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ψ_f OPTIMALITY OF MBGDD OF TYPE 1 UNDER MIXED EFFECTS MODEL WITHIN THE RESTRICTED CLASS OF BINARY DESIGNS

By SUNANDA MUKHOPADHYAY
Indian Statistical Institute

SUMMARY. In the present paper, Cheng's (1978) result regarding the ψ_f-optimality of the most balanced group divisible (GD) designs under a fixed effects model has been extended to the case of a mixed effects model when one is confined to the class of binary deigns.

1. Introduction

In the present paper, we prove the type 1 ψ_f optimality of the most balanced GD designs of type 1 within the restricted class of all proper, connected and binary block designs under the assumption of a mixed effects additive model with the treatment effects fixed and block effects random. The corresponding optimality result within the restricted class of equireplicate designs has been reported by Khatri and Shah (1981).

2. Optimality results

The coefficient matrix of the reduced normal equations for treatment effects based on a proper block design with block size k under mixed effects model is given by (Bose, 1975).

$$c_d^{(M)} = w(D_r - k^{-1}N_dN_d') + \bar{w}(k^{-1}N_dN_d' - n^{-1}rr')$$

where the symbols used are as in Bose (1975).

Let $Z = w - w$. Then

$$C_d^{(M)} = ZC_d^{(F)} + \bar{w}\bar{C}_d \quad \ldots \quad (2.1)$$

where

$$C_d^{(F)} = D_r - k^{-1}N_dN_d' \quad \ldots \quad (2.2)$$

which is the C-matrix based on the same design d under the assumption of a fixed effects model and

$$\bar{C}_d = D_r - n^{-1}rr'. \quad \ldots \quad (2.3)$$

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Let us now come to the proof of the optimality result.

Let $\mathcal{D}_1$ be the class of all proper, connected and binary designs with given b, v and k $(< v)$ and d^* be a most balanced GD design (MBGDD) of type 1, i.e. a GD design with $m = 2$ and $\lambda_2 = \lambda_1+1$.

Then in view of Theorem 2.2 of Cheng (1978) it is enough to verify the following conditions with d^* as an MBGDD of type 1 and $\mathcal{D}$ as $\mathcal{D}_1$ defined above.

Conditions are :

(i) $C_{d^*}^{(M)}$ has two positive eigenvalues μ and μ' $(\mu > \mu')$, the multiplicity of μ being 1,

(ii) d^* maximises tr $C_d^{(M)}$ over $d \in \mathcal{D}_1$,

(iii) $(P_{d^*}^{(M)})^2 < \{(v-1)(v-2)\}^{-1}(\text{tr } C_{d^*}^{(M)})^2$,

(iv) d^* maximises

$$\text{tr } C_d^{(M)} - \left(\frac{v-1}{v-2}\right)^{1/2} P_d^{(M)} \text{ over all } d \in \mathcal{D}_1$$

where

$$P_d^{(Q)} = \{\text{tr}(C_d^{(Q)})^2 - (v-1)^{-1}(\text{tr } C_d^{(Q)})^2\}^{1/2} \quad \ldots \quad (2.4)$$

with $Q = M$ or F.

Now for any equireplicate design d^0 with replication r,

$$\bar{C}_{d^0} = r(I_v - v^{-1}J_{v\times v}) \quad \ldots \quad (2.5)$$

and so the i-th eigenvalue of $C_{d^0}^{(M)}$

$$\mu_i^{(M)} = Z\mu_i^{(F)} + \bar{w}r$$

where $\mu_i^{(F)}$ is the corresponding i-th eigenvalue of $C_{d^0}^{(F)}$.

Hence condition (1) holds for $C_{d^*}^{(M)}$ as it is known to hold for $C_{d^*}^{(F)}$ (vide Cheng 1978), and d^* is an equireplicate design.

Condition (ii) will follow from the following lemma.

Lemma : *Let $\mathcal{D}$ be the class of all proper and connected block designs with given b, v and k $(< v)$. Then*

$$tr\ C_{d^*}^{(M)} = \max_{d \in \mathcal{D}} tr\ C_d^{(M)} \quad \ldots \quad (2.6)$$

is implied by a d^* such that

(a) $|n_{d^*_{ij}} - k/v| < 1$,

and (b) $r_1 = r_2 = \ldots = r_v = n/v$

where $n_{d^*_{ij}}$ is the (i, j)-th element of N_{d^*}.

Proof: Trivial.

Since $\mathcal{D}_1 \subset \mathcal{D}$ and $d^* =$ MBGDD of type 1 satisfies conditions (a) and (b) of the above lemma, condition (ii) of (2.4) follows.

For proving condition (iii) of (2.4), we note that for an equireplicate design d^0.

$$c_{d^0}^{(F)} \cdot \bar{c}_{d^0} = rc_{d^0}^{(F)} \quad \dots \quad (2.7)$$

and

$$(\bar{C}_{d^0})^2 = r\bar{C}_{d^0}$$

as

$$C_d^{(F)} . 1_v = \bar{C}_d \cdot 1_v = 0.$$

Hence

$$(P_{d^0}^{(M)})^2 = Z^2(P_{d^0}^{(F)})^2 + 2Z\bar{w}\{\text{tr}(C_{d^0}^{(F)}\,\bar{C}_{d^0}) - (v-1)^{-1}\,\text{tr}C_{d^0}^{(F)} \cdot \text{tr}\,\bar{C}_{d^0}\}$$

$$+\bar{w}^2\,\{\text{tr}\,\bar{C}_{d^0}^2 - (v-1)^{-2}\,(\text{tr}\,\bar{C}_{d^0})^2\}$$

$$= Z^2(P_{d^0}^{(F)})^2 \text{ since the 2nd term and 3rd term vanish by virtue of (2.5).}$$

In particular

$$P_{d^*}^{(M)} = ZP_{d^*}^{(F)}. \quad \dots \quad (2.8)$$

since d^* is an equireplicate design and so by Theorem (3.1) of Cheng (1978) condition (iii) of (2.4) is proved.

So, now we re left with the verification of condition (iv) of (2.4) which is equivalent to the following conditions.

$$\text{tr}\,C_{d^*}^{(M)} - \text{tr}\,C_d^{(M)} \geqslant \left(\frac{v-1}{v-2}\right)^{1/2} (P_{d^*}^{(M)} - P_d^{(M)}) \;\forall\; d \in \mathcal{D}_1 \quad \dots \quad (2.9)$$

Since $\text{tr}\,\bar{C}_{d^*} \geqslant \text{tr}\,\bar{C}_d \;\forall\; d \in \mathcal{D}_1$,

$$\text{L.H.S. of (2.9)} \geqslant Z\,.\,(\text{tr}\,C_{d^*}^{(F)} - \text{tr}\,C_d^{(F)}).$$

Hence if we can show that

$$(P_d^{(M)})^2 \geqslant Z^2(P_d^{(F)})^2$$

then the result will hold in view of Cheng (1978) and relation (2.8).

Now,

$$(P_d^{(M)})^2 = Z^2(P_d^{(F)})^2 + 2Z\bar{w}\{\text{tr}\,(C_d^{(F)}\,.\,\bar{C}_d) - (v-1)^{-1}\,\text{tr}\,C_d^{(F)}\,\text{tr}\,\bar{C}_d\}$$

$$+\bar{w}^2\{\text{tr}\,\bar{C}_d^2 - (v-1)^{-1}(\text{tr}\,\bar{C}_d)^2\} \quad \dots \quad (2.10)$$

3rd term of (2.10) is $\geqslant 0$ since it is of the form $\bar{w}^2.\ \sum_{i=1}^{v-1} (\mu_i - \bar{\mu})^2$ where $\mu, \mu_2, \dots, \mu_{v-1}$ are the positive eigenvalues of $\bar{C}_d$ and $\bar{\mu} = \text{tr}\,\bar{C}_d/(v-1)$.

So, if we can show that

$$\mathrm{tr}(C_d^{(p)} \,.\, \bar{C}_d) - (v-1)^{-1}\, \mathrm{tr}\, C_d^{(p)}\ \mathrm{tr}\, \bar{C}_d > 0, \qquad \ldots \quad (2.11)$$

we are through,

$$\text{L.H.S. of (2.11)} = \sum_{i=1}^{v} c_{d_{ii}} \bar{c}_{d_{ii}} + \sum_{i \neq j} c_{d_{ij}} \bar{c}_{d_{ij}} - (v-1)^{-1} \mathrm{tr}\, C_d^{(p)} \,.\, \mathrm{tr}\, \bar{C}_d$$

where $c_{d_{ij}}$ and $\bar{c}_{d_{ij}}$ represent the (i, j)-th elements of $C_d^{(p)}$ and $\bar{C}_d$ respectively.

But since

$$c_{d_{ii}} = \frac{r_i(k-1)}{k} \text{ and } \mathrm{tr}\, C_d^{(p)} = \frac{v\bar{r}(k-1)}{k}$$

where $\bar{r} = \sum_{i=1}^{v} r_i/v = n/v$, the above expression is

$$= \frac{k-1}{k} \sum_{i=1}^{v} (r_i - \bar{r}) \bar{c}_{d_{ii}} - \frac{1}{v(v-1)}\, \mathrm{tr}\, C_d^{(p)} \,.\, \mathrm{tr}\, \bar{C}_d + \sum_{i \neq j} c_{d_{ij}} \bar{c}_{d_{ij}}. \qquad \ldots \quad (2.12)$$

Now since $C_d^{(p)} \,.\, 1_v = \bar{C}_d \,.\, 1_v = 0$

$$\mathrm{tr}\, C_d^{(p)} = -\sum_{i \neq j} c_{d_{ij}} = \sum_{i \neq j} (\lambda_{ij}/k)$$

and

$$\mathrm{tr}\, \bar{C}_d = -\sum_{i \neq j} \bar{c}_{d_{ij}} = \sum_{i \neq j} (r_i r_j/n).$$

Hence (2.12) can be written as

$$\frac{k-1}{k} \sum_{i=1}^{v} (r_i - \bar{r})\, \bar{c}_{d_{ii}} + \frac{1}{kn} \sum_{i \neq j} (\lambda_{ij} - \bar{\lambda})\, r_i r_j. \qquad \ldots \quad (2.13)$$

Now the m trix $\Lambda = (\lambda_{ij})_{1 \leq i, j \leq v}$, where $\lambda_{ii} = 0$, $i = 1, 2, \ldots, v$, has all elements nonnegative. So, we can apply the inequality of Atkinson, Watterson and Moran (1960) and obtain

$$v^2 \sum_i \sum_j \lambda_{ij} \lambda_{i.} \lambda_{.j} \geq \lambda_{..}^3$$

where

$$\lambda_{i.} = \sum_{j=1}^{v} \lambda_{ij} = r_i(k-1)$$

$$= \lambda_{.i},\ i = 1, 2, \ldots, v$$

and

$$\lambda_{..} = \sum_i \sum_j \lambda_{ij} = v(v-1)\bar{\lambda}.$$

Hence, the 2nd term of (2.13)

$$= \frac{1}{kn(k-1)^2} \left\{ \sum_{i \neq j} \lambda_{ij} \lambda_{i.} \lambda_{j.} - \frac{\lambda_{..}}{v(v-1)} \sum_{i \neq j} \lambda_{i.} \lambda_{.j} \right\}$$

$$\geq \frac{\lambda_{..}}{kn(k-1)^2 v(v-1)} \sum_{i=1}^{v} (\lambda_{i.} - \lambda_{..}/v)^2 \geq 0. \qquad \ldots \quad (2.14)$$

Again the first term of (2.13)

$$= \sum^{v} (r_i-\bar{r})r_i(1-r_i/n)\cdot\frac{k-1}{k}$$

$$= \frac{k-1}{kn}\left\{(v-2)\bar{r}\sum_i (r_i-\bar{r})^2-\Sigma(r_i-\bar{r})^3\right\} \text{ on simplification.}$$

If d is such that $r_i \leqslant (v-1)\bar{r}$ $\forall\, i = 1, 2, \ldots, v$ then the quantity within second bracket is nonnegative and we are through. So, we assume that one of the r_i's (say r_1) is $>(v-1)\bar{r}$.

Now for a given value of r, such that

$$(v-1)\bar{r} < r_1 \leqslant v\bar{r},\ r_i-\bar{r} < 0, \quad i = 2, \ldots, v$$

and
$$\sum_{i=2}^{v} (r_i-\bar{r}) = -(r_1-\bar{r}).$$

So, we have

$$\sum_{i=2}^{v} (r_i-\bar{r})^2 \geqslant (r_1-\bar{r})^2/(v-1) \text{ and } -\sum_{i=2}^{v} (r_i-\bar{r})^3 = \sum_{i=2}^{v} (\bar{r}-r_i)^3 \geqslant \frac{(r_1-\bar{r})^3}{(v-1)^2}.$$

Hence
$$\sum_{i=1}^{v} (r_i-\bar{r})^2 \geqslant \frac{v}{v-1}(r_1-\bar{r})^2 \text{ and } \sum_{i=1}^{v} (r_i-\bar{r})^3 \geqslant -\frac{v(v-2)}{(v-1)^2}(r_1-\bar{r})^3.$$

So, the first term of (2.13)

$$\geqslant \frac{v(v-2)}{n.(v-1)^2}(r_1-\bar{r})^2\{(v-1)\bar{r}-(r_1-\bar{r})\}\cdot\frac{k-1}{k}$$

$$> 0.$$

Thus (2.13) holds and the result follows.

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