SATYA R. CHAKRAVARTY and DIGANTA MUKHERJEE

MEASURES OF DEPRIVATION AND THEIR MEANING IN TERMS OF SOCIAL SATISFACTION

ABSTRACT. This paper proposes relative and absolute measures of deprivation using social satisfaction functions. The relative (absolute) measure gives us the amount by which social satisfaction can be increased in proportional (absolute) terms by redistributing incomes equally. We also demonstrate the existence of a relationship between summary indices of deprivation (including the Gini coefficient, the maximin index, the coefficient of variation and their absolute counterparts) and social satisfaction.

KEY WORDS: Deprivation, Satisfaction, Social satisfaction functions, Measures of deprivation, Unified approach

1. INTRODUCTION

A person's feeling of deprivation in a society arises out of the comparison of his situation with those of better off persons. Runciman (1966) used the example of promotion to illustrate an individual's feeling of deprivation and argued that the extent of deprivation felt by an individual for not being promoted is an increasing function of the number of persons who have been promoted. Yitzhaki (1979) considered deprivation in terms of income and showed that one plausible index of deprivation in a society is the product of the mean income and the Gini coefficient for the society.

Hey and Lambert (1980) provided an alternative characterization of the Yitzhaki index. Essential to their alternative derivation is Runcimans's remark "The magnitude of a relative deprivation is the extent of the difference between the desired situation and that of the person desiring it" (1966: 10). (See also Temkin (1986) who argued that aggregate inequality in a society should be measured in terms of such differences. Chakravarty (1990) and Amiel and Cowell (1994) provide further discussions along this line). Kakwani (1984) plotted the sum of income share shortfalls of different individuals from richer individuals against the cumulative proportions of persons to

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generate the relative deprivation curve and demonstrated that the area under this curve is the Gini coefficient. Alternatives and variations of the Gini index of deprivation have been suggested by many authors including Chakravarty and Chakraborty (1984), Berrebi and Silber (1985), Paul (1991) and Chakravarty and Chattopadhyay (1994).

However, most of the existing indices have been proposed on an ad hoc basis. The purpose of this paper is to develop a unified approach, which relies on the social satisfaction function, to the measurement of deprivation. Our approach borrows much from the theory of ethical inequality indices (see Atkinson 1970 and Kolm 1976). We suggest indices of both relative and absolute variety. (While a relative index is invariant to equiproportionate changes in all incomes, an absolute index does not alter under equal absolute changes in all incomes.) Our relative (absolute) index can be regarded as the size of proportional (absolute) deviation of social satisfaction from its maximum attainable value. The relative (absolute) index also determines the quantity by which we can increase social satisfaction in proportional (absolute) value by redistributing incomes equally. An alternative interpretation of the absolute index is that it measures the total cost per capita of the deprivation itself. Needless to say, the general deprivation indices proposed here are not meant to supplement the existing indices. Rather, we show how the existing indices (including the Gini coefficient, the maximin index, the coefficient of variation and their absolute counterparts) can be interpreted in our framework and hence can be related to social satisfaction functions in a negative monotonic way.

In Section 2 we discuss social satisfaction functions. The relative and absolute indices are proposed in Sections 3 and 4 respectively. Section 5 concludes.

2. SOCIAL SATISFACTION FUNCTIONS

For a population of size n, the set of income distributions is denoted by D^n, with a typical element x = $(x_1, x_2, \ldots, x_n)$, where D^n is the non-negative orthant of the *Euclidean* n-space R^n with the origin deleted. We will assume that all income distributions are illfare ranked, that is, for all x $\in D^n$, $x_1 \leqslant x_2 \leqslant \cdots \leqslant x_n$. For all x $\in$

D^n, the mean of x is denoted by $\lambda(x)$. By deleting origin from D^n, we ensure that for all x $\in D^n$, $\lambda(x) > 0$. An n-coordinated vector of ones will be denoted by 1^n.

In view of Runciman's remark mentioned in the introduction, the deprivation $d_{ij}(\mathbf{x})$ felt by an individual with income x_i relative to j^{th} person's income x_j is given by

$$d_{ij}(\mathbf{x}) = \frac{1}{n}(x_j - x_i), \quad \text{where } x_j \geqslant x_i,$$
$$= 0 \qquad \text{otherwise.} \tag{1}$$

Note that $d_{ij}(\mathbf{x})$ is increasing in x_j and decreasing in x_i. Now, an individual with income x_i is deprived of all incomes $x_{i+1}, \ldots, x_n$. Therefore the total deprivation felt by this person is

$$d_i(x) = \frac{1}{n} \sum_{j=i+1}^{n} (x_j - x_i). \tag{2}$$

We can rewrite d_i(x) in (2) as $\lambda(x)$ - $\sum_{j=1}^{i} x_j/n$ - $\frac{n-i}{n} x_i$. Following Yitzhaki (1979), Hey and Lambert (1980) and Stark and Yitzhaki (1988) we regard the complement

$$s_i(x) = \sum_{j=1}^{i} \frac{x_j}{n} + \frac{n-i}{n} x_i \tag{3}$$

of d_i(x) to the mean income $\lambda(x)$ as the satisfaction function of the person with income x_i. The function $s_i(x)$ can be interpreted as follows. Note that person i does not feel frustrated if he compares his own income x_i with lower incomes $x_1, x_2, \ldots, x_{i-1}$. His frustration about incomes $x_{i+1}, \ldots, x_n$ can be eliminated by replacing each of these $(n - i)$ higher incomes by x_i. Thus, in the censored income distribution $(x_1, \ldots, x_{i-1}, x_i, \ldots, x_i)$ corresponding to $(x_1, \ldots, x_{i-1}, x_i, x_{i+1}, \ldots, x_n)$ person i does not have any feeling of frustration. Now, given the position of a person in an income distribution, he may be regarded as either being satisfied or frustrated. Since $s_i(\mathbf{x})$ is based on the censored income distribution $(x_1, \ldots, x_{i-1}, x_i, \ldots, x_i)$ in which person i has no feeling of frustration, it can be considered as his satisfaction function.[1] It may be interesting to observe that higher incomes are taken into account sequentially in the definition of $s_i(\mathbf{x})$. Thus, in $s_1(\mathbf{x})$ we consider only x_1 and ultimately in $s_n(\mathbf{x})$ the entire distribution $(x_1, \dots, x_n)$ is incorporated.

The individual satisfaction function s_i possesses many interesting properties:

(i) s_i is increasing in x_i.

(ii) s_i is independent of incomes higher than x_i.

(iii) A rank preserving increase in any income smaller than x_i increases s_i.

(iv) A rank preserving transfer of income between any two persons with income smaller than x_i does not change s_i.

(v) s_i is unit-translatable – an equal absolute change in all incomes changes s_i by the same absolute amount. More precisely, $s_i(x + c1^n) = s_i(x) + c$, where c is any scalar such that $x + c1^n \in D^n$.

(vi) s_i is linearly homogeneous – $s_i(cx) = cs_i(x)$ for all c > 0.

(vii) Given that x_i's are illfare ranked, s_i's are also illfare ranked. That is, for any x $\in D^n$, $x_1 = s_1(x) \leqslant s_2(x) \leqslant \cdots \leqslant s_n(x) = \lambda(x)$. Therefore, for any $x \in D^n$, s(x)=$(s_1(x), \dots, s_n(x)) \in D^n$. Further, if x_i's are equally distributed, that is, if x_i = c > 0 for all i, then s_i = c for all i, $1 \leqslant i \leqslant n$.

(viii) s_i is population replication invariant – an income by income replication of the population leaves s_i unchanged.

(ix) s_i is a continuous function.[2]

Clearly, other specifications of satisfaction functions are also possible. For instance, we can express satisfaction in terms of income squares, income ratios and so on. However, this definition of individual satisfaction function allows us to obtain well-known measures of deprivation like the Gini coefficient and the coefficient of variation in our framework.

We now assume that social satisfaction function W^n is a real valued function of individual satisfaction levels, where W^n is ordinally significant and the superscript n of W^n indicates the dependence of the function on the population size n. That is, W^n, which given $x \in D^n$, associates to the corresponding satisfaction vector s(x) = $(s_1(x), \dots, s_n(x))$, a value $W^n(s(x))$ indicating the level of social satisfaction. This is analogous to the requirement that the

Bergson–Samuelson social welfare function denotes social welfare as a general function of individual utilities. It is supposed that W^n is (i) continuous, (ii) smooth (non-decreasing along the ray of equality and each satisfaction contour crosses the ray of equality), (iii) symmetric and (iv) quasi-concave. The continuity assumption ensures that minor changes in incomes (hence in satisfactions) will generate minor change in W^n. Therefore a continuous satisfaction function will not be over-sensitive to minor observational errors in incomes. Non-decreasingness along ray of equality means that if all individuals enjoy the same income, then less will not be preferred to more. This condition is weaker than any of the Pareto preference conditions. The second condition of smoothness means that each income distribution is socially indifferent (according to satisfaction) to some equal distribution of income. According to symmetry social satisfaction does not alter under any permutation of individual satisfactions. Thus, any information other than individual satisfaction levels (for instance, the names of the individuals) are irrelevant to the measurement of social satisfaction. An implication of symmetry is that W^n can be defined directly on ordered satisfaction vectors (as we have done). Finally, quasi-concavity is the requirement that satisfaction contours are convex to the origin. A satisfaction function W^n satisfying conditions (i)–(iv) will be called regular.

Let us now define the representative level of satisfaction $s_e(x)$ associated with s(x) as that level of satisfaction which, if enjoyed by everybody, will make the existing distribution s(x) socially indifferent. More precisely,

$$W^n(s_e(x)1^n) = W^n(s(x)). \tag{4}$$

Since W^n is regular, we can solve (4) for the unique representative satisfaction

$$s_e(x) = E^n(s(x)). \tag{5}$$

By continuity of W^n, $E^n(s(x))$ is continuous. Furthermore, $E^n(s(x))$ is a particular numerical representation of W^n, that is, for any x,y $\in D^n$,

$$W^n(s(x)) \geqslant W^n(s(y)) \Leftrightarrow E^n(s(x)) \geqslant E^n(s(y)). \tag{6}$$

Thus, one satisfaction profile is preferred to another with the same population size if and only if its representative satisfaction is higher. The indifference surfaces of E^n are numbered so that for all c > 0,

$$E^n(s(c1^n)) = c. \tag{7}$$

By symmetry and quasi-concavity of W^n, for any x $\in D^n$, $s_e(x) \leqslant \sum_{i=1}^{n} s_i(x)/n \leqslant \lambda(x) = s_n(x)$. Maximum social satisfaction is achieved when incomes are perfectly equalized which in turn implies that all individuals enjoy the maximum possible satisfaction $s_n(\mathbf{x}) = \lambda(\mathbf{x})$.

3. RELATIVE MEASURES OF DEPRIVATION

As a general measure of deprivation we suggest the use of $I^n(x)$, the proportionate gap between the representative satisfaction $E^n(s(x))$ and its maximum attainable value $\lambda(x)$, where the income distribution x is arbitrary. More precisely,

$$I^n(x) = 1 - \frac{E^n(s(x))}{\lambda(x)}. \tag{8}$$

For a regular satisfaction function, $I^n(x)$ is continuous,symmetric in incomes and bounded between zero and one, where the lower bound is achieved whenever income (hence satisfaction) levels are equal. When efficiency considerations are absent (that is, mean income is fixed), an increase in I^n is equivalent to a reduction in satisfaction and vice-versa. From policy point of view I^n is a measure of the amount (in proportional terms) by which social satisfaction could be increased if incomes were redistributed equally. Given a functional form for I^n, we can recover E^n as

$$E^n(s(x)) = \lambda(x)(1 - I^n(x)). \tag{9}$$

Next, using (9), we can retrieve W^n with the help of (5) and (4). (In fact, W^n is an ordinal transform of E^n, that is, $W^n(s(x)) = f(E^n(s(x))) = f(\lambda(x)(1 - I^n(x)))$, where f is increasing in its argument.)

In general, I^n is not a relative index. Since $\lambda(x)$ is linearly homogeneous, from (8) it follows that I^n will be a relative index, that is, it remains invariant under equiproportionate changes in all incomes, whenever $E^n(s(x))$ is linearly homogeneous in incomes. This is equivalent to the requirement that W^n is homothetic. The converse is also true, that is, given homotheticity of W^n, I^n becomes a relative index.

To illustrate the general formula in (8) let us assume that E^n is the symmetric mean of order r, that is,

$$E_r^n(s(x)) = \left(\frac{1}{n}\sum_{i=1}^{n}(s_i(x))^r\right)^{\frac{1}{r}}, \; r \leqslant 1, r \neq 0,$$
$$= \prod_{i=1}^{n}(s_i(x))^{\frac{1}{n}}, \quad r = 0. \tag{10}$$

The corresponding deprivation index becomes

$$I_r^n(x) = 1 - \frac{1}{\lambda(x)}\left(\frac{1}{n}\sum_{i=1}^{n}\left(\sum_{j=1}^{i}\frac{x_j}{n} + \frac{n-i}{n}x_i\right)^r\right)^{\frac{1}{r}}, \; r \leqslant 1, r \neq 0,$$
$$= 1 - \frac{1}{\lambda(x)}\prod_{i=1}^{n}\left(\sum_{j=1}^{i}\frac{x_j}{n} + \frac{n-i}{n}x_i\right)^{\frac{1}{n}}, \quad r = 0. \tag{11}$$

The parameter r in (10) determines the curvature of the social satisfaction contour.[3] As r decreases E_r^n (I_r^n) in (10) ((11)) becomes more sensitive to the satisfaction (deprivation) of the poorer persons. For r=1, E_r^n becomes the Gini satisfaction function

$$E_1^n(s(x)) = \frac{1}{n^2}\sum_{i=1}^{n}(2(n-i)+1)x_i, \tag{12}$$

whose associated index in (11) is the relative Gini deprivation index $G(x) = 1 - \frac{1}{n^2}\sum_{i=1}^{n}(2(n-i)+1)x_i/\lambda(x)$. On the other hand, as r $\to -\infty$, $E_r^n \to \min_i s_i(x) = x_1$, the Rawlsian (1971) maximin satisfaction function and the corresponding index becomes the relative maximin deprivation index $1 - x_1/\lambda(x)$.

We now wish to illustrate how one can identify the satisfaction functions associated with a given numerical measure of deprivation.

As the first example let us consider the deprivation index $C^n(x)$ suggested in Chakravarty and Chakraborty (1984), where

$$C^n(x) = \surd\left(\sum_{i=1}^{n}\sum_{j=i+1}^{n}(x_j - x_i)^2/(n^2\lambda^2(x))\right)$$
$$= \surd\left(\frac{n-1}{n}((v^n(x))^2 + 1) - 2\sum_{i=1}^{n} x_i \sum_{j=i+1}^{n} x_j/(n^2\lambda^2(x))\right). \quad (13)$$

Here $v^n(x) = \surd(\frac{1}{n}\sum(x_i/\lambda(x))^2 - 1)$ is the coefficient of variation. Given other things, an increase in the coefficient of variation increases C^n and vice-versa. Now, C^n is bounded above by $\surd(n-1)$. This bound is achieved when the entire income is monopolized by the richest person and all the other persons receive zero income. Therefore we consider the transformed index $C^n(x)/\surd(n-1)$ which is normalized over [0,1]. Then the satisfaction function corresponding to this normalized index becomes

$$E_c^n(s(x)) = \lambda(x)(1 - C^n(x)/(\surd(n-1))). \quad (14)$$

An alternative of interest arises from Paul's (1991) index given by

$$P_e^n(x) = \frac{1}{n^2}\sum_{i=1}^{n}\sum_{j=i+1}^{n}\left(\frac{x_j}{x_i}\right)^e - \sum_{i=1}^{n}\frac{n_i}{n^2},\ 0 < e < 1, \quad (15)$$

where n_i is the number of persons richer than i. This index is bounded above by $\frac{n(n-1)}{2n^2}(\frac{x_n}{x_1}) = U_n$ (say). Since for P_e^n to be defined we require positivity of all incomes, the upper bound depends on the specific income values. Therefore, the satisfaction function corresponding to the normalized index $P_e^n(x)/U_n$ becomes $\lambda(x)(1 - P_e^n(x)/U_n)$. Clearly, while in this case the individual satisfaction functions can be assumed to depend on income ratios, in (14) they depend on income squares.

Thus, given any homothetic satisfaction function we can generate the corresponding deprivation index. Conversely, we can associate a satisfaction function to any deprivation index in a negative monotonic way.

4. ABSOLUTE MEASURES OF DEPRIVATION

As an alternative to I^n in (8) we propose the use of A^n, the absolute shortfall of the representative satisfaction from its maximum attainable value as a general index of deprivation. More precisely,

$$A^n(x) = \lambda(x) - E^n(s(x)). \tag{16}$$

Given regularity of W^n, A^n is continuous and symmetric in incomes, and has zero as its greatest lower bound, which is achieved whenever incomes are equal. Given a form of A^n, we can recover E^n (hence W^n) with the help of (16),(5) and (4).

Using arguments similar to that employed on section 3 we can show that A^n is an absolute index, that is, it remains invariant under equal absolute changes in all incomes, if and only if E^n is unit-translatable (equivalently, W^n is translatable). From policy point of view the absolute index A^n tells us by how much social satisfaction could be increased (in absolute terms) if incomes were redistributed equally. It also gives us the amount of money that must be given to each person to achieve a distribution that generates the same level of social satisfaction as the distribution where each individual receives the present mean income. Thus, the absolute index calculates the per capita cost of deprivation. This is another policy interpretation of A^n.

To illustrate the general formula in (16), let us suppose that E^n is of the Kolm–Pollak type, that is,

$$E^n_\theta(s(x)) = -\frac{1}{\theta}\log\frac{\sum_{i=1}^n e^{-\theta s_i}}{n}, \tag{17}$$

whose associated index becomes

$$A^n_\theta(x) = \frac{1}{\theta}\log\frac{\sum_{i=1}^n e^{\theta(\lambda(x)-s_i)}}{n}, \tag{18}$$

where the parameter θ, which determines the curvature of satisfaction contours, is non-negative.[4] As $\theta \to 0$, $E^n_\theta(s(x)) \to$ Gini satisfaction function and the corresponding deprivation index becomes the Gini absolute index of deprivation $\lambda(x) - \frac{1}{n^2}\sum_{i=1}^n (2(n-i)+1)x_i$, the index proposed by Yitzhaki. On the other hand as $\theta \to \infty$, $A^n_\theta(x) \to \lambda(x) - x_1$, the absolute maximin deprivation index. Note that both the Gini and the maximin relative indices satisfy a compromise property - when multiplied by the mean income they become absolute indices. The associated satisfaction functions are both homothetic and translatable. The compromise property is also satisfied by the index considered in (13). In this case the absolute index $\lambda(x)C^n(x)$ becomes monotonically related to the standard deviation $\lambda(x)v^n(x)$, the absolute counterpart to the coefficient of variation.

Thus, we can determine a unique absolute measure of deprivation from a given translatable social satisfaction function. Given a deprivation index we can relate it to a satisfaction function in a negative monotonic way. Choice of a particular satisfaction function is essentially a matter of value judgment.

5. CONCLUSION

In this paper we have proposed general indices of relative (absolute) notions of deprivation using social satisfaction functions. We have shown that to every homothetic (translatable) social satisfaction function, there corresponds a unique relative (absolute) index of deprivation. Conversely, for each deprivation index a social satisfaction function can be found that imply the index. We have analyzed a number of indices of deprivation along this line. In particular, we take the Gini coefficient, the maximin index, the coefficient of variation and their absolute versions. The simple additive form of individual satisfaction we considered enabled us to obtain these well-known measures in our structure. Which social satisfaction function will be adopted becomes an issue of value judgment.

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NOTES

1. For further discussions on censored income distributions and their role in the construction of poverty indices see Takayama (1979) and Chakravarty (1990).
2. Notions of satisfaction (deprivation) ordering using $s_i(x)$ and $s_i(x)/\lambda(x)$ ($d_i(x)$ and $d_i(x)/\lambda(x)$) and their consequences in terms of alternative redistributive criteria have been examined in Chakravarty (1997) and Chakravarty, Chattopadhyay and Majumdar (1995).
3. E^n verifies an especially attractive aggregative property. Consider any partitioning of a population of size n into subgroups with respect to some homogeneous characteristic, say, age, sex, race, region etc. Now, consider the n-person income distribution in which an individual enjoys his subgroup representative income. Then E^n is the only linearly homogeneous representative income for which this distribution becomes socially indifferent to the actual distribution of income.
4. The Kolm-Pollak function is the only unit-translatable function which verifies the aggregation property discussed in Note 3.

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Address for correspondence: Satya R. Chakravarty, Economic Research Unit, Indian Statistical Institute, 203 B.T.Road, Calcutta 700 035, India
Fax: +91-33-577-6680; E-mail: satya@isical.ac.in