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On the Diophantine equation
$x(x+1)(x+2)\ldots(x+(m-1)) = g(y)$

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ABSTRACT

Let $g(y) \in \mathbf{Q}[Y]$ be an irreducible polynomial of degree $n \geq 3$. We prove that there are only finitely many rational numbers x, y with bounded denominator and an integer $m \geq 3$ satisfying the equation $x(x+1)(x+2)\ldots(x+(m-1)) = g(y)$. We also obtain certain finiteness results when $g(y)$ is not an irreducible polynomial.

1. INTRODUCTION

Let $f(x) = x(x+1)(x+2)\ldots(x+(m-1)), m > 2$ and $g(y)$ be any polynomial of degree $n \geq 2$ in $\mathbf{Q}[Y]$. In this paper we consider the equation

(1) $f(x) = g(y)$
with
(2) x, y *rational with bounded denominator.*

Here we say that the equation $f(x) = g(y)$ has infinitely many rational solutions with bounded denominator if there exists a positive integer Δ such that $f(x) = g(y)$ has infinitely many rational solutions x, y satisfying $\Delta x, \Delta y \in \mathbf{Z}$.

We obtain an explicit criterion for the solutions of equation (1) satisfying (2). When $g(y)$ is an irreducible polynomial, our methods here yield a bound C on m which is effective.

The equations of type $f(x) = g(y)$ for various polynomials $g(y)$ have been studied extensively during the last decade. One can see papers [9], [10], [11], [12], [13], [14] and [16] of R.Tijdeman et al. more details. Also a special case is when $g(y) = y^n - r$ where r is any rational number in $\mathbf{Q}$. Earlier in [6], we have proved that in this case there are effective finiteness results for $x \in \mathbf{Z}$ and $y \in \mathbf{Q}$. Note that, although our results in this paper apply to the special case, they do not give effective results as in [6].

MAIN RESULTS

Main results of this paper are as follows:

Theorem 1.1. *Suppose $f(x) = g(y)$ has infinitely many solutions x, y satisfying (2). Then we are in one of the following cases:*

1. $g(y) = f(g_1(y))$ *for some* $g_1(y) \in \mathbf{Q}[Y]$.
2. *m even and $g(y) = \phi(g_1(y))$ where $\phi(X) = (X - (\frac{1}{2})^2)(X - (\frac{3}{2})^2) \cdot \cdots \cdot (X - (\frac{(m-1)}{2})^2)$ and $g_1(y) \in \mathbf{Q}[Y]$ is a polynomial whose squarefree part has at most two zeroes.*
3. $m = 4$ *and* $g(y) = \frac{9}{16} + b\delta(y)^2$ *where δ is a linear polynomial.*

Theorem 1.2. (a) *Fix $m \geq 3$ such that $m \neq 4$ and let $g(y)$ be an irreducible polynomial in $\mathbf{Q}[y]$. Then there are only finitely many solutions (x, y) of equation (1) satisfying (2).*

(b) *Let $m = 4$ and $g(y)$ be an irreducible polynomial. Then equation (1) has infinitely many solutions only when $g(y) = \frac{9}{16} + b\delta(y)^2$, where $b \in \mathbf{Q}^*$ and $\delta(y) \in \mathbf{Q}[y]$ is a linear polynomial. Besides this, equation (1) has only finitely many solutions satisfying (2).*

Theorem 1.3. *Assume that $g(y)$ is an irreducible polynomial in $\mathbf{Q}[y]$ and Δ be a positive integer. Then there exists a constant $C = C(g, \Delta)$ such that for any $m \geq C$, equation (1) does not have any rational solution with bounded denominator Δ. Moreover, C can be calculated effectively.*

Theorem 1.4. *Let $g(y)$ be an irreducible polynomial in $\mathbf{Q}[y]$ of degree at least 3. Then there are only finitely many solutions x, y, m with $m \geq 3$ of equation (1) satisfying (2).*

When deg $g(y) = 2$, equation (1) has finitely many solutions $x, y, m \in \mathbf{Z}$ satisfying equation (1) for $m \neq 4$.

Remark 1.5. For each of the cases in theorem 1.1, a set of rational solutions (x, y) with bounded denominator can be given as follows.

In case (1), an infinite set of rational solutions x, y with bounded denominator can be given as $x = g_1(y)$ with y suitably chosen. Note that if we had insisted on integral solutions x, y we would not have any solutions for $g(y) = \frac{(y^2+1)}{3}$ or $\frac{(y^2+1)}{4}$ for instance.

To get an infinite set of solutions in case (2), we write $g_1(y) = a(y)p(y)^2$ where $a(y)$ has degree ≤ 2 and only simple zeroes, if any. Then we check if $a(t)$ can be square for infinitely many choices of t. For each such t, denote $a(t) = \beta^2$ for some β. Then $(\beta p(t) - \frac{m-1}{2}, t)$ is a solution of equation (1).

Same procedure as above gives a set of solutions in case (3). Here $f_1(x) = a(x)p(x)^2$ where $a(x)$ is a polynomial of degree two and $g_1(y) = q(y)^2$ where $q(y)$ is a linear polynomial. We check if $a(t)$ can be square for infinitely many choices of t. For each such t, denote $a(t) = \delta^2$ for some δ. Then for $q(y) = ry + s$ we get $(t, \frac{\delta p(t) - s}{r})$ as a solution of equation (1).

2. SOME KNOWN RESULTS

In this section we recall some useful results which will be used to prove the main theorems.

For a polynomial $f(x) \in \mathbf{C}[x]$, let S_f denote the set of stationary points of f (i.e. points where f' vanishes). For any $a \in C$, let $m_a = \# \{\alpha \in S_f / f(\alpha) = a\}$. Then, a theorem of Beukers et al. [3] asserts:

Theorem 2.1. *Let $f(x) = x(x+1)...(x+(m-1))$. Then, for all $a \in C, m_a \leq 2$. Moreover, $m_a \leq 1$ if m is odd.*

Now we mention the results of Tichy, Bilu et al. on equations of the form $f(x) = g(y)$.

Definition 2.2. *A decomposition of a polynomial $F(x) \in \mathbf{C}[x]$ is an equality of the form $F(x) = G_1(G_2(x))$, where $G_1(x), G_2(x) \in \mathbf{C}[x]$. The decomposition is called nontrivial if deg $G_1 > 1$, deg $G_2 > 1$.*

Two decompositions $F(x) = G_1(G_2(x))$ and $F(x) = H_1(H_2(x))$ are called *equivalent* if there exist a linear polynomial $l(x) \in \mathbf{C}[x]$ such that $G_1(x) = H_1(l(x))$ and $H_2(x) = l(G_2(x))$. The polynomial called *decomposable* if it has atleast one nontrivial decomposition, and *indecomposable* otherwise.

The following result is due to Bilu et al [2]:

Therorem 2.3. *Let $m \geq 3$ and $f_m(x) = x(x+1)...(x+(m-1))$. Then,*

(i) *$f_m(x)$ is indecomposable if* m *is odd and,*

(ii) *if $m = 2k$, then any nontrivial decomposition of $f_m(x)$ is equivalent to $f_m(x) = R_k((x + \frac{m-1}{2})^2)$ where $R_k = (x - \frac{1}{4})(x - \frac{9}{4}) \cdots\cdots (x - \frac{(2k-1)^2}{4})$.*

In particular, the polynomial R_k is indecomposable.

The relevance of this theorem is seen from a theorem of Bilu and Tichy [4]. To state it, we need to recall the following notions.

Definition 2.4. *The Dickson polynomial $D_n(x,a)$ is defined as,*

$$D_n(x,a) = \sum_{i=0}^{[\frac{n}{2}]} \frac{n}{n-i} \binom{n-i}{i} (-a)^i x^{n-2i}.$$

Note that $D_n(x,a)$ has degree n.

Definition 2.5. *For a complex polynomial P, a complex number c is said to be an extremum if $P(x) - c$ has multiple roots.*

Definition 2.6. *The P-type of c is defined to be the tuple $(\mu_1, \cdots, \mu_s)$ of the multiplicities of the distinct roots of $P(x) - c$.*

With this definition, one has the following well known property of Dickson polynoimial. For a detail proof one can see [5]:

Theorem 2.7. (a) *The Dickson polynomial $P(x) = D_t(x,0)$ has exactly one extremum 0 of P-type (t).*
(b) *If $a \neq 0$ and $t \geq 3$ then $D_t(x,a)$ has exactly two extrema $\pm 2a^{\frac{t}{2}}$. If t is odd, then both are of P-type $(1,2,2...,2)$. If t is even, then $2a^{\frac{t}{2}}$ is of P-type $(1,1,2...,2)$ and $-2a^{\frac{t}{2}}$ is of P-type $(2,2,..,2)$.*

In what follows, a and b are nonzero elements of some field, m and n are positive integers, and $p(x)$ is a nonzero polynomial (which may be constant).

STANDARD PAIRS

A standard pair of the first kind is

$$(x^t, ax^r p(x)^t) \quad \text{or} \quad (ax^r p(x)^t, x^t)$$

where $0 \leq r < t$, $(r,t) = 1$ and $r + deg\, p(x) > 0$.

A standard pair of the second kind is

$$(x^2, (ax^2+b)p(x)^2) \quad \text{or} \quad ((ax^2+b)p(x)^2, x^2).$$

A standard pair of the third kind is

$$(D_k(x,a^t), D_t(x,a^k))$$

where $(k,t) = 1$. Here D_t is the t-th Dickson polynomial.

A standard pair of the fourth kind is

$$(a^{-t/2} D_t(x,a), b^{-k/2} D_k(x,a))$$

where $(k,t) = 2$.

A standard pair of the fifth kind is

$$((ax^2-1)^3, 3x^4 - 4x^3) \quad \text{or} \quad (3x^4 - 4x^3, (ax^2-1)^3).$$

By a standard pair over a field k, we mean that $a, b \in k$, and $p(x) \in k[x]$.

The following theorem is by Tichy & Bilu [4].

Theorem 2.8. *For non-constant polynomials $f(x)$ and $g(x) \in \mathbf{Q}[x]$, the following are equivalent:*

(a) *The equation $f(x) = g(y)$ has infinitely many rational solutions with a bounded denominator.*

(b) *We have $f = \phi(f_1(\lambda))$ and $g = \phi(g_1(\mu))$ where $\lambda(x), \mu(x) \in \mathbf{Q}[X]$ are linear polynomials, $\phi(x) \in \mathbf{Q}[X]$, and $(f_1(x), g_1(x))$ is a standard pair over $\mathbf{Q}$ such that the equation $f_1(x) = g_1(y)$ has infinitely many rational solutions with a bounded denominator.*

We use the following result in the proofs of the main theorems.

Theorem 2.9. *Let $f(x) = x(x+1)\cdots(x+m-1)$ and $c \in \mathbf{Q}$. Then*

(a) *When $m > 6$, the polynomial $f(x) + c$ has atleast three simple roots.*

(b) *If $m = 3, 6$ then, the polynomial $f(x) + c$ has only simple roots.*

(c) *When $m = 4$, $f(x) + c$ has multiple root only for $c = 1$ and $c = \frac{-9}{16}$. In fact $f(x) + 1 = (x^2 + 3x + 1)^2$ and has no rational root.*

Also $f(x) - \frac{9}{16} = (x + \frac{3}{2})^2(x^2 + 3x - \frac{1}{4})$ and the factor $x^2 + 3x - \frac{1}{4}$ has distinct, non-rational roots.

(d) *When $m = 6$, $f(x) + c$ has multiple root only if $c = (\frac{15}{8})^2$. In this case,*

$$f(x) + (\frac{15}{8})^2 = (x + \frac{5}{2})^2(x^4 + 10x^3 + \frac{115}{4}x^2 + \frac{75}{4}x + \frac{9}{16}).$$

The factor $\theta(x) = x^4 + 10x^3 + \frac{115}{4}x^2 + \frac{75}{4}x + \frac{9}{16}$ has no multiple root and also $\theta(\frac{-5}{2}) \neq 0$.

Note that part (a) is a direct consequence of Theorem 2.1 and parts (b), (c) and (d) can be verified by direct and straightforward computation.

3. PROOFS OF THE THEOREMS

Let $f(x) = x(x+1)(x+2)...(x+(m-1))$, $m \geq 3$ and let $g(y)$ be any polynomial in $\mathbf{Q}[Y]$. Fix m and suppose the equation (1) has infinitely many solutions satisfying (2). Then by theorem 2.8, there exists linear polynomials $\lambda(x), \mu(x) \in \mathbf{Q}[X]$, and a standard pair $(f_1(x), g_1(x))$ such that for some $\phi(x) \in \mathbf{Q}[X]$, we have $f(x) = \phi(f_1(\lambda(x)))$ and $g(x) = \phi(g_1(\mu(x)))$.

Now as degree of $f(x)$ is m, degree of $\phi(x)$ can be atmost m. This together with theorem 2.3 implies that degree of $\phi(x)$ can be m or $\frac{m}{2}$ or 1. We will deal separately with each of these cases in the following propositions.

Proposition 3.1. *Suppose deg $\phi = m$. Then equation (1) has infinitely many solutions (x, y) satisfying (2) implies $g(x)$ is as in case (1) of Theorem 1.1.*

Proof. Suppose $f(x) = g(y)$ has infinitely many solutions. Then, as above, there are linear polynomials $\lambda(x), \mu(x) \in \mathbf{Q}[X]$, and a standard pair $(f_1(x), g_1(x))$ such that $f(x) = \phi(f_1(\lambda(x)))$ and $g(x) = \phi(g_1(\mu(x)))$.

Since deg $\phi(x) = m = deg f(x)$, we get that $\phi(x) = f(\delta(x))$ for some linear polynomial $\delta(x) = ux + v \in \mathbf{Q}[X]$. Therefore deg $f_1 = 1$. So the standard pair (f_1, g_1) can only be of first kind or third kind.

If the standard pair is of first kind then $(f_1, g_1) = (x, acx^r)$ or (acx, x^t) for some $t \in \mathbf{Z}$ and $a, c \in \mathbf{Q}$. In both possibilities we get $g(x) = \phi(g_1(\mu(x))) = f(\delta(g_1(\mu(x))))$. By denoting $h(x) = \delta(g_1(\mu(x)) \in \mathbf{Q}[X]$, we get $g(x) = f(h(x))$. Therefore, in this case $(m, g(x)) = (m, f(h(x))$. Which is case (1) of Theorem 1.1. We get similar result if (f_1, g_1) is a standard pair of third kind. This completes the proof. □

Proposition 3.2. *Suppose deg $\phi = \frac{m}{2}$. Then equation(1) has infinitely many solutions (x, y) satisfying (2) implies $(m, g(x))$ is as in case (2) of Theorem 1.1.*

Proof. Let deg $\phi(x) = \frac{m}{2}$ and suppose equation(1) has infinitely many solutions. Then by theorem 2.8, there are linear polynomials $\lambda(x), \mu(x) \in \mathbf{Q}[X]$ and a standard pair $(f_1(x), g_1(x))$ such that $f(x) = \phi(f_1(\lambda(x)))$, $g(x) = \phi(g_1(\mu(x)))$ and $f_1(x) = g_1(y)$ has infinitely many rational solutions x, y with bounded denominator.

Since deg $\phi(x) = \frac{m}{2}$ and deg $f = m$ we have deg $f_1(x) = 2$. Therefore, $f(x)$ has a nontrivial decomposition. Then theorem 2.3 and theorem 2.8 together imply $f(x) = \phi(\delta(f_1(\lambda(x))))$ and $g(y) = \phi(\delta(g_1(\mu(y))))$ where $\phi(x) = (x - \frac{1}{4})(x - \frac{9}{4}) \cdots\cdots (x - \frac{(m-1)^2}{4})$ and δ is a linear polynomial. Choose $\lambda(x) = x$, $\mu(y) = y$ and denote $h_1(x) = \delta(f_1(x))$, $h_2(y) = \delta(g_1(y))$. Then, $(f(x), g(y))$ can be written as $(\phi(h_1(x)), \phi(h_2(y))$.

We have to show that the squarefree part of $h_2(y)$ has at most two zeroes. But, since $h_1(x)$ is the square of a linear polynomial (by theorem 2.3) and $h_1(x) = h_2(y)$ has infinitely many rational solutions with bounded denominator, it follows immediately from Siegel's theorem that $h_1(x)$ has at most two zeroes of odd multiplicity. □

Proposition 3.3. *Suppose deg $\phi = 1$. Then equation (1) has finitely many solutions satisfying (2) except in the case $m = 4$.*

When $m = 4$ the equation has infinitely many solutions only when $g(y) = \frac{9}{16} + b(\mu(y))^2$ for some $b \in \mathbf{Q}^*$ and linear polynomial $\mu(y) \in \mathbf{Q}[Y]$.

(Note that $m = 4$ and $g(y) = \frac{9}{16} + b(\mu(y))^2$ is case (3) of theorem 1.1).

Proof. Suppose, if possible, that $f(x) = g(y)$ has infinitely many solutions. Then, by theorem 2.8, there are linear polynomials $\lambda, \mu \in \mathbf{Q}[X]$, and a standard pair $(f_1, g_1) \in \mathbf{Q}[X]$ such that for some $\phi \in \mathbf{Q}[X]$, we have $f(x) = \phi(f_1(\lambda(x)))$ and $g(x) = \phi(g_1(\mu(x)))$. As deg $\phi(x) = 1$, we write $\phi(x) = \alpha + \beta x$ for some $\alpha \in \mathbf{Q}, \beta \in \mathbf{Q}^*$. Therefore, we can write

(3) $\quad f(x) = \alpha + \beta f_1(\lambda(x)), \quad g(x) = \alpha + \beta(g_1(\mu(x))$

Note that here, $\deg f_1 = \deg f = m$. Let $\lambda(x) = lx + l'$ for some $l, l' \in \mathbf{Q}$.

Let (f_1, g_1) be a standard pair of first kind.
Then $(f_1(x), g_1(x)) = (x^t, ax^r p(x)^t)$ or switched with $r < t, (r, t) = 1, r + deg\, p > 0$.

Let $(f_1(x), g_1(x)) = (x^t, ax^r p(x)^t), r < t, (r, t) = 1, r + deg\, p > 0$. By equations (3), we have $f(x) = \alpha + \beta(\lambda(x))^t$ i.e. $f(x) - \alpha = \beta(\lambda(x))^t$. But, this implies all roots of $f(x) - \alpha$ are rational and no root is simple. By theorem 2.9, it is not possible.

Now let $(f_1(x), g_1(x)) = (ax^r p(x)^t, x^t), r < t, (r, t) = 1, r + \deg p > 0$. Since $t = \deg g_1 = \deg g \geq 2, r \neq 0$. Thus, $f(x) - \alpha = \beta a(\lambda(x))^r p(\lambda(x))^t$.

As $r \neq 0$, we have $t \geq 2$. So every root of $p(x)$ is a multiple root of $f(x)$. Since $\lambda(x) = lx + l'$, $x = \frac{-l'}{l}$ is a rational root of $f(x) - \alpha$ with multiplicity r. By theorem 2.9, $f(x) - \alpha$ has at least one simple root unless $m = 4$ and $\alpha = -1$(in which case the roots are irrational). This forces $r = 1$. But, this means $f(x) - \alpha$ has exactly one simple root. However, theorem 2.9 imples that $f(x) - \alpha$ can never have exactly one simple root.

Therefore, $(f_1(x), g_1(x))$ can not be a standard pair of first kind.

Let (f_1, g_1) be a standard pair of second kind.
Then, $(f_1(x), g_1(x)) = (x^2, (ax^2 + b)p(x)^2)$ or switched. Let $(f_1(x), g_1(x)) = (x^2, (ax^2 + b)p(x)^2)$. This is not possible as it implies that $m = \deg f = \deg f_1 = 2$.

Now let $(f_1(x), g_1(x)) = ((ax^2 + b)p(x)^2, x^2)$.

Here $\deg f_1 = m = 2 + 2 \deg p$. Therefore, m is even and also $\deg g = 2$. Now $m > 2$ gives $\deg p > 0$. Therefore, $f(x) - \alpha = \beta[a(\lambda(x))^2 + b]p(\lambda(x))^2$. This implies that $f(x) - \alpha$ has at most two simple roots. By theorem 2.9 this is possible only when $m = 4$ and $\alpha = \frac{9}{16}$. By substituting this values, we get $g(y)$ is of the form $g(y) = \frac{9}{16} + \beta\mu(y)^2$. By denoting $b = \beta$, we get in this case $(m, g(x)$ is as in case (3) of Theorem 1.1.

Let (f_1, g_1) be a standard pair of fifth kind.
Then $(f_1(x), g_1(x)) = ((ax^2 - 1)^3, 3x^4 - 4x^3)$ or switched.

If $(f_1(x), g_1(x)) = ((ax^2 - 1)^3, 3x^4 - 4x^3)$ then $f(x) - \alpha = \beta[a(\lambda(x))^2 - 1]^3$. This implies degree of $f(x) - \alpha = 6$ and all roots of $f(x) - \alpha$ are multiple roots. But this is impossible by theorem 2.9.

Now let $(f_1(x), g_1(x)) = (3x^4 - 4x^3, (ax^2 - 1)^3)$. Once again, we have, $f(x) - \alpha = \beta(\lambda(x))^3[3\lambda(x) - 4]$. This implies $m = 4$ and that $f(x) - \alpha$ has only one simple root and the other root is multiple root with multiplicity 3. This again contradicts theorem 2.9.

Therefore, $(f_1(x), g_1(x))$ can not be a standard pair of fifth kind.

Let (f_1, g_1) be a standard pair of third kind.

Then $(f_1(x), g_1(x)) = (D_m(x, a^n), D_n(x, a^m)), (m, n) = 1$. As before, we have α, β in $\mathbf{Q}$, $\beta \neq 0$ so that $f(x) - \alpha = \beta D_m(\lambda(x), a^n)$ for some rational linear polynomial $\lambda(x) = lx + l'$. By theorem 2.1, we know that for any complex number c, the polynomial $f(X) - c$ can have atmost one multiple root if $\deg f = m$ is odd and atmost two multiple roots if m is even.

If $a = 0$ then $f_1(x) = D_m(x, 0) = x^m$ which is not possible as we have checked when $(f_1(x), g_1(x))$ is a standard pair of first kind. Therefore $a \neq 0$ and $f_1(x) = (D_m(x, a^n))$. By theorem 2.7, $f(x)$ has two extrema and $f(x) - \alpha = \beta D_m(\lambda(x), a^n)$ also has two extrema.

If m is an odd integer then by theorem 2.7, both extrema are of the type $(1, 2, 2 \ldots)$. But, by theorem 2.1, $f(x) - \alpha$ can have atmost one multiple root if m is odd. This implies that in this case the only type an extremum can have is $(1, 2)$. Therefore, if m is odd then $m = 3$. But, by theorem 2.9, when $m = 3$, all roots of $f(x) - \alpha$ are simple. This is a contradiction.

If m is even, then by theorem 2.7, the extrema are of the type $(1, 1, 2, 2, \ldots, 2)$ and $(2, 2, \ldots, 2)$. Now, since $f(x) - \alpha$ can have atmost two multiple roots, the above types can only be either $(1, 1, 2, 2)$ or $(1, 1, 2)$ or $(2, 2)$. Thus, if m is even then it must be 4 or 6.

If $m = 6$, the type $(1, 1, 2, 2)$ is ruled out by theorem 2.9 because when $m = 6$, $f(x) - \alpha$ has four simple roots and only one multiple root.

We now deal with the case when $m = 4$. In this case equation (1) can be wriiten as

$$x(x+1)(x+2)(x+3) = \alpha + \beta D_4(l' + lx, a^n)$$
$$= \alpha + \beta[(l' + lx)^4 - 4a^n(l' + lx)^2 + 2a^{2n}].$$

Evaluating at $x = 0, -1, -2, -3$, and using the fact that $\beta \neq 0$, we obtain

$$l'^4 - 4a^n l'^2 = (l' - l)^4 - 4a^n(l' - l)^2 = (l' - 2l)^4 - 4a^n(l' - 2l)^2$$
$$= (l' - 3l)^4 - 4a^n(l' - 3l)^2.$$

Since $l \neq 0$, this equation does not have a solution. Hence the case $m = 4$ is also ruled out. Therefore (f_1, g_1) can not be a standard pair of third kind.

Let (f_1, g_1) be a standard pair of fourth kind.
Then $(f_1, g_1) = (a^{-m/2} D_m(x, a), b^{-n/2} D_n(x, b))$ where $(m, n) = 2$.

As $a \neq 0$ we can argue as in the previous case and since here, m is even, the only case possible is $m = 4$ or 6. This gives a contradiction exactly as in the previous case. Therefore (f_1, g_1) can not be a standard pair of fourth kind.

Hence we have shown that when deg $\phi = 1$, equation (1) has only finitely many solutions except for the case $m = 4$. In this case it has infinitely many solutions only when $g(y) = \frac{9}{16} + b\mu(y)^2$. This completes the proof of the proposition. □

Proof of theorem 1.1. Fix $m \geq 3$. We have to prove that if equation (1) has infinitely many solutions then the pair $(m, g(x))$ is as in case (1) or (2) or (3). This is clear from the discussion in the begining of this section and propositions 3.1, 3.2 and 3.3.

Proof of theorem 1.2. Let $g(y)$ be an irreducible polynomial $\mathbf{Q}[Y]$. We have to prove that there are only finitely many solutions x, y of equation (1) satisfying (2) except for the case $m = 4$ and $g(y) = \frac{9}{16} + b\mu(y)^2$.

Suppose equation (1) has infinitely many solutions. Then, as before by theorem 2.8, there are linear polynomials $\lambda, \mu \in \mathbf{Q}[X]$, and a standard pair $(f_1, g_1) \in \mathbf{Q}[X]$ such that for some $\phi \in \mathbf{Q}[X]$, we have $f(x) = \phi(f_1(\lambda(x)))$ and $g(x) = \phi(g_1(\mu(x)))$. As we have observed before, deg ϕ can be m, $\frac{m}{2}$ or 1.

When deg $\phi = m$, by proposition 3.1, $g(y)$ is of the form $f(h(y)$ for some polynomial $h(y) \in \mathbf{Q}[Y]$. As $f(x)$ is reducible, in this case $g(y)$ is also reducible which violates the assumption.

When deg $\phi = \frac{m}{2}$, from proposition 3.2 we have $g(y) = \phi(h(y))$ for some polynomial $h(y) \in \mathbf{Q}[Y]$ and $\phi(x) = (x - \frac{1}{4})(x - \frac{9}{4}) \cdots\cdots (x - ((2k-1)^2/4))$. As $\phi(x)$ is reducible, we get $g(y)$ is also reducible, which again violates the assumption.

Therefore, the only possibility is deg $\phi = 1$. In this case, proposition 3.3 gives the required result.

Proof of theorem 1.3. Let $g(y)$ be an irreducible polynomial of degree n in $\mathbf{Q}[Y]$ and Δ be a positive integer. Write $g(y) = \frac{g_1(y)}{A}$ for some nonzero integer A and a polynomial $g_1(y) \in \mathbf{Z}[Y]$. Assume that $f(x) = g(y)$ (i.e. $Af(x) = g_1(y)$) has a rational solution (x, y) with bounded denominator Δ.

Since $g(y)$ (and hence $g_1(y)$) is an irreducible polynomial, by the Chebotarev density theorem, there are infinitely many primes $p \in \mathbf{Z}$ such that $g_1(y)$ has no root modulo p. Choose a prime P such that $(A\Delta, P) = 1$, and $g_1(y)$ does not have root modulo P. We will show that when $m \geq P$ there does not exist any rational solution with denominator Δ of equation (1).

Suppose $m \geq P$ and assume that $f(x) = g(y)$ (i.e. $Af(x) = g_1(y)$) has a solution satisfying (2). Let $\frac{x_0}{d}, \frac{y_0}{d_1}$ be a solution where d, d_1, x_0, y_0 are integers such that d and d_1 divide Δ. Then we have, $Ax_0(x_0 + d)(x_0 + 2d)...(x_0 + (m-1)d) = d^m g_1(\frac{y_0}{d_1})$. Clearing the denominator on the right hand side, we get ${d_1}^n Ax_0(x_0 + d)(x_0 + 2d)...(x_0 + (m-1)d) = d^m h(y_0)$ where $h(y) = {d_1}^n g_1(\frac{y}{d_1})$ is in $\mathbf{Z}[y]$. Since $m \geq P$ and $(d, P) = 1$, P divides $x_0(x_0 + d)(x_0 + 2d)...(x_0 + (m-1)d)$. Therefore P divides $d^m h(y_0)$ and so $h(y_0) \equiv 0 \bmod P$. Since $(d_1, P) = 1$, this implies $g_1(z_0) \equiv 0 \bmod P$ where $d_1 z_0 \equiv y_0 \bmod P$. Hence g_1 has a root modulo P, which is a contradiction. Therefore when $m \geq P$, equation (1) does not have any rational solution with bounded denominator Δ. This completes the proof of the theorem.

Note that the bound $C = P$ in the theorem can be made effective by using the effective versions of Cebotarev density theorem in [7] and [8] by Lagarias et al.

Proof of theorem 1.4. It is clearly a consequence of theorems 1.2 and 1.3.

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