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Sequential Testing for Simple Hypotheses for Processes Driven by Fractional Brownian Motion

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Abstract: We prove the existence of an optimal sequential-test procedure for a simple null hypothesis that the observed process is a noise modeled by a fractional Brownian motion against the simple alternate hypothesis that the observed process is the sum of an unobserved signal and the noise.

Keywords: Fractional Brownian motion; Optimal test; Stochastic differential equations; Sequential test.

Subject Classification: 62M09; 60G15.

1. INTRODUCTION

Statistical inference for diffusion-type processes satisfying stochastic differential equations driven by Wiener processes has been studied earlier, and a comprehensive survey of various methods is given in Prakasa Rao (1999a). There has been a recent interest in studying similar problems for stochastic processes driven by a fractional Brownian motion (fBm). Le Breton (1998) studied parameter estimation and filtering in a simple linear model driven by a fractional Brownian motion.

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In a recent paper, Kleptsyna and Le Breton (2002) studied parameter-estimation problems for a fractional Ornstein-Uhlenbeck-type process. This is a fractional analogue of the Ornstein-Uhlenbeck process, that is, a continuous-time first-order autoregressive process $X = \{X_t, t \geq 0\}$, which is the solution of a one-dimensional homogeneous linear stochastic differential equation driven by a fBm $W^H = \{W_t^H, t \geq 0\}$ with Hurst parameter $H \in [1/2, 1)$. Such a process is the unique Gaussian process satisfying the stochastic integral equation

$$X_t = \theta \int_0^t X_s ds + \sigma W_t^H, \quad t \geq 0. \tag{1.1}$$

They investigate the problem of estimation of the parameters θ and σ^2 based on the observation $\{X_s, 0 \leq s \leq T\}$ and prove that the maximum likelihood estimator $\hat{\theta}_T$ is strongly consistent as $T \to \infty$.

We discussed more general classes of stochastic processes satisfying linear stochastic differential equations driven by an fBm and studied the asymptotic properties of the maximum likelihood and the Bayes estimators for parameters involved in such processes in Prakasa Rao (2003a,b). It is well known that sequential procedures can be used for estimation and testing problems leading to shorter expected period of observation time as compared to fixed sample procedures. Novikov (1972) investigated the asymptotic properties of a sequential maximum likelihood estimator for the drift parameter in the Ornstein-Uhlenbeck process. We have discussed analogous results for a fractional Ornstein-Uhlenbeck type process in Prakasa Rao (2004a).

We study the sequential-testing problem for a simple null hypothesis that an observable process is a special case of the noise modeled by an fBm against the simple alternate hypothesis that the process also contains an unobservable signal along with the noise. Self-similar processes and fBm have been used for modeling phenomena with long-range depedence. It was recently observed that such a phenomenon occurs in problems connected with traffic patterns of packet flows in high-speed data networks such as the Internet and in the study of economic behavior in finance (cf. Prakasa Rao, 2004b). The motivation for the present study comes from such observations, which in turn can be looked as modeling in the branch of signal processing. Suppose we surmise that a signal (which is unobserved) is possibly transmitted over a channel corrupted by an fBm. We are interested in testing the simple hypothesis that there is no transmitted signal, but only a noise modeled by an fBm that is transmitted through the channel, against the hypothesis that a signal is transmitted corrupted by a noise modeled by the fBm. We prove the existence of an optimal sequential-testing procedure for such a problem. Results obtained are analogues of similar results for diffusion processes derived in Liptser and Shiryayev (2001b).

2. PRELIMINARIES

Let $(\Omega, \mathcal{F}, (\mathcal{F}_t), P)$ be a stochastic basis satisfying the usual conditions. The natural filtration of a stochastic process is understood as the P-completion of the filtration generated by this process.

Let $W^H = \{W_t^H, t \geq 0\}$ be a normalized fBm with Hurst parameter $H \in (0, 1)$, that is, a Gaussian process with continuous sample paths such that $W_0^H = 0$, $E(W_t^H) = 0$ and

$$E(W_s^H W_t^H) = \frac{1}{2}[s^{2H} + t^{2H} - |s-t|^{2H}], \quad t \geq 0, \; s \geq 0. \tag{2.1}$$

Let us consider a stochastic process $Y = \{Y_t, t \geq 0\}$ defined by the stochastic integral equation

$$Y_t = \int_0^t C(s)ds + \int_0^t B(s)dW_s^H, \quad Y_0 = 0, \; t \geq 0 \tag{2.2}$$

where $C = \{C(t), t \geq 0\}$ is an $(\mathcal{F}_t)$-adapted process and $B = \{B(t), t \geq 0\}$ is a nonvanishing nonrandom function. For convenience, we write (2.2) in the following stochastic differential equation form:

$$dY_t = C(t)dt + B(t)dW_t^H, \quad Y_0 = 0, \; t \geq 0 \tag{2.3}$$

driven by the fBm W^H. The integral

$$\int_0^t B(s)dW_s^H \tag{2.4}$$

is not a stochastic integral in the Ito sense, but one can define the integral of a deterministic function with respect to the fBM in a natural sense (cf. Norros et al., 1999). Even though the process Y is not a semimartingale, one can associate a semimartingale $Z = \{Z_t, t \geq 0\}$, which is called a *fundamental semimartingale*, such that the natural filtration $(\mathcal{Z}_t)$ of the process Z coincides with the natural filtration $(\mathcal{Y}_t)$ of the process Y (Kleptsyna et al., 2000a). Define, for $0 < s < t$,

$$k_H = 2H\Gamma\left(\frac{3}{2} - H\right)\Gamma\left(H + \frac{1}{2}\right), \tag{2.5}$$

$$\kappa_H(t, s) = k_H^{-1} s^{\frac{1}{2}-H}(t-s)^{\frac{1}{2}-H}, \tag{2.6}$$

$$\lambda_H = \frac{2H\Gamma(3-2H)\Gamma(H+\frac{1}{2})}{\Gamma(\frac{3}{2}-H)}, \tag{2.7}$$

$$m_t^H = \lambda_H^{-1} t^{2-2H}, \tag{2.8}$$

where the function $\Gamma(.)$ is the Euler gamma function and

$$M_t^H = \int_0^t \kappa_H(t,s)dW_s^H, \quad t \geq 0. \tag{2.9}$$

The process M^H is a Gaussian martingale, called the *fundamental martingale* (cf. Norros et al., 1999), and its quadratic variation $\langle M_t^H \rangle = m_t^H$. Furthermore, the natural filtration of the martingale M^H coincides with the natural filtration of the fBm W^H. In fact, the stochastic integral

$$\int_0^t B(s)dW_s^H \tag{2.10}$$

can be represented in terms of the stochastic integral with respect to the martingale M^H. For a measurable function f on $[0, T]$, let

$$K_H^f(t,s) = -2H\frac{d}{ds}\int_s^t f(r)r^{H-\frac{1}{2}}(r-s)^{H-\frac{1}{2}}\,dr, \quad 0 \leq s \leq t \tag{2.11}$$

when the derivative exists in the sense of absolute continuity with respect to the Lebesgue measure (see Samko et al., 1993 for sufficient conditions). The following result is due to Kleptsyna et al. (2000b).

Theorem 2.1. *Let M^H be the fundamental martingale associated with the fBm W^H as given by* (2.9). *Then*

$$\int_0^t f(s)dW_s^H = \int_0^t K_H^f(t,s)dM_s^H, \quad t \in [0,T] \tag{2.12}$$

a.s $[P]$ *whenever both sides are well defined.*

Suppose the sample paths of the process $\{\frac{C(t)}{B(t)}, t \geq 0\}$ are smooth enough (see Samko et al., 1993) so that the process

$$Q_H(t) = \frac{d}{dm_t^H}\int_0^t \kappa_H(t,s)\frac{C(s)}{B(s)}ds, \quad t \in [0,T] \tag{2.13}$$

is well defined almost everywhere where w^H and k_H are as defined in (2.8) and (2.6) respectively and the derivative is understood in the sense of absolute continuity. Let the process $Z = (Z_t, t \in [0,T])$ be defined by

$$Z_t = \int_0^t \kappa_H(t,s)[B(s)]^{-1}dY_s \tag{2.14}$$

where the function $\kappa_H(t,s)$ is as defined in (2.6). The process Z defines a semimartingale associated with the process Y, and the natural filtration $(\mathcal{Z}_t)$ of Z coincides with the natural fitration $(\mathcal{Y}_t)$ of Y. The following theorem is due to Kleptsyna et al. (2000a).

Theorem 2.2. *Suppose the sample paths of the process* Q_H *defined by* (2.13) *belong P-a.s to* $L^2([0,T], dw^H)$ *where* w^H *is as defined by* (2.8). *Define the process Z as in* (2.14). *Then the following results hold.*

(i) *The process Z is an* $(\mathcal{F}_t)$ *-semimartingale with the decomposition*

$$Z_t = \int_0^t Q_H(s)dw_s^H + M_t^H \tag{2.15}$$

where M^H *is the fundamental martingale given by* (2.9).

(ii) *The process Y admits the representation*

$$Y_t = \int_0^t K_H^B(t,s)dZ_s \tag{2.16}$$

where the function $K_H^B(.,.)$ *is as in* (2.11).

(iii) *The natural fitrations of* $(\mathcal{Z}_t)$ *and* $(\mathcal{Y}_t)$ *coincide.*

Kleptsyna et al. (2000a) derived the following Girsanov-type formula as a consequence of Theorem 2.2.

Theorem 2.3. *Suppose the assumptions of Theorem* 2.2 *hold. Define*

$$\Lambda_H(T) = \exp\left\{-\int_0^T Q_H(t)dM_t^H - \frac{1}{2}\int_0^T Q_H^2(t)dm_t^H\right\}. \tag{2.17}$$

Suppose that $E(\Lambda_H(T)) = 1$. *Then the measure* $P^* = \Lambda_H(T)P$ *is a probability measure and the probability measure of the process Y under* P^* *is the same as that of the process V defined by*

$$V_t = \int_0^t B(s)dW_s^H, \quad 0 \le t \le T. \tag{2.18}$$

3. MAIN RESULTS

Suppose that $\theta = \{\theta_t, t \ge 0\}$ is an unobservable $\mathcal{F}_t$-adapted process independent of the fBm $W = \{W_t^H, t \ge 0\}$. Suppose that one of the following two hypotheses hold for the $\mathcal{F}_t$-adapted observable process $\psi = \{\psi_t, t \ge 0\}$:

$$H_0 : d\psi_t = dW_t^H, \quad \psi_0 = 0, \ t \ge 0, \tag{3.1}$$

and

$$H_1 : d\psi_t = \theta_t dt + dW_t^H, \quad \psi_0 = 0, \ t \ge 0. \tag{3.2}$$

If we interpret the process θ as a signal and the fBm W^H as the noise, then we are interested in testing the simple hypothesis H_1 indicating the presence of the signal in the observation of the process ψ against the simple hypothesis H_0 that the signal θ is absent. Assume that the sample paths of the process $\{\theta_t, t \geq 0\}$ are smooth enough so that the process

$$Q(t) = \frac{d}{dm_t^H} \int_0^t \kappa_H(t,s)\theta_s\, ds, \quad t \geq 0 \tag{3.3}$$

is well defined almost everywhere where m_t^H and $\kappa_H(t,s)$ are as defined in (2.8) and (2.6) respectively. Suppose the sample paths of the process $\{Q(t),\ 0 \leq t \leq T\}$ belong almost surely to $L^2([0,T], dm_t^H)$ for every $T \geq 0$. Define

$$Z_t = \int_0^t \kappa_H(t,s) d\psi_s, \quad t \geq 0. \tag{3.4}$$

Then the process $Z = \{Z_t, t \geq 0\}$ is an $(\mathcal{F}_t)$-semimartingale with the decomposition

$$Z_t = \int_0^t Q(s) dw_s^H + M_t^H \tag{3.5}$$

where M^H is the fundamental martingale defined by (2.9), and the process ψ admits the representation

$$\psi_t = \int_0^t K_H(t,s) dZ_s. \tag{3.6}$$

Here the function $K_H(.,.)$ is given by (2.11) with $f \equiv 1$. We denote the probability measure of the process ψ under H_i as P_i for $i = 0, 1$. Let E denote the expectation under the probability measure P and E_i denote the expectation under the hypothesis $H_i, i = 0, 1$. Let P_i^T be the measure induced by the process $\{\psi_t, 0 \leq t \leq T\}$ under the hypothesis H_i. Following Theorem 2.3, we get that the Radon-Nikodym derivative of P_1^T with respect to P_0^T is given by

$$\frac{dP_1^T}{dP_0^T} = \exp\left[\int_0^T Q(s) dZ_s - \frac{1}{2}\int_0^T Q^2(s) dw_s^H\right]. \tag{3.7}$$

Let us consider the sequential plan $\Delta = \Delta(\tau, \delta)$ for testing H_0 versus H_1 characterized by the stopping time τ and the decision function δ. We assume that τ is a stopping time with respect to the family of σ-algebras $\mathcal{B}_t = \sigma\{x : x_s, s \leq t\}$ where $x = \{x_t, t \geq 0\}$ are continuous functions with $x_0 = 0$. The decision function $\delta = \delta(x)$ is $\mathcal{B}_\tau$-measurable and takes the values 0 and 1. Suppose x is the observed sample path. If $\delta(x)$ takes the value 0, then it amounts to the decision that the hypothesis H_0 is accepted, and if $\delta(x)$ takes the value 1, then it will indicate the acceptance of the hypothesis H_1. For any sequential plan $\Delta = \Delta(\tau, \delta)$, define

$$\alpha(\Delta) = P_1(\delta(\psi) = 0), \quad \beta(\Delta) = P_0(\delta(\psi) = 1).$$

Observe that $\alpha(\Delta)$ and $\beta(\Delta)$ are the first and second kind of errors respectively. Let $\Delta_{\alpha,\beta}$ be the class of sequential plans for which

$$\alpha(\Delta) \leq \alpha, \quad \beta(\Delta) \leq \beta$$

where $0 < \alpha + \beta < 1$, and

$$E_i\left(\int_0^{\tau(\psi)} m_t^2(\psi)dm_t^H\right) < \infty, \quad i = 0, 1. \tag{3.8}$$

We now state the main theorem giving the optimum sequential plan subject to the conditions stated above.

Theorem 3.1. *Suppose the process $Q = \{Q_t, \mathcal{F}_t, t \geq 0\}$ defined above satisfies the condition*

$$E|Q_t| < \infty, \quad 0 \leq t < \infty. \tag{3.9}$$

Let

$$m_t(\psi) = E_1(Q_t|\mathcal{F}_t^{\psi}). \tag{3.10}$$

Suppose that

$$P_i\left\{\int_0^{\infty} m_t^2(\psi)dm_t^H = \infty\right\} = 1, \quad i = 0, 1. \tag{3.11}$$

Then there exists a sequential plan $\tilde{\Delta} = \Delta(\tilde{\tau}, \tilde{\delta})$ in the class $\Delta_{\alpha,\beta}$ that is optimal in the sense that for any other sequential plan $\Delta = \Delta(\tau, \delta)$ in $\Delta_{\alpha,\beta}$,

$$E_i\left(\int_0^{\tilde{\tau}(\psi)} m_t^2(\psi)dm_t^H\right) \leq E_i\left(\int_0^{\tau(\psi)} m_t^2(\psi)dm_t^H\right), \quad i = 0, 1. \tag{3.12}$$

The sequential plan $\tilde{\Delta} = \Delta(\tilde{\tau}, \tilde{\delta})$ is defined by the relations

$$\tilde{\tau}(\psi) = \inf\{t : \lambda_t(\psi) \geq B \text{ or } \lambda_t(\psi) \leq A\},$$

and

$$\begin{aligned}\tilde{\delta}(\psi) &= 1 \text{ if } \lambda_{\tilde{\tau}(\psi)} \geq B,\\ &= 0 \text{ if } \lambda_{\tilde{\tau}(\psi)} \leq A,\end{aligned}$$

where

$$\lambda_t(\psi) = \int_0^t m_s(\psi)dZ_s - \frac{1}{2}\int_0^t m_s^2(\psi)dw_s^H$$

and

$$A = \log\frac{\alpha}{1-\beta}, \quad B = \log\frac{1-\alpha}{\beta}.$$

Furthermore,

$$E_0\left(\int_0^{\tilde{\tau}(\psi)} m_t^2(\psi)dm_t^H\right) = 2\ \ V(\beta,\alpha), \tag{3.13}$$

and

$$E_1\left(\int_0^{\tilde{\tau}(\psi)} m_t^2(\psi)dm_t^H\right) = 2\ \ V(\alpha,\beta), \tag{3.14}$$

where

$$V(x,y) = (1-x)\log\frac{1-x}{y} + x\log\frac{x}{1-y}. \tag{3.15}$$

We first derive three lemmas that will be used to prove the main result.

Lemma 3.1. *The sequential plan* $\tilde{\Delta} = \Delta(\tilde{\tau},\tilde{\delta})$ *satisfies the properties*

$$P_i(\tilde{\tau}(\psi) < \infty) = 1, \quad i = 0, 1. \tag{3.16}$$

Proof. Note that

$$P_0(\tilde{\tau}(\psi) < \infty) = P(\tilde{\tau}(W^H) < \infty)$$

since $\psi_t = W_t^H$ under H_0. Let

$$\sigma_n(W^H) = \inf\left\{t : \int_0^t m_s^2(W^H)dw_s^H \geq n\right\}.$$

Then

$$\lambda_{\tilde{\tau}(W^H)\wedge\sigma_n(W^H)}(W^H) = \int_0^{\tilde{\tau}(W^H)\wedge\sigma_n(W^H)} m_s(W^H)dM_t^H$$
$$- \frac{1}{2}\int_0^{\tilde{\tau}(W^H)\wedge\sigma_n(W^H)} m_s^2(W^H)dw_s^H$$

and

$$A \leq \lambda_{\tilde{\tau}(W^H)\wedge\sigma_n(W^H)}(W^H) \leq B.$$

Hence

$$A \leq E(\lambda_{\tilde{\tau}(W^H)\wedge\sigma_n(W^H)}(W^H)) \leq B,$$

which implies that

$$E\left(\int_0^{\tilde{\tau}(W^H)\wedge\sigma_n(W^H)} m_s^2(W^H)dw_s^H\right) \leq 2(B-A) < \infty,$$

since $0 < \alpha + \beta < 1$. In particular, we have

$$E\left(\int_0^{\tilde{\tau}(W^H)} m_s^2(W^H)dw_s^H\right) \leq 2(B-A) < \infty. \tag{3.17}$$

Since

$$E\left(\int_0^{\tilde{\tau}(W^H)} m_s^2(W^H)dw_s^H) \geq E(I_{\{\tilde{\tau}(W^H)=\infty\}}\int_0^\infty m_s^2(W^H)dw_s^H\right),$$

it follows that $P(\tilde{\tau}(W^H) < \infty) = 1$ from equation (3.11). Applying an analogous argument, we can prove that $P_1(\tilde{\tau}(\psi) < \infty) = 1$. This completes the proof. □

Let

$$v_t = Z_t - \int_0^t m_s(\psi)dw_s^H. \tag{3.18}$$

Then

$$dZ_t = m_s(\psi)dw_s^H + dv_t, \quad t \geq 0 \tag{3.19}$$

where $\{v_t, \mathcal{F}_t^\psi, t \geq 0\}$ is a continuous Gaussian martingale with $\langle v\rangle_t = m_t^H$. Furthermore, under H_1,

$$\lambda_t(\psi) = \int_0^t m_s(\psi)dv_s + \frac{1}{2}\int_0^t m_s^2(\psi)dw_s^H. \tag{3.20}$$

This can be seen from Theorem 2 in Kleptsyna et al. (2000a).

Remark 3.1. The random variable $\lambda_{\tilde{\tau}(\psi)}$ takes the values A and B only almost surely under the probability measures P_0 as well as P_1.

Lemma 3.2. *The sequential plan* $\tilde{\Delta} = \Delta(\tilde{\tau}, \tilde{\delta})$ *defined in Theorem* 3.1 *has the property*

$$\alpha(\tilde{\Delta}) = \alpha;\ \beta(\tilde{\Delta}) = \beta.$$

Proof. Since

$$\alpha(\tilde{\Delta}) = P_1(\tilde{\delta}(\psi) = 0) = P_1(\lambda_{\tilde{\tau}(\psi)}(\psi) = A)$$

and

$$\beta(\tilde{\Delta}) = P_0(\tilde{\delta}(\psi) = 1) = P_1(\lambda_{\tilde{\tau}(\psi)}(\psi) = B),$$

it is sufficient to prove that

$$P_1(\lambda_{\tilde{\tau}(\psi)}(\psi) = A) = \alpha; \quad P_0(\lambda_{\tilde{\tau}(\psi)}(\psi) = A) = \beta. \tag{3.21}$$

Following the techniques in Liptser and Shiryayev (2001b, p. 251), let $a(x)$ and $b(x)$, $A \le x \le B$ be the solutions of the differential equations

$$a''(x) + a'(x) = 0, \quad a(A) = 1,\ a(B) = 0 \tag{3.22}$$

and

$$b''(x) + b'(x) = 0, \quad b(A) = 0,\ b(B) = 1. \tag{3.23}$$

It can be checked that

$$a(x) = \frac{e^A(e^{B-x} - 1)}{e^B - e^A}, \quad b(x) = \frac{e^x - e^A}{e^B - e^A} \tag{3.24}$$

and

$$a(0) = \alpha; \quad b(0) = \beta. \tag{3.25}$$

□

We will first prove that

$$P_1(\lambda_{\tilde{\tau}(\psi)}(\psi) = A) = \alpha. \tag{3.26}$$

Let

$$\sigma_n(\psi) = \inf\left\{t : \int_0^t m_s^2(\psi)dw_s^H \ge n\right\}.$$

Applying the generalized Ito-Ventzell formula for continuous local martingales (cf. Prakasa Rao, 1999b, p. 76), we obtain that

$$\begin{aligned}
a(\lambda_{\bar{\tau}(\psi)\wedge\sigma_n(\psi)}(\psi)) &= a(0) + \int_0^{\bar{\tau}(\psi)\wedge\sigma_n(\psi)} a'(\lambda_t(\psi))m_s(\psi)dv_s \\
&\quad + \frac{1}{2}\int_0^{\bar{\tau}(\psi)\wedge\sigma_n(\psi)} [a'(\lambda_t(\psi)) + a''(\lambda_t(\psi))]m_s^2(\psi)dw_s^H \\
&= \alpha + \int_0^{\bar{\tau}(\psi)\wedge\sigma_n(\psi)} a'(\lambda_t(\psi))m_s(\psi)dv_s. \qquad (3.27)
\end{aligned}$$

But

$$\begin{aligned}
&E_1\int_0^{\bar{\tau}(\psi)\wedge\sigma_n(\psi)} [a'(\lambda_t(\psi))m_s(\psi)]^2 dw_s^H \\
&\leq \sup_{A\leq x\leq B}[a'(x)]^2 E_1\left(\int_0^{\bar{\tau}(\psi)\wedge\sigma_n(\psi)} m_s^2(\psi)dw_s^H\right) \\
&\leq n\sup_{A\leq x\leq B}[a'(x)]^2 < \infty.
\end{aligned}$$

Hence

$$E_1\left(\int_0^{\bar{\tau}(\psi)\wedge\sigma_n(\psi)} a'(\lambda_t(\psi))m_s(\psi)dv_s\right) = 0.$$

Taking the expectation under the probability mesasure P_1 on both sides of (3.27), we get that

$$E_1(a(\lambda_{\bar{\tau}(\psi)\wedge\sigma_n(\psi)}(\psi)) = \alpha.$$

Observe that the function $a(x)$ is bounded on the interval $[A, B]$ and $\sigma_n(\psi) \to \infty$ a.s. under P_1 as $n \to \infty$. An application of the dominated convergence theorem proves that

$$E_1[a(\lambda_{\bar{\tau}(\psi)\wedge\sigma_n(\psi)}(\psi))] = \alpha. \qquad (3.28)$$

Applying Lemma 3.1, noting that $\lambda_{\bar{\tau}(\psi)}$ takes only the values A and B a.s under the probability measure P_1 and observing that $a(A) = 1$ and $a(B) = 0$, we obtain that

$$\begin{aligned}
\alpha &= E_1[a((\lambda_{\bar{\tau}(\psi)})] \\
&= 1.P_1(\lambda_{\bar{\tau}(\psi)} = A) + 0.P_1(\lambda_{\bar{\tau}(\psi)} = B) \\
&= P_1(\lambda_{\bar{\tau}(\psi)} = A). \qquad (3.29)
\end{aligned}$$

Similar arguments show that

$$P_0(\lambda_{\bar{\tau}(\psi)} = B) = \beta. \qquad (3.30)$$

Lemma 3.3. *The relations* (3.13) *and* (3.14) *hold for the sequential plan* $\tilde{\Delta} = \Delta(\tilde{\tau}, \tilde{\delta})$.

Proof. Proof of this lemma is analogous to the proof of Lemma 17.9 in Liptser and Shiryayev (2001b) as an application of a generalized Ito-Ventzell formula for continuous local martingales. We give a detailed proof here for completeness.

Let $g_i(x), A \le x \le B, i = 0, 1$ be the solutions of the differential equations

$$g_i''(x) + (-1)^{i+1} g_i'(x) = -2, \quad g_i(A) = g_i(B) = 0, \; i = 0, 1.$$

It can be checked that

$$g_0(x) = 2\left[\frac{(e^B - e^{A+B-x})(B - A)}{e^B - e^A} + A - x\right],$$

$$g_1(x) = 2\left[\frac{(e^B - e^x)(B - A)}{e^B - e^A} - B + x\right],$$

and

$$g_0(0) = -2\,V(\beta, \alpha); \quad g_1(0) = 2\,V(\alpha, \beta).$$

Suppose the hypothesis H_0 holds. Define

$$\sigma_n(W^H) = \inf\left\{t : \int_0^t m_s^2(W^H)\,dw_s^H \ge n\right\}, \quad n \ge 1.$$

Applying the generalized Ito-Ventzell formula to $g_0(\lambda_t(W^H))$, we obtain that

$$\begin{aligned}
&g_0(\lambda_{\tilde{\tau}(W^H)\wedge\sigma_n(W^H)}(W^H)) \\
&= g_0(0) + \int_0^{\tilde{\tau}(W^H)\wedge\sigma_n(W^H)} g_0'(\lambda_t(W^H))m_s(W^H)\,dM_s^H \\
&\quad - \frac{1}{2}\int_0^{\tilde{\tau}()\wedge\sigma_n(W_s^H)} [g_0'(\lambda_t(W_s^H)) - g_0''(\lambda_t(W_s^H))]m_s^2(W_s^H)\,dw_s^H \\
&= g_0(0) + \int_0^{\tilde{\tau}(W_s^H)\wedge\sigma_n(W_s^H)} g_0'(\lambda_t(W_s^H))m_s(W_s^H)\,dM_s^H \\
&\quad + \int_0^{\tilde{\tau}(W_s^H)\wedge\sigma_n(W_s^H)} m_s^2(W_s^H)\,dw_s^H. \qquad (3.31)
\end{aligned}$$

Since

$$E_0\left(\int_0^{\tilde{\tau}(W_s^H)\wedge\sigma_n(W_s^H)} g_0'(\lambda_t(W_s^H))m_s(W_s^H)\,dM_s^H\right) = 0,$$

taking expectations with respect to the probability meaure P_0 on both sides of equation (3.31), we have

$$E_0\left(\int_0^{\bar{\tau}(W_s^H)\wedge\sigma_n(W_s^H)} m_s^2(W_s^H)dw_s^H\right) = -g_0(0) + E_0(g_0(\lambda_{\bar{\tau}(W^H)\wedge\sigma_n(W^H)}(W^H))).$$

Taking limit as $n \to \infty$, we obtain that

$$E_0\left(\int_0^{\bar{\tau}(\psi)} m_t^2(\psi)dm_t^H\right) = -g_0(0) = 2\,V(\beta,\alpha), \tag{3.32}$$

Similarly, we can prove that

$$E_1\left(\int_0^{\bar{\tau}(\psi)} m_t^2(\psi)dm_t^H\right) = -g_1(0) = 2\,V(\alpha,\beta). \tag{3.33}$$

This completes the proof. □

We now prove Theorem 3.1.

Proof of Theorem 3.1. Let $\Delta = \Delta(\tau,\delta)$ be any sequential plan in the class $\Delta_{\alpha,\beta}$. Let P_i^τ be the restriction of the probability measure P_i restricted to the σ-algebra $\mathcal{B}_\tau$ for $i = 0, 1$. In view of the conditions (3.8), (3.9), (3.11) and the representation (3.20), it follows that the probability measures $P_i^\tau, i = 0, 1$ are equivalent by Theorem 7.10 in Liptser and Shiryayev (2001a). Furthermore,

$$\log\frac{dP_1^\tau}{dP_0^\tau}(\tau, W^H) = \int_0^{\tau(W^H)} m_s(W^H)dM_s^H - \frac{1}{2}\int_0^{\tau(W^H)} m_s^2(W^H)dw_s^H,$$

and

$$\log\frac{dP_1^\tau}{dP_0^\tau}(\tau,\psi) = \int_0^{\tau(\psi)} m_s(\psi)dZ_s - \frac{1}{2}\int_0^{\tau(\psi)} m_s^2(\psi)dw_s^H.$$

Therefore

$$\begin{aligned} E_0\left(\log\frac{dP_0^\tau}{dP_1^\tau}(\tau,\psi)\right) &= \frac{1}{2}E_0\left(\int_0^{\tau(\psi)} m_s^2(\psi)dw_s^H\right) \\ &= \frac{1}{2}E_0\left(\int_0^{\tau(W^H)} m_s^2(W^H)dw_s^H\right) \end{aligned} \tag{3.34}$$

and

$$E_1\left(\log\frac{dP_1^\tau}{dP_0^\tau}(\tau,\psi)\right) = \frac{1}{2}E_1\left(\int_0^{\tau(\psi)} m_s^2(\psi)dw_s^H\right). \tag{3.35}$$

Applying the Jensen's inequality and following the arguments similar to those in Liptser and Shiryayev (2001b, p. 254–255), it can be shown that

$$\frac{1}{2}E_1\left(\int_0^{\tau(\psi)} m_s^2(\psi)dw_s^H\right) \geq (1-\alpha)\log\frac{1-\alpha}{\beta} + \alpha\log\frac{\alpha}{1-\beta}$$
$$= \frac{1}{2}E_1\left(\int_0^{\tilde{\tau}(\psi)} m_s^2(\psi)dw_s^H\right). \tag{3.36}$$

by using Lemma 3.3. Hence

$$E_1\left(\int_0^{\tilde{\tau}(\psi)} m_s^2(\psi)dw_s^H\right) \leq E_1\left(\int_0^{\tau(\psi)} m_s^2(\psi)dw_s^H\right). \tag{3.37}$$

Similarly, we can prove that

$$E_0\left(\int_0^{\tilde{\tau}(\psi)} m_s^2(\psi)dw_s^H\right) \leq E_0\left(\int_0^{\tau(\psi)} m_s^2(\psi)dw_s^H\right). \tag{3.38}$$

This completes the proof of the Theorem 3.1. □

Remark 3.2. As a special case of the above result, suppose that $\theta_t = h(t)$ where $h(t)$ is a nonrandom but differentiable function such that

$$\int_0^{\infty} h^2(t)dt = \infty, \quad h(t)h'(t) \geq 0, \quad t \geq 0. \tag{3.39}$$

Let α, β be given such that $0 < \alpha + \beta < 1$.

Let $\Delta_{\alpha,\beta}$ be the class of sequential plans as discussed earlier for given α, β with $0 < \alpha + \beta < 1$. Consider the plan $\Delta_T = (T, \delta_T)$ having the fixed observation time T for $0 < T < \infty$ and belonging to the class $\Delta_{\alpha,\beta}$. Then the optimal sequential plan $\tilde{\Delta} = (\tilde{\tau}, \tilde{\delta}) \in \Delta_{\alpha,\beta}$ has the properties

$$E_i(\tilde{\tau}) \leq T, \quad i = 0, 1. \tag{3.40}$$

This can be seen by checking that, for $i = 0, 1$,

$$E_i\left(\int_0^{\tilde{\tau}(\psi)} h^2(t)dt\right) \leq E_i\left(\int_0^{T} h^2(t)dt\right)$$
$$= \int_0^T h^2(t)dt = \Phi(T) \text{ (say)}, \tag{3.41}$$

which in turn implies that

$$\Phi(T) \geq E_i\left(\int_0^{\tilde{\tau}(\psi)} h^2(t)dt\right)$$
$$= E_i(\Phi(\tilde{\tau}(\psi)))$$
$$\geq \Phi(E_i(\tilde{\tau}(\psi))) \tag{3.42}$$

by observing that the function $\Phi(.)$ is convex and by applying the Jensen's inequality. The above inequality in turn proves that

$$E_i(\tilde{\tau}(\psi)) \leq T, \quad i = 0, 1. \tag{3.43}$$

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