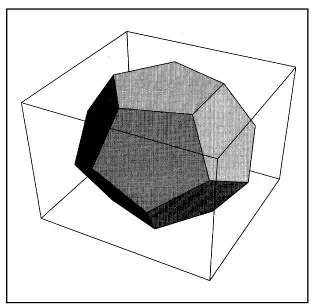


Figure 3 Octahedron.

Figure 4 Dodecahedron.



Box 5. Archimidean Solids (also called semi-regular solids)

These are convex polyhedra such that (i) all their faces are regular polygons, and (ii) the symmetry group is transitive on the vertices (i.e., any vertex can be mapped to any other vertex by a suitable isometry of the body). The square anti-prism is an example. It may be described as follows. Begin with the cube and rotate one of its faces (in its plane) around its center by half a right angle. Then move this face closer to the opposite face by translating it parallel to the original position. For the correct choice of the distance between these two faces, the sides of the convex hull of the eight vertices (in their new positions) are all equal and the square anti-prism results.

vertices $\frac{1}{\sqrt{\tau+2}}(0, \pm 1, \pm\tau)$, where $\tau = (1+\sqrt{5})/2$ is the 'golden ratio'. (Exercise: from this information, can you find the co-ordinates of the vertices of the dual polyhedron, namely the dodecahedron?) You may try to build your own model of these two solids by pasting together 20 equilateral triangles / 12 regular pentagons of the same size, preferably made of cardboard.

If you wanted to place n points on (the circumference of) a circle so that they are as far away from each other as possible, then clearly you would have to place them at the vertices of a regular n-gon. (Exercise : Why ?) Since the platonic solids are the three dimensional analogues of regular polygons, it is natural to conjecture that all five of them are (unique ?) solutions of Tammes' problem for the relevant values of n. In fact, the tetrahedron, octahedron and the icosahedron are indeed the unique solutions for $n = 4$, 6 and 12 respectively. (Notice that these are the platonic solids with triangular faces. We shall present a proof of these three results in Part 2 of this article.) But the conjecture is false for $n = 8$: the cube is not optimal in the sense of Tammes' problem ! The unique solution in this case is provided by the vertices of the square anti-prism with 8 vertices, 16 edges, 2 square faces and 8 triangular faces. We do not know if the dodecahedron is a solution for $n = 20$, perhaps it is not! For $n = 11$ the unique solution is obtained by deleting a vertex of the icosahedron. For $n = 5$ one solution

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is the octahedron minus a single point, but a second solution is obtained by taking three equidistant points on the equator together with the two poles. In fact, one has, in this case, infinitely many solutions 'in between' these two extreme solutions. Any $n \leq 3$ points on the sphere lie on a circle, so that the solutions in these cases are obvious. The solutions for $n = 7,\ 9$ and 10, though unique, are too difficult to describe here. (At least, I must so declare since I do not understand them myself!) In a remarkable paper in *Mathematische Annalen*, 1961, R M Robinson settled a conjecture of van der Waerden in the affirmative by showing that the unique solution for $n = 24$ is given by the vertices of the snub cube (*Figure 5*). This is a so-called archimidean solid (the square anti-prism (*Figure 6*) is another) with 24 vertices, 60 edges, 6 square faces and 32 triangular faces. If the reader is particularly brave and curious, he or she may try to go through the monumental paper by L Danzer which occupies 64 pages of the 60th volume (1986) of *Discrete Mathematics.* In this paper, the cases $n \leq 12$ of Tammes' problem are given a uniform treatment. (The cases $n = 10$ and 11 are solved for the first time in this paper.) The wonderful book on regular polytopes by Coxeter (see Suggested Reading) is highly recommended to any one wishing to know more about the regular and semi-regular solids alluded to above.

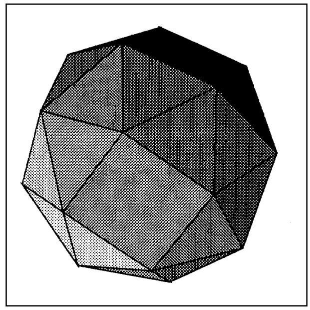
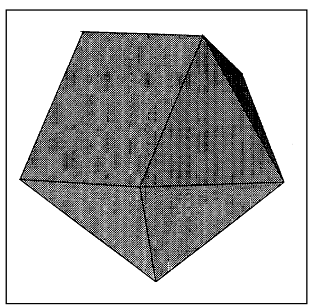


Figure 5 Snub cube.

Higher Dimensional Analogues: Optimal Spherical Codes

In the formulation of Tammes' problem it was implicit that the sphere in question was in three dimensional euclidean space; further, by a suitable choice of coordinates, it could be taken to be the unit sphere (i.e., the sphere of radius 1 centered at the origin). More generally, let S^{d-1} denote the unit sphere in the d dimensional euclidean space $\mathbb{R}^d$. By a spherical code of size n and rank d one means a set of n points in S^{d-1}. This terminology is fashionable because the points in the spherical code are thought of as the words in a code language. The minimum distance of a code X is by definition the minimum of $\{||x-y|| : x \neq y,\ x, y \in X\}$. (Here $||.||$ is the usual euclidean norm: $\|x\|^2 = \sum_{i=1}^{d} x_i^2$. So $\|x-y\|$ is the euclidean distance between the points x

Figure 6 Square anti-prism.



To be able to detect and correct errors committed during transmission, it is desirable that the minimum distance of a spherical code be as large as possible.

and y. We have, $S^{d-1} = \{x \in I\!R^d : \ \|x\| = 1\}$.) Clearly, to be able to detect and correct errors committed during transmission, it is desirable that the minimum distance of a spherical code should be as large as possible. Let $\rho(n,d)$ denote the maximum of the minimum distance of X where the maximum is over all spherical codes X of size n and rank d. Also, by an optimal spherical code of size n and rank d we mean such a spherical code whose minimum distance equals $\rho(n,d)$. So the problem is to compute $\rho(n,d)$ and find the optimal spherical codes for various values of n and d. Exercise: if $n = e+1 < d+1$, then show that the spherical code is 'actually' a code of rank (at most) e, so that one is reduced to a case of lower rank. So, in the following, one may as well assume $n \geq d+1$.

The Regular Polytopes

Given the paucity of solutions to this problem in the case of rank 3, it is not surprising that very little is known about the higher ranks. We may as well begin with the regular polytopes which generalize the platonic solids of three dimension. In d dimension there are three regular polytopes with $n = d+1$, $2d$ and 2^d respectively. These generalize the tetrahedron, octahedron and the cube; they are called the (regular) simplex, the cross polytope (or orthoplex) and the hypercube. (The regular simplex is the convex hull of $d+1$ equidistant vertices. The hypercube is the convex hull of the 2^d vertices with zero-one coordinates. The orthoplex is the convex hull of the $2d$ vertices with one coordinate equal to ± 1 and remaining coordinates zero.) Apart from these three families, there are five sporadic regular polytopes, two in three dimension (which we have already met) and three in four dimension. The sporadic regular polytopes in $I\!R^4$ are: (i) the 24-cell with 24 vertices, 96 edges, 96 triangles and 24 octahedral faces; in co-ordinates, its vertices may be taken to be the permutations of the points $(\pm 1/\sqrt{2}, \pm 1/\sqrt{2}, 0, 0)$; (ii) the 600-cell with 120 vertices, 720 edges, 1200 triangles and 600 tetrahedral faces, (iii) the 120-cell with 600 vertices, 1200 edges, 720 triangles and 120 dodecahedral faces. The 24-cell is self dual while the other two are duals of each other. The vertices of the 600-cell may be obtained from the points

$(1,0,0,0)$, $\frac{1}{2}(1,1,1,1)$ and $\frac{1}{2}(\tau,1,\tau^{-1},0)$ by arbitrary sign changes and even permutations of the coordinates. (Here, as in the description of the icosahedron, τ is the golden ratio.) Other interesting details of the 600-cell and its dual may be found in Coxeter's book. In certain quarters, the 24-cell and the 600-cell are better known as the root systems D_4 and H_4. In this avatar, they may be located in *Reflection Groups and Coxeter Groups* by J E Humphreys. A polytope is called *simplicial* if its maximal faces are simplexes. Thus, among the regular polytopes, the simplicial ones are the simplexes, the cross polytopes, the icosahedron and the 600-cell. In a paper published in volume 32 of *Acta Mathematica Hungarica* (1978), K Boroczky proved that all these simplicial regular polytopes are unique optimal codes. The new result here is the optimality of the 600-cell.

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The Kissing Number Problem

The known solutions to this problem constitute yet another source of optimal spherical codes. The kissing number $\kappa(d)$ in dimension d is the largest number of solid balls in $\mathbb{R}^d$; all of the same size, which can touch a given ball of the same size. (Of course, solid balls are not allowed to penetrate each other. Technically, these balls must have pairwise disjoint interiors.) The problem is to determine this number. The solution is known only for $d = 2$, 3, 8 and 24, and the corresponding values of κ are $6 = 2\binom{3}{2}$, $12 = 2\binom{4}{2}$, $240 = 2\binom{10}{3}$ and $196560 = 2\binom{28}{5}$. What is its relation to optimal spherical codes? Well, given $\kappa(d)$ balls of radius half each touching a ball of radius half centered at the origin, the centers of these balls form a spherical code of size $\kappa(d)$ and rank d with minimum distance ≥ 1. Clearly this construction can be reversed, so that $\kappa(d)$ is the largest value of n for which $\rho(n,d) \geq 1$. Since the regular hexagon is the unique optimal code of size 6 and rank 2, and its minimum distance is exactly equal to 1, this proves $\kappa(2) = 6$. Since the icosahedron is the optimal code of size 12 and rank 3, and since its minimum distance is $\sqrt{2-2/\sqrt{5}} > 1$, we see that $\kappa(3) \geq 12$. It is not difficult to see that $\kappa(3) \leq 13$. The Newton-Gregory controversy already alluded to was on whether the true value in this case was twelve or thirteen. Newton believed it was 12 but Gregory thought otherwise. Since $\rho(12,3) > 1$, centering the balls at the vertices of an icosahedron is not the only way to place twelve balls touching a given ball. In this arrangement, no two of the 12 balls touch each other, so that one may well think that they may be pushed around to make place for a thirteenth ball. In fact, a better way to place the twelve balls is to take them to be the balls of radius half centered at the permutations of the points $(\pm 1/\sqrt{2}, \pm 1/\sqrt{2}, 0)$ (an arrangement reminiscent of the 24-cell) in which case each of the twelve surrounding balls touch four others apart from the central ball. It was only in 1874 that R Hoppe proved that $\rho(13,3) < 1$; thus Newton was right. To date, no really accessible proof of this fact is available. The case of four dimensions is very analogous. It is known that $\kappa(4)$ is either 24 or 25, but nobody knows the correct value. It is very tempting to conjecture that the 24-cell is the (unique?) optimal spherical code of size 24 and rank 4. If true, this will of course imply $\kappa(4) = 24$. Though one knows next to nothing about the intervening dimensions, one knows everything about the cases $d = 8$ and 24: not only is the kissing number known in these two cases, but one also knows that the arrangements in these two cases are unique. One must take the balls centered at the roots of the E_8 root system in one case, and at the minimum norm vectors of the Leech lattice in the other case. These are two objects which play very important roles in other parts of mathematics. For one thing, they are awfully symmetric structures : their automorphism groups are, respectively, the group $O_8^+(2) : 2$ of order $348,364,800$ and the largest of the sporadic simple groups of Conway, of order $4,157,776,806,543,360,000$. They provide us with unique optimal codes with $(n,d) = (240,8)$, $(196560,24)$ and minimum distance 1. The usual proofs of these assertions involve moderately heavy dozes of analytic number theory, invoking, as they do, Theta and modular functions. The reader wishing to see these proofs may consult the book by J H Conway and N J A Sloane (see Suggested Reading). Alternative proofs may be constructed following the outline to be presented in Part 2.

Suggested Reading

- H S M Coxeter. ***Regular Polytopes***. Dover. New York, 1973.
- J H Conway and N J A Sloane. ***Sphere Packings, Lattices and Groups***. Springer Verlag. New York, 1988.

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