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## MULTIPLICATIVELY SPECTRUM-PRESERVING MAPS OF FUNCTION ALGEBRAS

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**ABSTRACT.** Let  $X$  be a compact Hausdorff space and  $\mathcal{A} \subset C(X)$  a function algebra. Assume that  $X$  is the maximal ideal space of  $\mathcal{A}$ . Denoting by  $\sigma(f)$  the spectrum of an  $f \in \mathcal{A}$ , which in this case coincides with the range of  $f$ , a result of Molnár is generalized by our Main Theorem: If  $\Phi : \mathcal{A} \rightarrow \mathcal{A}$  is a surjective map with the property  $\sigma(fg) = \sigma(\Phi(f)\Phi(g))$  for every pair of functions  $f, g \in \mathcal{A}$ , then there exists a homeomorphism  $\Lambda : X \rightarrow X$  such that

$$\Phi(f)(\Lambda(x)) = \tau(x)f(x)$$

for every  $x \in X$  and every  $f \in \mathcal{A}$  with  $\tau = \Phi(1)$ .

### 1. INTRODUCTION

Molnár [M, Theorem 5] proved the following theorem: If  $X$  is a first-countable compact Hausdorff space and  $C(X)$ , the algebra of complex-valued continuous functions on  $X$ , and

$$\Phi : C(X) \rightarrow C(X)$$

a surjective mapping such that

$$\text{for every pair of functions } f, g \in C(X), \sigma(fg) = \sigma(\Phi(f)\Phi(g))$$

where  $\sigma(f)$  denotes the spectrum of  $f$ , which in this case would be simply  $f(X)$ , the range of  $f$ , then there exists a homeomorphism  $\varphi$  of  $X$  onto itself and a function  $\tau$ , whose range is  $\{-1, 1\}$  such that

$$\Phi(f)(x) = \tau(x)f(\varphi(x)) \text{ for all } x \in X \text{ and all } f \in C(X).$$

In this paper we deal with a function algebra  $\mathcal{A}$  in place of  $C(X)$  and regard  $X$  as the maximal ideal space of  $\mathcal{A}$ .  $X$  is of course compact Hausdorff but not necessarily first-countable. For this purpose, we need to recall some results of Bishop and de Leeuw [BL] concerning function algebras, peaking functions, generalized peak points etc., for which a readable exposition may be found in [Br, Chapter 2] and [P, Chapter 8].

**1.1 Peaking function.** A function  $f$  in  $\mathcal{A}$  is said to be a peaking function if for any  $x$  in  $X$ , either  $f(x) = 1$  or  $|f(x)| < 1$  and the set  $\{x : x \in X, f(x) = 1\}$ , denoted by  $P(f)$  and referred to as the **peaking set**, is non-empty.

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**1.2. Generalized peak point.** A point  $x$  in  $X$  is said to be a generalized peak point for the algebra  $\mathcal{A}$  if, given any neighborhood  $V$  of  $x$ , there exists a peaking function  $f$  in  $\mathcal{A}$  such that  $P(f) \subset V$ ,  $f(x) = 1$ .

The set of all generalized peak points is called the *Choquet boundary* of  $\mathcal{A}$  and denoted by  $\partial_{\mathcal{A}}(X)$ . Its closure is the so-called Shilov boundary of  $\mathcal{A}$ . Since any  $f \in \mathcal{A}$  assumes its maximum modulus  $\|f\|_{\infty} := \sup_{x \in X} |f(x)|$  on the Choquet boundary (see [P, Prop. 6.3]), we see that

$$(1.3) \quad \text{any peaking set meets } \partial_{\mathcal{A}}(X).$$

Also, given any  $x \in X$ , there exists a probability measure  $\mu$ , a representing measure for  $x$ , supported on the Shilov boundary  $S = \overline{\partial_{\mathcal{A}}(X)}$  such that for every  $f \in \mathcal{A}$ ,

$$(1.4) \quad f(x) = \int_S f d\mu.$$

The following theorem will be invoked several times in the proof of our *Main Theorem* in the next section.

**1.5. Theorem (Bishop).** *Given any peaking set  $E$  and any  $f \in \mathcal{A}$ , there exists a peaking function  $h$  in  $\mathcal{A}$  with  $P(h) = E$  and  $|f(z)h(z)| < \max_E |f|$  for any  $z \notin E$ .*

A proof may be found in [Br, page 102]. At one point in the next section, we shall need the fact contained in the following proposition.

**1.6. Proposition.** *Any family of peaking sets  $E_{\alpha}$ , with finite intersection property, has a common intersection with  $\partial_{\mathcal{A}}(X)$ .*

*Proof.* The proof is a convexity argument. Let  $S_{\mathcal{A}} = \{L \in \mathcal{A}^* : \|L\| = L(1) = 1\}$  be the state space of  $\mathcal{A}$ . We know that (see [P, page 37])  $\varphi(\partial_{\mathcal{A}}(X)) = \text{ext}(S_{\mathcal{A}})$  where  $\text{ext}(S_{\mathcal{A}})$  denotes the set of extreme points of the compact convex set  $S_{\mathcal{A}} \subset \mathcal{A}^*$ , non-empty by the Krein-Milman theorem, and  $\varphi$  denotes the evaluation map  $x \rightsquigarrow \varphi(x)$  that imbeds  $X$  homeomorphically into  $S_{\mathcal{A}}$  with weak\* topology. Each  $F_{\alpha} := \text{weak}^*$  closed convex hull of  $\varphi(E_{\alpha})$ , where  $E_{\alpha} = \{x \in X : h_{\alpha}(x) = 1\}$  and each  $h_{\alpha} \in \mathcal{A}$  is the associated peaking function, is a weak\* closed face of  $S_{\mathcal{A}}$ —in fact,  $F_{\alpha} = \{L \in S_{\mathcal{A}} : L(h_{\alpha}) = 1\}$ . Consequently by the finite intersection property,  $F := \bigcap_{\alpha} F_{\alpha}$  is a non-empty weak\* closed face of  $S_{\mathcal{A}}$  and therefore has an extreme point  $p$  that necessarily belongs to  $\text{ext}(S_{\mathcal{A}})$  and is therefore of the form  $\varphi(x)$  for some  $x \in \partial_{\mathcal{A}}(X)$ . But  $p \in \text{ext}(F_{\alpha}) \subset \varphi(E_{\alpha})$  for every  $\alpha$  by the Milman theorem; hence  $x \in \bigcap_{\alpha} E_{\alpha}$ , and we are done.  $\square$

## 2. PROOF OF THE MAIN THEOREM

In the sequel  $f, g, h, k$ , etc. denote functions from  $\mathcal{A}$  and  $c$  denotes a generic constant. Also for any  $f \in \mathcal{A}$ , we shall sometimes abbreviate  $\|f\|_{\infty}$  to  $\|f\|$ . It is convenient to present the proof of our theorem as a sequence of remarks. We point out that the proofs of these remarks, though modelled in several instances on [M], are rendered somewhat complicated by the more general situation that is being considered here.

*Remark 1. Reduction.* Since  $\sigma(1^2) = \sigma(\Phi(1)^2)$ , we have  $\Phi(1)^2 = 1$ , and so by defining  $\Psi f = \Phi(1)\Phi(f)$ , we see that  $\Psi(1) = (\Phi(1))^2 = 1$ . Furthermore,  $\Psi(f)\Psi(g) = \Phi(1)\Phi(f)\Phi(1)\Phi(g) = \Phi(f)\Phi(g)$  and, consequently,

$$\sigma(fg) = \sigma(\Psi(f)\Psi(g)).$$



Now if we prove the existence of a homeomorphic self-map  $\Lambda$  of  $X$  such that

$$\Psi(f)(\Lambda(x)) = f(x)$$

for every  $x \in X$ , we would have proved the theorem mentioned in the abstract. So from now on, we assume that  $\Phi(1) = 1$  and so

$$(2.1) \quad \sigma(f) = \sigma(\Phi(f)) \quad \forall f \in \mathcal{A},$$

from which it immediately follows that

$$(2.2) \quad \|f\|_\infty = \|\Phi(f)\|_\infty.$$

*Remark 2.* If  $f, g \in \mathcal{A}$ , then  $|f| \leq |g|$  on  $\partial_{\mathcal{A}}(X)$  if and only if

$$(2.3) \quad \text{for every } c \geq 0 \text{ and every } h, |gh| \leq c \text{ implies } |fh| \leq c.$$

*Proof.* That  $|f| \leq |g|$  on  $\partial_{\mathcal{A}}(X)$  implies (2.3) is obvious by (1.4). Assume that (2.3) is true but  $|f| \not\leq |g|$  on  $\partial_{\mathcal{A}}(X)$ . Hence there must exist an  $x_0$  in  $\partial_{\mathcal{A}}(X)$  such that

$$|f(x_0)| > |g(x_0)|;$$

for, otherwise,  $|f| \leq |g|$  on  $\partial_{\mathcal{A}}(X)$ .

Let  $\gamma = \frac{1}{2}(|f(x_0)| + |g(x_0)|)$ . So  $|g(x_0)| < \gamma < |f(x_0)|$ , and there exists an open neighborhood  $V$  of  $x_0$  such that  $|g(x)| < \gamma$  in  $V$  and a function  $h$  such that  $h(x_0) = 1 = \|h\|$ , and  $|g(x)h(x)| < \gamma$  in  $X \setminus V$ . Such an  $h$  exists, because  $x_0$  is a generalized peak point for  $\mathcal{A}$ . Therefore  $|gh| < \gamma$  on all of  $X$ , but  $|f(x_0)h(x_0)| = |f(x_0)| > \gamma$ , a contradiction. This proves the assertion (2.3).

From (2.3), we can deduce the following:

$$(2.4) \quad \text{if } \sigma(fh) = \sigma(gh) \text{ for every } h, \text{ then on } \partial_{\mathcal{A}}(X), |f| = |g|.$$

Since  $\sigma(fh) = \sigma(gh) \forall h \in \mathcal{A}$  we see that for any constant  $c \geq 0$  and any  $h \in \mathcal{A}$ ,  $|gh| \leq c$  implies  $|fh| \leq c$  and so (2.3) gives  $|f| \leq |g|$  on  $\partial_{\mathcal{A}}(X)$ . Since the hypothesis is symmetric in  $f, g$ , we obtain also  $|g| \leq |f|$  on  $\partial_{\mathcal{A}}(X)$ . Combining, we have (2.4).

As a consequence we have

*Remark 3.*

$$(2.5) \quad \text{On } \partial_{\mathcal{A}}(X), |f| \leq |g| \Leftrightarrow |\Phi(f)| \leq |\Phi(g)| \quad \forall f, g \in \mathcal{A}.$$

*Proof.* Assume that  $|f| \leq |g|$  on  $\partial_{\mathcal{A}}(X)$  and  $|\Phi(g)k| \leq c$  for some  $k \in \mathcal{A}$  and  $c \geq 0$ .  $\Phi$  being surjective, there exists an  $h \in \mathcal{A}$  such that  $\Phi(h) = k$ . Hence we have

$$|\Phi(g)\Phi(h)| \leq c.$$

But since

$$\sigma(gh) = \sigma(\Phi(g)\Phi(h)),$$

we obtain  $|gh| \leq c$  and so by (2.3),  $|fh| \leq c$ . Since

$$\sigma(fh) = \sigma(\Phi(f)\Phi(h)),$$

we obtain  $|\Phi(f)\Phi(h)| = |\Phi(f)k| \leq c$ . Now since  $k, c$  are arbitrary, from Remark 2; it follows that

$$|\Phi(f)| \leq |\Phi(g)| \text{ on } \partial_{\mathcal{A}}(X).$$

Now the other implication has a similar proof.

*Remark 4.* For any fixed  $x \in \partial_{\mathcal{A}}(X)$ ,

$$(2.6) \quad E := \bigcap_{f \in \mathcal{F}_x} P(f) = \{x\},$$

where  $\mathcal{F}_x$  denotes the family of all peaking functions  $f \in \mathcal{A}$  such that  $f(x) = 1$ .

*Proof.* Assume  $E$  contains a point  $y$  other than  $x$ . From (1.2) it follows that every point of  $\partial_{\mathcal{A}}(X)$  is a generalized peak point for  $\mathcal{A}$ , which means that, given any neighborhood  $V$  of  $x$ , there exists a peaking function  $h$  in  $\mathcal{A}$  such that  $h(x) = 1 = \|h\|$  and  $|h| < 1$  outside  $V$ , which means  $P(h) \subset V$ . So if we choose a neighborhood  $V$  of  $x$  that does not contain  $y$ , since  $E \subset V$ ,  $y \notin E$ , a contradiction.

We now have the important

*Remark 5.* If  $x \in \partial_{\mathcal{A}}(X)$ ,

$$(2.7) \quad \bigcap_{f \in \mathcal{F}_x} P(\Phi(f)) \text{ contains one and only one generalized peak point.}$$

First, because of (2.1),  $\Phi(f)$  is a peaking function if and only if  $f$  is a peaking function. Also, each  $P(\Phi(f))$  is compact.

Secondly, if  $f_1, f_2, \dots, f_n$  belong to  $\mathcal{F}_x$ , then  $g = f_1 f_2 \dots f_n$  belongs to  $\mathcal{F}_x$ . Since  $|g| \leq |f_i|$ , we obtain in view of (2.5),

$$|\Phi(g)| \leq |\Phi(f_i)| \text{ for each } 1 \leq i \leq n \text{ on } \partial_{\mathcal{A}}(X).$$

Since  $g$  is a peaking function, so is  $\Phi(g)$ , and so  $\Phi(g)(\xi) = 1$  for some  $\xi$  in  $\partial_{\mathcal{A}}(X)$ . Then  $\Phi(f_i)(\xi) = 1$  for  $1 \leq i \leq n$  or

$$\bigcap_{1 \leq i \leq n} P(\Phi(f_i)) \neq \emptyset.$$

This proves that the family of sets  $\{P(\Phi(f)) : f \in \mathcal{F}_x\}$  has the finite intersection property, and since each of them is compact, it must be that

$$E' = \bigcap_{f \in \mathcal{F}_x} P(\Phi(f)) \neq \emptyset.$$

Thus,  $E'$  being a non-empty intersection of peaking sets must intersect  $\partial_{\mathcal{A}}(X)$  by Proposition 1.6.

Thirdly, if  $y \in E' \cap \partial_{\mathcal{A}}(X)$ , let  $k$  be a peaking function such that  $k(y) = 1$ . By surjectivity of  $\Phi$ ,  $k = \Phi(h)$  for some peaking function  $h \in \mathcal{A}$  (recall that  $\sigma(k) = \sigma(h)$ ). We claim that  $h(x) = 1$ . To show this, choose any neighborhood  $V$  of  $x$  and a peaking function  $g$  such that  $g(x) = 1$  and  $|g| < 1$  outside  $V$ . So  $g \in \mathcal{F}_x$  and hence  $\Phi(g)(y) = 1$ . Consider  $\Phi(g)\Phi(h) = \lambda \in \mathcal{A}$ .  $\Phi(g), \Phi(h)$  being both peaking functions that take the value 1 at  $y$ , we see that  $\lambda(y) = 1$  and  $\lambda$  is a peaking function. Again  $\Phi$  being surjective, there exists a peaking function  $\mu \in \mathcal{A}$  such that  $\Phi(\mu) = \lambda$ . Since  $|\lambda| \leq |\Phi(g)| \wedge |\Phi(h)|$  on  $\partial_{\mathcal{A}}(X)$ , by (2.5) it follows that  $|\mu| \leq |g| \wedge |h|$  on  $\partial_{\mathcal{A}}(X)$ . Hence there exists a  $\xi \in \partial_{\mathcal{A}}(X)$  such that  $\mu(\xi) = 1$ , and so  $g(\xi) = h(\xi) = 1$ , which implies that  $\xi \in V$ . Since  $V$  is an arbitrary neighborhood of  $x$  and  $h$  is continuous, we get

$$h(x) = 1.$$

Lastly, if there is a generalized peak point  $z$  other than  $y$  in  $E'$ , we can choose  $k$  in such a way that  $k(y) = 1, |k(z)| < 1$ .  $\Phi$  being surjective, we obtain  $h'$  such that  $\Phi(h') = k$ . So by the previous paragraph, we see that  $h'$  belongs to  $\mathcal{F}_x$  and so

$\Phi(h') = 1$  on  $E'$  and consequently  $k(z) = 1$ , which is a contradiction. This proves Remark 5. □

Let the unique point  $y$  given by Remark 5 be denoted by  $\tau(x)$  since it depends on  $x$  and nothing else. We sum up what we established above as follows:

*Remark 6.* If  $x \in \partial_{\mathcal{A}}(X)$  and  $f \in \mathcal{F}_x$ , then  $\tau(x) \in \partial_{\mathcal{A}}(X)$  and  $\Phi(f)$  belongs to  $\mathcal{F}_{\tau(x)}$ . Conversely, if  $k \in \mathcal{F}_{\tau(x)}$  and  $\Phi(h) = k$ , then  $h \in \mathcal{F}_x$ .

We now have

*Remark 7.*  $\Phi$  is injective and homogeneous, i.e.,  $\Phi(cf) = c\Phi(f)$  for any  $f \in \mathcal{A}$  and  $c \in \mathbb{C}$ .

*Proof.* Suppose if possible that  $\Phi(f) = \Phi(g)$  for some  $f \neq g$ . For any  $h \in \mathcal{A}$ ,  $\Phi(f)\Phi(h) = \Phi(g)\Phi(h)$  and consequently,

$$\sigma(\Phi(f)\Phi(h)) = \sigma(\Phi(g)\Phi(h)),$$

from which we see that

$$\sigma(fh) = \sigma(gh).$$

We deduce from (2.4) that  $|f| = |g|$  on  $\partial_{\mathcal{A}}(X)$ . Since  $f \neq g$ , there exists a  $y \in \partial_{\mathcal{A}}(X)$  such that  $f(y) \neq g(y)$ ; for otherwise  $f - g$  would vanish on  $\partial_{\mathcal{A}}(X)$ , and so  $f = g$  on  $X$  by (1.4). We may assume that  $f(y) \neq 0$  because if  $f(y) = 0$ , then, since  $|f(y)| = |g(y)|$ , it would follow that  $g(y) = 0 = f(y)$ . Therefore we can choose a neighborhood  $V$  of  $y$  and a peaking function  $h$  such that  $1 = h(y)$ ,  $|h(z)| < 1$  outside  $V$ . Then  $E := P(h) \subset V$ . By (1.5), we can modify  $h$  so that it would still be a peaking function that peaks on  $E$  and moreover satisfies the following:

$$(2.8) \quad \begin{aligned} |f(z)h(z)| &< \max_E |f| = \max_X |fh|, \\ |g(z)h(z)| &< \max_E |g| = \max_X |gh| \end{aligned}$$

for all  $z$  outside  $E$ .

There exists  $\xi \in E$  such that  $|f(\xi)| = \max_E |f| = \|fh\|_{\infty}$ . Since  $\sigma(fh) = \sigma(gh)$ ,  $f(\xi) = f(\xi)h(\xi) = g(z)h(z)$  for some  $z \in X$ . If  $z \notin E$ , then  $|g(z)h(z)| < \max_E |g| = \|gh\|_{\infty} = \|fh\|_{\infty} = |f(\xi)|$ , a contradiction. So  $z \in E$  and  $f(\xi) = g(z)$  where both  $\xi, z$  lie in  $V$ . Since  $V$  is an arbitrary neighborhood of  $y$  and  $f, g$  are continuous, we get  $f(y) = g(y)$ , again a contradiction.

Thus

$$\sigma(fh) = \sigma(gh) \forall h \Leftrightarrow f = g$$

and  $\Phi$  is injective.

Now for the homogeneity. Notice that

$$\sigma(\Phi(cf)\Phi(h)) = \sigma(cf\Phi(h)) = c\sigma(f\Phi(h)) = c\sigma(\Phi(f)\Phi(h)) = \sigma(c\Phi(f)\Phi(h)).$$

Since  $\Phi$  is bijective, we see that  $\Phi(cf) = c\Phi(f) \forall f \in \mathcal{A}$ . □

*Remark 8.*

$$(2.9) \quad |f(x)| = |\Phi(f)(\tau(x))| \quad \forall f \in \mathcal{A}, \quad \forall x \in \partial_{\mathcal{A}}(X).$$

*Proof.* Take  $f \in \mathcal{A}$  and assume first that  $f(y) \neq 0, y \in \partial_{\mathcal{A}}(X)$ . In this case, for any given neighborhood  $V$  of  $y$ , we can find a function  $h$  such that  $h(y) = 1 = \|h\|$  and  $fh$  attains its maximum modulus in  $V$ . (To find  $h$ , let  $k$  be a peaking function

with  $P(k) \subset V$ , and let  $h = k^n$  for some sufficiently large positive integer  $n$ .) There exists a  $\xi$  in  $V$  such that

$$|f(\xi)h(\xi)| = \|fh\|_\infty.$$

But  $\sigma(\Phi(f)\Phi(h)) = \sigma(fh)$  from which it follows that  $|\Phi(f(\tau(y)))\Phi(h(\tau(y)))| \leq |f(\xi)h(\xi)|$ . Since  $\Phi(h(\tau(y))) = h(y) = 1$  (Remark 6),  $|h(\xi)| \leq 1$ , we get

$$|\Phi(f)(\tau(y))| \leq |f(\xi)|.$$

$V$  being arbitrary and  $f$  continuous, we have

$$|\Phi(f)(\tau(y))| \leq |f(y)|.$$

If, on the other hand,  $f(y) = 0$ , we could ensure that  $h$  satisfies  $h(y) = 1 = \|h\|_\infty$  and  $\|fh\|_\infty < \epsilon$  for some preassigned  $\epsilon > 0$ . Hence once again because  $\sigma(\Phi(f)\Phi(h)) = \sigma(fh)$ , we see that  $\|\Phi(f)\Phi(h)\| < \epsilon$  by (2.2) and so

$$|\Phi(f)(\tau(y))\Phi(h)(\tau(y))| < \epsilon,$$

and since  $\Phi(h)(\tau(y)) = 1$ , we get

$$|\Phi(f)(\tau(y))| < \epsilon,$$

which proves  $f(y) = \Phi(f)(\tau(y)) = 0$ .

Now let  $V$  be any neighborhood of  $\tau(y)$ , and assume that  $\Phi(f)(\tau(y)) \neq 0$ . We can, as before, choose  $h'$  with  $h'(\tau(y)) = 1 = \|h'\|$  and  $\Phi(f)h'$  attains its maximum modulus at a point  $\xi$  in  $V$ . Since  $\Phi$  is surjective, let  $\Phi(h) = h'$ . By Remark 6,  $h(y) = 1$  and since  $f(y)h(y)$  belongs to  $\sigma(fh) = \sigma(\Phi(f)\Phi(h))$ , we get

$$f(y) = \Phi(f)(\xi')\Phi(h)(\xi')$$

for some  $\xi'$  in  $X$ . So  $|f(y)| \leq |\Phi(f)(\xi)|$ . By continuity, we see that

$$|f(y)| \leq |\Phi(f)(\tau(y))|.$$

If  $\Phi(f)(\tau(y)) = 0$ , we can repeat an argument similar to the one in the last paragraph and obtain  $f(y) = 0$ .

Putting all these facts together, we see that the proof of Remark 8 is complete.  $\square$

*Remark 9.*  $\tau$  is a homeomorphism of  $\partial_{\mathcal{A}}(X)$  onto itself.

*Proof.* We observe first that  $\tau$  is injective: if  $\tau(x) = \tau(y)$ , then  $|\Phi(f)(\tau(x))| = |\Phi(f)(\tau(y))|$  and this implies that  $|f(x)| = |f(y)|$  for all  $f \in \mathcal{A}$  by Remark 8. Since  $\mathcal{A}$  separates points of  $X$ , it is easily seen that there exist functions  $f$  such that  $f(x) = 0, f(y) = 1$  proving that  $x = y$ . Next we show that  $\tau$  is continuous. Choose any  $x \in X$  and a neighborhood  $V$  of  $\tau(x)$  and a peaking function  $h$  such that

$$h(\tau(x)) = 1, |h(y)| \leq 1/2 \quad \forall y \in X \setminus V.$$

$\Phi$  being surjective, there exists a  $g$  such that  $\Phi(g) = h$ . Since  $|g| \equiv |\Phi(g(\tau))|$  by Remark 8, if we let  $W = \{\xi : |g(\xi)| > 1/2\}$ , then  $\tau(W) \subset V$  because if  $\xi \in W$ , then

$$|h(\tau(\xi))| = |\Phi(g)(\tau(\xi))| = |g(\xi)| > 1/2.$$

Since  $|h(\tau(x))| = |\Phi(g)(\tau(x))| = |g(x)| = 1$ ,  $W$  is a neighborhood of  $x$  in  $\partial_{\mathcal{A}}(X)$ . Thus we have proved that  $\tau$  is injective and continuous.

Now since  $\Phi$  is a bijection, we see that  $\Phi^{-1}$  has the same properties as  $\Phi$ . Thus there would exist an injective continuous map  $\psi : \partial_{\mathcal{A}}(X) \rightarrow \partial_{\mathcal{A}}(X)$  such that

$$|g(x)| \equiv |\Phi^{-1}(g)(\psi(x))| \quad \forall x \in \partial_{\mathcal{A}}(X), \forall g \in \mathcal{A}.$$

Now let  $g = \Phi(h)$ . Then  $|\Phi(h)(x)| = |g(\psi(x))|$ . Now let  $x = \tau(y)$ . Then  $|g(y)| = |\Phi(h)(\tau(y))| = |g(\psi(\tau(y)))|$  by Remark 8. Since functions of type  $|g|$  separate points of  $\partial_{\mathcal{A}}(X)$ , we get  $\psi(\tau(y)) \equiv y$  and by a similar argument, we also obtain  $\tau(\psi(y)) \equiv y$ . Thus we proved that  $\tau$  is a self-homeomorphism of  $\partial_{\mathcal{A}}(X)$ .  $\square$

*Remark 10.*

$$(2.10) \quad f(x) = \Phi(f)(\tau(x)) \text{ for all } x \text{ in } \partial_{\mathcal{A}}(X) \text{ and for all } f \text{ in } \mathcal{A}.$$

Choose any point  $x$  in  $\partial_{\mathcal{A}}(X)$ . Let  $V$  be any open neighborhood of  $x$ . Since  $x$  is in  $\partial_{\mathcal{A}}(X)$ , there exists a peaking function  $h$  such that  $h(x) = 1$  and the peaking set  $P(h) = E$  is contained in  $V$ . Now by Bishop's theorem 1.5, we can modify  $h$  so that it has the same properties as before but, in addition,

$$(2.11) \quad |f(z)h(z)| < \max_E |f| \text{ for all } z \text{ outside } E.$$

Thus there exists a  $\xi$  in  $E$  such that  $|f(\xi)| = \max_E |f| = \|fh\|_{\infty}$ . Since  $\sigma(fh) = \sigma(\Phi(f)\Phi(h))$ , we have  $\|fh\| = \|\Phi(f)\Phi(h)\|$  and so there exists a point  $z$  such that  $f(\xi)h(\xi) = \Phi(f)(z)\Phi(h)(z)$ . We may assume that  $z \in \partial_{\mathcal{A}}(X)$  since the set where  $\Phi(f)\Phi(h)$  assumes the value  $f(\xi)h(\xi)$  is a peaking set and every peaking set meets  $\partial_{\mathcal{A}}(X)$ .

Since  $\tau$  is surjective,  $z = \tau(\eta)$  for some  $\eta$  in  $\partial_{\mathcal{A}}(X)$ . Now by (2.9) we notice that

$$|\Phi(f)(\tau(\eta))\Phi(h)(\tau(\eta))| = |f(\eta)h(\eta)|.$$

Now  $\eta$  must be in  $E$  because otherwise  $|f(\eta)h(\eta)| < |f(\xi)|$  by (2.11). Thus we have found  $\xi, \eta$  in  $E$  such that  $f(\xi) = \Phi(f)(\tau(\eta))$ , since  $\Phi(h)(\tau(\eta)) = h(\eta) = 1$  by Remark 6. Since  $\xi, \eta$  lie in  $V$  and  $V$  is an arbitrary open neighborhood of  $x$ , we get by continuity of  $\tau, f$ , and  $\Phi(f)$  that  $f(x) = \Phi(f)(\tau(x))$ . This completes the proof of (2.10).

*Remark 11.*  $\Phi$  is an algebra isomorphism of  $\mathcal{A}$  onto itself.

*Proof.* We already saw that it is a bijection and homogeneous. Let  $f, g \in \mathcal{A}$ . By (2.10) for any  $x$  in  $\partial_{\mathcal{A}}(X)$ ,

$$f(x) = \Phi(f)(\tau(x)), g(x) = \Phi(g)(\tau(x))$$

and

$$f(x)g(x) = \Phi(fg)(\tau(x)), f(x) + g(x) = \Phi(f + g)(\tau(x)).$$

Thus

$$\Phi(fg)(\tau(x)) = \Phi(f)(\tau(x))\Phi(g)(\tau(x)), \Phi(f + g)(\tau(x)) = \Phi(f)(\tau(x)) + \Phi(g)(\tau(x)).$$

Since  $\tau$  is surjective, we get

$$\Phi(f)(x)\Phi(g)(x) = \Phi(fg)(x), \Phi(f + g)(x) = \Phi(f)(x) + \Phi(g)(x)$$

on all of  $\partial_{\mathcal{A}}(X)$  and then by the maximum principle on all of  $X$ . This completes the proof of Remark 11. The algebraic isomorphism  $\Phi : \mathcal{A} \rightarrow \mathcal{A}$  gives rise to a weak\* homeomorphism  $\Phi^* : \mathcal{A}^* \rightarrow \mathcal{A}^*$ , which in turn induces a homeomorphism  $\Lambda$  of  $X$  (the maximal ideal space of  $\mathcal{A}$ ) onto itself and hence we can state

*Remark 12.* There exists a self-homeomorphism  $\Lambda$  of  $X$  onto itself such that

$$\Phi(f)(\Lambda(x)) = f(x) \text{ on all of } X.$$

But in view of (2.10), we see that  $\Lambda(x) = \tau(x)$  for all  $x$  in  $\partial_{\mathcal{A}}(X)$ . This completes the proof of the Main Theorem announced in the abstract.  $\square$



**Conclusion.** We conclude this paper by observing:

If  $X$  is a compact Hausdorff space (not necessarily first countable), then our Main Theorem clearly holds for  $C_{\mathbb{R}(X)}$  — the Choquet boundary being  $X$  and the peaking functions being those given by Urysohn's lemma — and it follows that Theorem 6 in [M] is valid in this general setting with the same proof as given there.

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