On Lipschitzian Q_0 and INS Matrices

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ABSTRACT

A problem posed by Murthy, Parthasarathy, and Sriparna is settled in this note, viz., a nondegenerate matrix satisfying Property (**) introduced by Murthy, Parthasarathy, and Sabatini is shown to be a Lipschitizian matrix. The analysis is based on the results recently derived on INS matrices. We also prove that the class INS under the assumption of nondegeneracy is complete.

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1. INTRODUCTION

Given a matrix $M \in R^{n \times n}$ and a vector $q \in R^n$, the linear complementarity problem LCP(q, M) is to find a vector $z \in R^n$ such that

$$Mz + q \geqslant 0, \qquad z \geqslant 0, \quad \text{and} \quad z^t(Mz + q) = 0. \tag{1}$$

Let $S(q, M)$ denote the set of all solutions to (1). We refer to the books by Cottle, Pang, and Stone [1] and Murty [5] for a detailed study of linear complementarity problems. If the multivalued mapping $\Phi_M : q \to S(q, M)$ is Lipschitizian, then it is known (Gowda and Sznajder [2], Murthy, Parthasarathy, and Sriparna [4]) that the corresponding matrix M is nondegenerate, i.e., every principal minor of M is nonzero. Also, from Stone [8] and Sridhar [6], it is known that M belongs to the class INS (invariant number of solutions) introduced by Stone [7].

Murthy, Parthasarathy, and Sabatini [3] introduced Property $(**)$ (defined below) in connection with Lipschitzian Q_0-matrices, and in [4], the following problem was posed: If a nondegenerate matrix $M \in R^{n \times n}$ satisfies Property $(**)$, then is the multivalued mapping Φ_M Lipschitzian? In this note, we answer this affirmatively.

To start with, we present below the necessary definitions and results.

DEFINITION 1. A matrix $M \in R^{n \times n}$ is said to be *Lipschitzian* if the multivalued mapping $\phi_M : R^n \to R^n_+$ satisfies the following: there exists a real number λ such that for any two vectors $p, q \in R^n$, with $\Phi_M(q) \neq \phi$ and $\Phi_M(p) \neq \phi$,

$$\Phi_M(p) \subseteq \Phi_M(q) + \lambda \|q - p\| B,$$

where $\|\cdot\|$ denotes the Euclidean norm and B denotes the closed unit ball in R^n.

Let $K(M)$ denote the set of vectors $q \in R^n$ for which LCP(q, M) has a solution. Using $K(M)$, the class with invariant number of solutions [7], denoted by INS, is defined as follows. A matrix M is said to be an INS matrix if there exists a positive integer k such that for every $q \in \operatorname{int} K(M)$, $|S(q, M)| = k$.

As mentioned earlier, it has been observed by Stone [8] and Sridhar [6] that a Lipschitzian matrix is an INS matrix. The converse of this result is proved by Stone [8] under the assumption of arc-connectedness of $K(M)$. The definition of Lipschitzian arc-connectedness of a set, as defined in [8], is given below.

DEFINITION 2. A set $S \subseteq R^n$ is said to be *Lipschitz arc-connected* if there exists a constant L such that, for all $x, y \in S$, the set S contains a polygonal arc between x and y whose length does not exceed $L\|x - y\|$.

The following theorem is due to Stone [8].

THEOREM 1. *If $M \in$ INS is a nondegenerate matrix and if* int $K(M)$ *is Lipschitz arc-connected, then M is Lipschitzian.*

A matrix M is said to belong to the class Q_0 if for every $q \in R^n$, the existence of a solution to $Mz + q \geqslant 0$ and $z \geqslant 0$ implies the existence of a solution to (1). For $M \in R^{n \times n}$, if for some $J \subseteq \{1, \dots, n\}$ one has $\det M_{JJ} \neq 0$, then the *principal pivot transform* of M with respect to J is defined as $A \in R^{n \times n}$, where $A_{JJ} = M_{JJ}^{-1}$, $A_{J\bar{J}} = -M_{JJ}^{-1}M_{J\bar{J}}$, $A_{\bar{J}J} = M_{\bar{J}J}M_{JJ}^{-1}$, and $A_{\bar{J}\bar{J}} = M_{\bar{J}\bar{J}} - M_{\bar{J}J}M_{JJ}^{-1}M_{J\bar{J}}$. In connection with Lipschitzian Q_0-matrices, the following property was introduced in [3].

DEFINITION 3. Let $M \in R^{n \times n}$. Then M is said to satisfy *Property* $(**)$ if for every principal pivot transform A of M, the rows corresponding to nonpositive diagonal entries of A are nonpositive.

Murthy et al. [3, 4] and Sridhar [6] showed the following.

THEOREM 2. *Let $M \in R^{n \times n}$. If M satisfies Property $(**)$, then M is Q_0. Conversely, if M is Lipschitzian and Q_0, then M satisfies Property $(**)$.*

For $M \in R^{n \times n}$, let us consider a complementary cone pos $C(J)$ relative to M, where the matrix $C(J) \in R^{n \times n}$ for $J \subseteq \{1, \dots, n\}$ is defined as $C(J)_{\cdot j} = -M_{\cdot j}$ if $j \in J$ and $C(J)_{\cdot j} = I_{\cdot j}$ otherwise (cf. [1, 5]). We denote by pos $C(J)_{\cdot \bar{i}}$ the facet relative to M for some $i \in \{1, \dots, n\}$. The following definitions of proper and reflecting facets relative to M are needed in the sequel.

DEFINITION 4. For $M \in R^{n \times n}$, $J \subseteq \{1, \dots, n\}$ and $i \in \{1, \dots, n\}$, consider the product

$$(\det M_{JJ})(\det M_{KK})$$

for $K \subseteq \{1, \dots, n\}$ such that $J \Delta K = \{i\}$. The common facet pos $C(J)_{\cdot \bar{i}}$ is *proper* (*reflecting*) if the above product is positive (negative). If the product is zero, then the common facet pos $C(J)_{\cdot \bar{i}}$ relative to M is said to be *degenerate*.

When a facet $F = \text{pos}\, C(J)_{\cdot \bar{i}}$ is reflecting, there exists a nonzero vector $p \in R^n$ such that $p^t F = 0$ and the columns $I_{\cdot i}$ and $-M_{\cdot i}$ lie on the same side of the hyperplane.

2. THE MAIN RESULT

For $M \in R^{n \times n}$, if int $K(M)$ is connected and all the reflecting and degenerate facets relative to M lie on the boundary of $K(M)$, then from Corollary 6.6.22 of Cottle, Pang, and Stone [1] it follows that M is an INS matrix. Using this, we prove the following result.

THEOREM 3. *Let $M \in R^{n \times n}$ be a nondegenerate matrix. If M satisfies Property $(**)$, then M is in the class* INS $\cap$ Q_0.

Proof. From Theorem 2, we at first note that M is a Q_0-matrix. Since M is a nondegenerate matrix, there are no degenerate facets relative to M. We prove that every reflecting facet relative to M lies on the boundary of $K(M)$. Suppose M satisfies Property $(**)$. Let $m_{11} < 0$. It is clear that the facet $F = \text{pos}(I_{\cdot 2}, \dots, I_{\cdot n})$ is reflecting. Since M satisfies Property $(**)$, $m_{1j} \leqslant 0$ for all $j \in \{2, \dots, n\}$. For the vector $p = I_{\cdot 1} \in R^n$, we have $p^t F = 0$ and $p^t(-M_{\cdot j}) \geqslant 0$ for all j. This implies that $p^t q \geqslant 0$, $\forall q \in K(M)$. Hence, $K(M)$ fully lies on one side on the hyperplane containing F. So the facet F lies on the boundary of $K(M)$.

Suppose $F = \text{pos}\, C(J)_{\bar{i}}$ for $J \subseteq \{1, \dots, n\}$, $|J| \neq 1$, is a reflecting facet for some $i \in \{1, \dots, n\}$. Since the complementary matrix $C(J)$ is nonsingular, we can consider a principal pivot transform with respect to J. The resulting matrix $A = C(J)^{-1} \overline{C}(J)$, where $\overline{C}(J)$ contains columns of $[I : -M]$ not in $C(J)$, has its (i, i)th diagonal entry negative. Since M satisfies Property $(**)$, we notice as before that the reflecting facet $F = \text{pos}(I_{\cdot 1}, \dots, I_{i-1}, I_{i+1}, \dots, I_{\cdot n})$ lies on the boundary of $K(A)$. From the one-to-one correspondence existing between the complementary cones relative to M and the complementary cones relative to A, we can conclude that the reflecting facet F relative to M lies on the boundary of $K(M)$. As every reflecting facet relative to M lies on the boundary of $K(M)$ and int $K(M)$ is connected, M belongs to the class INS. This concludes the proof. ∎

As a corollary to the above theorem, we answer a question raised in [4].

COROLLARY 1. *Let $M \in R^{n\times n}$ be a nondegenerate matrix. If M satisfies Property $(**)$, then M is Lipschitzian.*

Proof. From the above theorem, it follows that M is an INS $\cap$ Q_0 matrix. Since any Q_0-matrix is Lipschitzian arc-connected (Proposition 3.2.1 in [1]), the result follows from Theorem 1. ■

Combining the results given here with the already known results on Lipschitzian matrices, we state the following theorem.

THEOREM 4. *Let $M \in R^{n\times n}$ be a nondegenerate Q_0-matrix. Then the following are equivalent:*

(i) *M is Lipschitzian.*
(ii) *M satisfies Property $(**)$.*
(iii) *M is an* INS *matrix.*

Furthermore, if any one of the above conditions holds for M, then it holds for every principal submatrix of M.

It has been observed by Murthy, Parthasarathy, and Sriparna [4] that the class of Lipschitzian matrices is complete, i.e., if M is Lipschitzian, then all principal submatrices of M are Lipschitzian. In [8], Stone raises the question whether the class of nondegenerate INS matrices is complete. From Theorem 4, it is clear that the class of nondegenerate INS $\cap$ Q_0 matrices is complete. In the general case, we answer this affirmatively, in the next theorem.

THEOREM 5. *The nondegenerate* INS *class is complete.*

Proof. Let $M \in$ INS be a nondegenerate matrix. It is enough to prove that A, the principal submatrix of M leaving out the first row and the first column, is an INS matrix. Let M be partitioned as

$$M = \begin{bmatrix} m_{11} & a^t \\ b & A \end{bmatrix}, \tag{2}$$

where $a, b \in R^{n-1}$ correspond to the first row and first column of M leaving out the diagonal entry m_{11}.

It is clear that, if $\bar{F} = \text{pos}(I_{\cdot 2}, \dots, I_{\cdot k-1}, -A_{\cdot k+1}, \dots, -A_{\cdot n})$ is a reflecting facet relative to $K(A)$ for some $2 \leqslant k \leqslant n$, then the facet F defined by $F = \text{pos}(I_{\cdot 1}, I_{\cdot 2}, \dots, I_{\cdot k-1}, -M_{\cdot k+1}, \dots, -M_{\cdot m})$ is a reflecting facet relative to $K(M)$. We claim that if $\bar{F} \cap \text{int } K(A) \neq \phi$ then $F \cap \text{int } K(M) \neq \phi$; this will imply that the reflecting facet F relative to $K(M)$ does not lie on the boundary of $K(M)$, contradicting the hypothesis that $M \in \text{INS}$.

Now, to prove our claim, first let us consider reflecting facets relative to $K(A)$ of the form $\text{pos}(I_{\cdot 2}, \dots, I_{\cdot k-1}, I_{\cdot k+1}, \dots, I_{\cdot n})$ for some $2 \leqslant k \leqslant n$. Suppose $\bar{F} = \text{pos}(I_{\cdot 2}, \dots, I_{\cdot n-1})$ is reflecting and $\bar{F} \cap \text{int } K(A) \neq \phi$. Then, for some $\bar{q} = (\bar{q}_2, \dots, \bar{q}_n)^t \in \bar{F}$, there exists an $\epsilon > 0$ such that for any $\bar{q}' \in R^{n-1}$ with $\|\bar{q}' - \bar{q}\| < \epsilon$, LCP$(\bar{q}, A)$ has a solution. We can assume without loss of generality that $\bar{q}$ is of the form $(\bar{q}_2, \dots, \bar{q}_{n-1}, 0)^t$ where $\bar{q}_i > 0$ for $i = 2, \dots, n-1$. Let $U = \{q : \|q - \bar{q}\| < \epsilon\}$.

Since A is a nondegenerate matrix, $S(q, A)$ is uniformly bounded for q varying over a bounded set. Now, from Corollary 7.2.3 of Cottle, Pang, and Stone [1], it follows that there exists a constant $\lambda > 0$ and a neighborhood V of $\bar{q}$ in R^{n-1} such that

$$\|z\| \leqslant \lambda \qquad \forall z \in S(q', A), \quad q' \in V.$$

We can assume without loss of generality that $U \subseteq V$. Now, let $q \in R^n$ be defined as $q_i = \bar{q}_i$ for $i = 2, \dots, n$, and q_1 be chosen such that $q_1 > \|a\|\lambda + \epsilon$, where a is as given in (2). Clearly, $q \in F$. Choose $r \in R^n$ such that $\|r - q\| < \epsilon$. Then $\|\bar{r} - \bar{q}\| < \|r - q\|$, where $\bar{r}$ is the $(n-1)$-vector obtained from r on leaving out the first coordinate. Hence, LCP$(\bar{r}, A)$ has a solution (u, v), where $u, v \in R^{n-1}$. Let

$$z_i = v_{i-1}, \quad w_i = u_{i-1} \qquad \text{for} \quad i = 2, \dots, n,$$

$$z_1 = 0, \qquad\qquad w_1 = a^t z + r_1.$$

If $a^t z > 0$, then $w_1 > 0$. Otherwise, $-a^t z = |a^t z| \leqslant \|a\| \cdot \|z\| \leqslant \|a\|\lambda < q_1 - \epsilon$. Since $r_1 \geqslant q_1 - \epsilon$, we have $w_1 = r_1 + a^t z \geqslant 0$. Hence, (w, z) forms a solution for the LCP(r, M). This implies that for every $\|r - q\| < \epsilon$, $r \in K(M)$, which implies that $F \cap \text{int } K(M) \neq \phi$. This leads to the contradiction that $M \in \text{INS}$. Hence, the reflecting facet $\bar{F}$ relative to $K(A)$ does not intersect int $K(A)$.

Now, if $\bar{F}$ is any other reflecting facet relative to A, we can consider a principal pivot transform B of M with respect to a complementary cone relative to $K(A)$, incident on $\bar{F}$. Then B is an INS matrix, and $\bar{B}$, the principal submatrix of B leaving out its first row and first column, is a principal pivot transform of A. From the one-to-one correspondence existing between the facets relative to A and the facets relative to $\bar{B}$, we notice that $\bar{F}$ corresponds to a reflecting facet relative to $K(\bar{B})$ of the form, $\text{pos}(I_{\cdot 2}, \ldots, I_{\cdot k-1}, I_{\cdot k+1}, \ldots, I_{\cdot n})$ for some $2 \leqslant k \leqslant n$. Therefore, we can appeal to our earlier argument to conclude that $\bar{F}$ does not intersect the interior of $K(A)$. This along with A being nondegenerate implies that A is an INS matrix. This concludes the proof. ■

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Received 22 January 1996; final manuscript accepted 16 September 1996