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ADMISSIBILITY IN THE GAMMA DISTRIBUTION : TWO EXAMPLES

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SUMMARY. Hwang (1981)—type bounds on admissible estimators of the vector of scale parameters in independent simple exponential distributions are obtained. The result suggests that Berger's (1980) improved estimate of the scale parameters stands as a dividing line between admissible and inadmissible estimates in the simple exponential distribution. Also, simultaneous estimation of the vector of scale parameters in the gamma family is considered, and it is shown that the natural estimate is inadmissible for $p \geqslant 3$ under the loss $\sum\limits_{i=1}^{p} a_i\theta_i - \sum\limits_{i=1}^{p} \log a_i\theta_i - p$. Some concluding remarks are made.

1. Introduction

Admissibility in multiparameter problems and its interrelations with certain differential inequalities have recently received enormous attention. Since the pioneering works of Stein, multiparameter admissibility vis-a-vis differential inequalities has been studied by many statisticians, notably Brown (1979), Berger (1980), Ghosh and Parsian (1980) and Hwang (1981). Brown (1971) showed that the question of admissibility of an estimate of the multinormal mean can often be settled by 'comparing' it with the celebrated James-Stein estimate,

$$\delta_c(X) = \left(1 - \frac{c}{\sum\limits_{j=1}^{p} X_j^2}\right).X, \qquad \dots \quad (1.1)$$

with $c = p-2$.

Formally, Brown (1971) showed that (a) an estimate $\delta(x)$ of the mean θ is inadmissible if for some $0 < c < p-2$, and some $M > 0$,

$$\sum_{i=1}^{p} x_i\,\delta_i(x) \geqslant \sum_{i=1}^{p} x_i\delta_{ci}(x) \quad \text{for } \|x\| \geqslant M. \qquad \dots \quad (1.2)$$

(b). If $\delta(x)$ is a generalized Bayes estimate of θ with uniformly bounded risk, then $\delta(x)$ is admissible if for $c = p-2$, and some $M > 0$,

$$\sum_{i=1}^{p} x_i\delta_i(x) \leqslant \sum_{i=1}^{p} x_i\delta_{0i}(x) \quad \text{for } \|x\| \geqslant M. \qquad \dots \quad (1.3)$$

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The implications of (a) and (b) above are very strong. They virtually imply that the James-Stein estimate with $c = p-2$ stands as a dividing line between admissible and inadmissible estimates of the multinormal mean. Recently, Hwang (1981) extended Brown's phenomenon (a) to the general continuous exponential family and gave necessary conditions for admissibility of an estimate of the vector of natural parameters under quadratic loss. For the special normal case, Hwang's result is somewhat stronger than Brown's indicated in (1.2).

In order to obtain these necessary conditions, Hwang (1981) extended Berger's (1980) identity to step-functions and gave unbiased estimates of $E_\theta[\theta h(X)]$, where h is a step-function, and θ denotes the natural parameter. In Section 2, we consider the problem of giving Hwang-type bounds on admissible estimates of the vector of reciprocals $(\theta_1^{-1}, \ldots, \theta_p^{-1})$ of the natural parameters in the continuous exponential family. As an application, it has been shown in the spirit of Hwang that if $X_1, \ldots, X_p$ are independent simple exponential random variables with scale-parameters $\theta_1, \ldots, \theta_p$, then any estimate $\delta(x) = (\delta_1(x), \ldots, \delta_p(x))$ of the mean vector $(\theta_1^{-1}, \ldots, \theta_p^{-1})$ is inadmissible if for some $0 < c < 2(p-1)$, and some $M > 0$,

$$\sum_{i=1}^{p} x_i^{-3}\delta_i(x) \leqslant \sum_{i=1}^{p} x_i^{-3}\delta_{c,i}^{B}(x) \text{ for every } x \in (0, M]^p$$

where
$$\delta_{c,i}^{B}(x) = \frac{x_i}{2}\left[1+\frac{c x_i^{-4}}{2.\left(\sum_{j=1}^{p} x_j^{-2}\right)^2}\right]$$

is the improved estimate suggested by Berger (1980).

In Section 3, we address the problem of simultaneous estimation of independent gamma scale-parameters. In contrast to the weighted quadratic losses of Berger (1980) and Ghosh and Parsian (1980), we have considered another invariant loss $L(\theta^{-1}, a) = \sum_{i=1}^{p} a_i\theta_i - \sum_{i=1}^{p} \log a_i\theta_i - p$, and have shown that the 'natural' estimate for this loss is inadmissible for $p \geqslant 3$. Finally, we make some concluding remarks relating this result to some earlier observations of Berger (1980) and Brown (1980), Brown (1966).

2. Estimation of the vector of reciprocals of natural parameters

In this section, we need to extend Berger's identity for $E_\theta[\theta^{-1}h(X)]$, where h is a step-function $g(x)I_A(x)$, A typically an interval (rectangle), and g absolutely continuous. We shall quote below the essential preliminaries without proof in the notations of Hwang (1981). The notations are clearly explained in Hwang (1981). As pointed out by Hwang (1981), his notations serve the purpose of putting the calculations in a simple and compact form.

Lemma 2.1 : *Let $X_1, \ldots, X_p$ be independent and let X_i have density (Lebesgue) $f_{\theta_i}(x_i) = e^{-\theta_i r_i(x_i)} p(\theta_i) t_i(x_i)$ on $[a, b]$. Let $g(X) = (g_1(X), \ldots, g_p(X))$ be an absolutely continuous function such that*

$$\lim_{\substack{x_i \to b \\ (x_i \to a)}} g_i(x) e^{-\theta_i r_i(x_i)} = 0 \text{ for every } \theta_i,\ i \geqslant 1.$$

Then,

$$E_\theta \left[\theta_i^{-1} \frac{\frac{\partial}{\partial X_i} g_i(X) I_A(X)}{t_i(X_i)} \right] = E_\theta \left[\frac{g_i(X) r_i'(X_i)}{t_i(X_i)} I_A(X) \right]. \quad \ldots \quad (2.1)$$

Remark : Suppose $h_i(x)$ is a given function of x and $g_i(x)$ is defined as the indefinite integral of $h_i(x) t_i(x_i)$ (with respect to x_i). Then (2.1) can be rewritten as

$$E_\theta[\theta_i^{-1} h_i(X) I_A(X)] = E_\theta \left[\frac{g_i(X) r_i'(X_i)}{t_i(X_i)} I_A(X) \right] - E_\theta \left[\theta_i^{-1} \frac{g_i(X)}{t_i(X_i)} \frac{\partial}{\partial X_i} I_A(X) \right] \quad \ldots \quad (2.2)$$

(2.2) is frequently useful in expressing the difference in risk of two estimators as the expectation of a differential operator plus a negative quantity. Next we quote a slightly modified version of a result due to Hwang (1981), without the proof.

Theorem 2.1 : *Let $X = (X_1, \ldots, X_p)$ have an arbitrary multivariate distribution depending on a parameter θ. Let $\gamma(\theta) = (\gamma_1(\theta), \ldots, \gamma_p(\theta))$ be any parametric function and let $\boldsymbol{\delta}_1(X)$, $\boldsymbol{\delta}_2(X)$ be two estimators of $\gamma(\theta)$ such that $R(\theta), \boldsymbol{\delta}_2(X)) \leqslant R(\theta, \boldsymbol{\delta}_1(X)) \ \forall \ \theta$ (with strict inequality for some θ). Let $\mathbf{d}(X) = \boldsymbol{\delta}_2(X) - \boldsymbol{\delta}_1(X)$. Then any other estimator $\boldsymbol{\delta}(X)$ of $\gamma(\theta)$ is inadmissible if*

$$\mathbf{d}(X).\boldsymbol{\delta}(X) \leqslant \mathbf{d}(X).\boldsymbol{\delta}_1(X) \text{ for all } x.$$

Remark : Theorem 2.1 asserts that Hwang's (1981) basic lemma is applicable for any parameter function so long as the loss is squared-error.

With these preliminaries, we now go into actually obtaining Hwang-type bounds on admissible estimators of $(\theta_1^{-1}, \ldots, \theta_p^{-1})$, in the general continuous exponential family.

Let $\lambda^*(X)$ and $\lambda(X)$ be two estimators of $(\theta_1^{-1}, \ldots, \theta_p^{-1})$ defined by

$$\lambda_i^*(x) = \frac{x_i}{\alpha_i+1}\,[1+\phi_i^*(x)I_A(x)], \quad 1 \leqslant i \leqslant p \qquad \ldots \quad (2.3)$$

$$\lambda_i(x) = \frac{x_i}{\alpha_i+1}\,[1+\phi_i(x)I_A(x)], \quad 1 \leqslant i \leqslant p \qquad \ldots \quad (2.4)$$

where $\alpha_1, \ldots, \alpha_p$ are any constants, and ϕ, ϕ^* are functions satisfying certain conditions indicated later in this section, and A is a p-dimensional rectangle. The calculations closely resemble that of Hwang (1981) The idea is to express $R(\theta, \lambda^*)-R(\theta, \lambda)$ as $E(\Delta\phi^*(x)-\Delta\phi(x))$ plus a negative quantity, where $\Delta(\cdot)$ is a differential operator, and choose ϕ, ϕ^* suitably such that $\Delta\phi^*(x) < \Delta\phi(x)$.

Define $\delta^0(X)$ by

$$\delta_i^0(x) = \frac{x_i}{\alpha_i+1}, \quad 1 \leqslant i \leqslant p. \qquad \ldots \quad (2.5)$$

Now,

$$R(\theta, \lambda^*)-R(\theta, \delta^0)$$

$$= E\sum_{i=1}^{p}(\lambda_i^*(x)-\theta_i^{-1})^2-E\sum_{i=1}^{p}(\delta_i^0(x)-\theta_i^{-1})^2$$

$$= E\sum_{i=1}^{p}\left(\lambda_i^*(x)-\frac{x_i}{\alpha_i+1}\right)^2+2E\sum_{i=1}^{p}\left(\frac{x_i}{\alpha_i+1}-\theta_i^{-1}\right)\left(\lambda_i^*(x)-\frac{x_i}{\alpha_i+1}\right)$$

$$= E\Big[\sum_{i=1}^{p}\frac{x_i^2}{(\alpha_i+1)^2}\,\phi_i^{*2}(x)I_A(x)+2\sum_{i=1}^{p}\frac{x_i^2}{(\alpha_i+1)^2}\,\phi_i^*(x)I_A(x)$$

$$-2\sum_{i=1}^{p}\theta_i^{-1}\frac{x_i}{\alpha_i+1}\,\phi_i^*(x)I_A(x)\Big]$$

$$= E\Big[\sum_{i=1}^{p}\frac{x_i^2}{(\alpha_i+1)^2}\,\phi_i^{*2}(x)I_A(x)+2\sum_{i=1}^{p}\frac{x_i^2}{(\alpha_i+1)^2}\,\phi_i^*(x)I_A(x)$$

$$-2\sum_{i=1}^{p}\frac{g_i^*(x)r_i'(x_i)}{t_i(x_i)}\,I_A(x)+2\sum_{i=1}^{p}\theta_i^{-1}g_i^*(x)\,\frac{\partial}{\partial x_i}\,I_A(x)/t_i(x_i). \qquad \ldots \quad (2.6)$$

The last equality in (2.6) is a consequence of (2.2), where $g_i^*(x)$ is to be taken as an indefinite integral (with respect to x_i) of $\frac{x_i}{\alpha_i+1}\,\phi_i^*(x)t_i(x_i)$. At this stage it is worth noting that $\phi_i^*(x)$ (and similarly $\phi_i(x)$, for (2.8) to be valid) should be such that

$$\lim_{\substack{x_i \to b \\ (x_i \to a)}} g_i^*(x) e^{-\theta_i r_i(x_i)} = 0$$

$$\lim_{\substack{x_i \to b \\ (x_i \to a)}} g_i(x) e^{-\theta_i r_i(x_i)} = 0, \quad \ldots \quad (2.7)$$

for every θ_i; $g_i(x)$ is formally defined below.

Similarly,

$$R(\theta, \lambda) - R(\theta, \delta^0)$$

$$= E\Big[\sum_{i=1}^{p} \frac{X_i^2}{(\alpha_i+1)^2} \phi_i^2(x) I_A(x) + 2 \sum_{i=1}^{p} \frac{X_i^2}{(\alpha_i+1)^2} \phi_i(x) I_A(x)$$

$$-2 \sum_{i=1}^{p} \frac{g_i(x) r_i'(x_i) I_A(x)}{l_i(x_i)} + 2 \sum_{i=1}^{p} \theta_i^{-1} g_i(x) \frac{\partial}{\partial x_i} I(x)/l_i(x_i) \quad \ldots \quad (2.8)$$

where $g_i(x)$ is an indefinite integral (with respect to x_i) of $\frac{x_i}{\alpha_i+1} \phi_i(x) l_i(x_i)$. (2.6) and (2.8) now enable us to write

$$R(\theta, \lambda^*) - R(\theta, \lambda)$$

$$= \{R(\theta, \lambda^*) - R(\theta, \delta^0)\} - \{R(\theta, \lambda) - R(\theta, \delta^0)\}$$

$$= E\Big[(\Delta\phi^*(x) - \Delta\phi(x)) I_A(x) + 2 \sum_{i=1}^{p} B_i(x, \theta)\Big] \quad \ldots \quad (2.9)$$

where

$$\Delta\phi^*(x) = \sum_{i=1}^{p} \frac{x_i^2}{(\alpha_i+1)^2} \phi_i^{*2}(x) + 2 \sum_{i=1}^{p} \frac{x_i^2}{(\alpha_i+1)^2} \phi_i^*(x) - 2 \sum_{i=1}^{p} \frac{g_i^*(x) r_i'(x_i)}{l_i(x_i)} \quad \ldots \quad (2.10)$$

$$\Delta\phi(x) = \sum_{i=1}^{p} \frac{x_i^2}{(\alpha_i+1)^2} \phi_i^2(x) + 2 \sum_{i=1}^{p} \frac{x_i^2}{(\alpha_i+1)^2} \phi_i^*(x) - 2 \sum_{i=1}^{p} \frac{g_i(x) r_i'(x_i)}{l_i(x_i)} \quad \ldots \quad (2.11)$$

and

$$B_i(x, \theta) = \theta_i^{-1} (g_i^*(x) - g_i(x)) \frac{\partial}{\partial x_i} I_A(x)/l_i(x_i). \quad \ldots \quad (2.12)$$

Therefore, if there exists a rectangle A such that $\Delta\phi^*(x) < \Delta\phi(x)$ for every $x \in A$, and $E_\theta(B_i(x, \theta)) \leqslant 0 \; \forall \; \theta$, then $R(\theta, \lambda^*) < R(\theta, \lambda) \; \forall \; \theta$, and Hwang's lemma (Theorem 2.1) is applicable.

Hence, any estimator $\delta(X) = (\delta_1(X), \ldots, \delta_p(X))$ of $(\theta_1^{-1}, \ldots, \theta_p^{-1})$ will be inadmissible if

$$\sum_{i=1}^{p} \delta_i(x) \frac{x_i}{\alpha_i+1} (\phi_i^*(x) - \phi_i(x)) \leqslant \sum_{i=1}^{p} \lambda_i(x) \frac{x_i}{\alpha_i+1} (\phi_i^*(x) - \phi_i(x)) \quad \ldots \quad (2.13)$$

for almost all x in A.

We can now state the following theorem.

Theorem 2.2 : *Let $\lambda^*(X)$ and $\lambda(X)$ be two estimators as defined in* (2.3) *and* (2.4), *satisfying the conditions* (2.7). *Let $\Delta\phi^*(x)$, $\Delta\phi(x)$, $B_1(x, \theta)$ be as in* (2.10), (2.11) *and* (2.12) *respectively. If there exists an A such that*

$$\Delta\phi^*(x) < \Delta\phi(x) \qquad \textit{for (almost all) } x \in A$$

and
$$E_\theta(B_1(x, \theta)) \leqslant 0 \qquad \textit{for every } \theta,$$

then any estimate $\delta(x)$ of $(\theta_1^{-1}, \ldots, \theta_p^{-1})$, satisfying (2.13) *for (almost all) $x \in A$, is inadmissible.*

Remark : Theorem 2.2 could also be stated by starting with an arbitrary estimate $\delta^0(X)$ rather than $\delta_i^0(X) = \dfrac{X_i}{\alpha_i+1}$. But since the motivation behind Theorem 2.2 is in estimating scale-parameters in independent gamma distributions, in which case $\dfrac{X_i}{\alpha_i+1}$ is the standard estimate of θ_i^{-1}, we have conveniently chosen $\delta_i^0(X)$ as $\dfrac{X_i}{\alpha_i+1}$, for some constants $\alpha_1, \alpha_2, \ldots, \alpha_p$.

Example : Let $X_1, \ldots, X_p$ be independent with $f_{\theta_i}(x_i) = \theta_i e^{-\theta_i x_i}$, $\theta_i > 0$, $x_i > 0$, $i = 1, 2, \ldots, p$. In this case $E_{\theta_i}(X_i) = \theta_i^{-1}$, $i = 1, 2, \ldots, p$. The natural estimate of $(\theta_1^{-1}, \ldots, \theta_p^{-1})$ is $\left(\dfrac{X_1}{2}, \ldots, \dfrac{X_p}{2}\right)$.

Let

$$\phi_i(x) = \frac{c x_i^{-4}}{2\left(\sum_{j=1}^{p} x_j^{-2}\right)^2}, \quad 1 \leqslant i \leqslant p, \; 0 < c < 2(p-1).$$

By definition, $g_i(x)$ is the indefinite integral of

$$\frac{c x_i^{-3}}{4\left(\sum_{j=1}^{p} x_j^{-2}\right)^2} \quad \text{with respect to } x_i.$$

Hence,
$$g_i(x) = \frac{c}{8\left(\sum_{j=1}^{p} x_j^{-2}\right)^2}, \quad 1 \leqslant i \leqslant p.$$

Indeed,
$$\frac{\partial}{\partial x_i} g_i(x) = \frac{2c x_i^{-3}}{8\left(\sum_{j=1}^{p} x_j^{-2}\right)^2} = \frac{c x_i^{-3}}{4\left(\sum_{j=1}^{p} x_j^{-2}\right)^2}, \quad 1 \leqslant i \leqslant p.$$

Now, using (2.11),

$$\Delta(\phi(x)) = \sum_{i=1}^{p} \frac{x_i^2}{4} \frac{c^2 x_i^{-6}}{4\left(\sum_{j=1}^{p} x_j^{-2}\right)^4} + 2\sum_{i=1}^{p} \frac{x_i^2}{4} \frac{c x_i^{-4}}{2\left(\sum_{j=1}^{p} x_j^{-2}\right)^2} - 2\sum_{i=1}^{p} \frac{c}{8\sum_{j=1}^{p} x_j^{-2}}$$

$$= \frac{1}{16}\left[-4pcD^{-1} + 4cD^{-1} + c^2 \frac{\sum_{i=1}^{p} x_i^{-6}}{\left(\sum_{j=1}^{p} x_j^{-2}\right)^4} \right]$$

$$= \frac{1}{16}\left[(c^2+4c-4pc)D^{-1} + c^2 \left\{ \frac{\sum_{i=1}^{p} x_i^{-6}}{\left(\sum_{i=1}^{p} x_i^{-2}\right)^4} - \frac{1}{\sum_{i=1}^{p} x_i^{-2}} \right\} \right], \quad \ldots \quad (2.14)$$

where $D = \sum_{i=1}^{p} x_i^{-2}$.

It is easy to see that $(c^2+4c-4pc)D^{-1}$ is minimized at $c = 2(p-1)$. Also, as $\left(\sum_{i=1}^{p} x_i^{-6}\right)\left(\sum_{i=1}^{p} x_i^{-2}\right) < \left(\sum_{i=1}^{p} x_i^{-2}\right)^4$ for every x, it follows that

$$c^2 \left\{ \frac{\sum_{i=1}^{p} x_i^{-6}}{\left(\sum_{i=1}^{p} x_i^{-2}\right)^4} - \frac{1}{\sum_{i=1}^{p} x_i^{-2}} \right\} > 4(p-1)^2 \left\{ \frac{\sum_{i=1}^{p} x_i^{-6}}{\left(\sum_{i=1}^{p} x_i^{-2}\right)^4} - \frac{1}{\sum_{i=1}^{p} x_i^{-2}} \right\} \quad \ldots \quad (2.15)$$

for every $0 < c < 2(p-1)$.

Therefore, if $\phi_i^*(x)$ is taken as $\phi_i(x)$ with $c = 2(p-1)$, then (2.15) gives $\Delta(\phi^*(x)) < \Delta(\phi(x))$ for every x.

Also, with this choice of $\phi_i^*(x)$,

$$g_i^*(x) - g_i(x) = \frac{2(p-1)-c.}{8\sum_{i=1}^{p} x_i^{-2}}. \quad \ldots \quad (2.16)$$

If now, $A = (0, M]^p$, then by using Hwang (1981), (2.16) and the fact that

$$c < 2(p-1),\ E_\theta\left[\theta_i^{-1}\{g_i^*(x) - g_i(x)\} \frac{\partial}{\partial x_i} I_A(x)\right] \leqslant 0 \ \forall\ \theta > 0.$$

Therefore, we have,

Corollary : *Let* $X_1, \ldots, X_p$ *be independent simple exponentials, with* $E_{\theta_i}(X_i) = \theta_i^{-1}$. *Then any estimate* $\delta(X) = (\delta_1(X), \ldots, \delta_p(x))$ *of* $(\theta_1^{-1}, \ldots, \theta_p^{-1})$ *is inadmissible, provided for some* $0 < c < 2(p-1)$, *some* $M > 0$,

$$\sum_{i=1}^{p} x_i^{-3}\delta_i(x) \leqslant \sum_{i=1}^{p} x_i^{-3}\delta_{c,i}^{B}(x) \text{ for every } x \in (0, M]^p \qquad \ldots \quad (2.17)$$

where $\delta_{c,i}^{B}(x) = \frac{x_i}{2}\left[1 + cx_i^{-4}/2\left(\sum_{i=1}^{p} x_i^{-2}\right)^2\right]$ = *Berger's estimate* (1980).

Remark : The corollary above seems to give evidence that Berger's estimate with $c = 2(p-1)$ stands as the dividing line between admissible and inadmissible estimates of the mean-vector in independent simple exponential distribution. In particular, this corollary also shows Berger's observation that the standard estimate $X/2$ is inadmissible if $p \geqslant 2$. For the general gamma case, however, the calculations corresponding to (2.14) and (2.15) get complicated and it is not clear if a similar results holds there too. Hwang (1981) obtained a similar bound for admissible estimates of natural parameters in independent gamma distributions.

3. Estimation of the gamma scale parameter

Let $X_1, X_2, \ldots, X_p$ be independent, with X_i having density

$$f_{\theta_i}(x_i) = e^{-\theta_i x_i}\theta_i^{\alpha_i} x_i^{\alpha_i - 1} / \Gamma(\alpha_i), \; x_i > 0, \; 1 \leqslant i \leqslant p,$$

where $\alpha_i > 0$ are known, and θ_i's (> 0) are considered unknown. Berger (1980) considered weighted quadratic losses $\sum_{i=1}^{p} \theta_i^{-m}(\delta_i\theta_i - 1)^2$ for $m = 0, 2, 1, -1$, and showed that the standard estimate of $(\theta_1^{-1}, \theta_2^{-1}, \ldots, \theta_p^{-1})$, namely, $\left(\frac{X_1}{\alpha_1+1}, \ldots, \frac{X_1}{\alpha_p+1}\right)$ is inadmissible for $p \geqslant 2$ except when $m = 0$, in which case it is inadmissible for $p \geqslant 3$. Ghosh and Parsian (1980) also discussed this problem for the same weighted quadratic losses. In this section, we consider a typically different loss $\sum_{i=1}^{p} \delta_i\theta_i - \sum_{i=1}^{p} \log \delta_i\theta_i - p$; the vector of unbiased estimates $\left(\frac{X_1}{\alpha_1}, \ldots, \frac{X_p}{\alpha_p}\right)$ is a natural estimate of the mean-vector for this loss. We show that this estimate is inadmissible for $p \geqslant 3$ and relate this inadmissibility result to some observations of Berger and Brown.

The usual technique of integration by parts (Berger's 1980 identity) and a theorem of Ghosh and Parsian (1980) are stated below for future reference. Also, a technical lemma, to be used subsequently, is also proved.

Lemma 3.1 : (Berger, 1980) : *Let* $h(x) = (h_1(x), \ldots, h_p(x))$ *be a function such that*

$$\lim_{x_i \to 0} h_i(x) x_i^{\alpha_i - 1} e^{-\theta_i x_i} = 0$$

and

$$\lim_{x_i \to \infty} h_i(x) x_i^{\alpha_i - 1} e^{-\theta_i x_i} = 0$$

for every $\theta_i > 0$. *Assume* $h_i(x)$ *has all partial derivatives of first order. Then,*

$$E_\theta[\theta_i h_i(X)] = E_\theta \left[h_i^{i(1)}(X) + \frac{(\alpha_i - 1) h_i(X)}{X_i} \right],$$

where

$$h_i^{i(1)}(X) = \frac{\partial}{\partial X_i} h_i(X), \ 1 \leqslant i \leqslant p.$$

It is implicitly assumed in the above identity that

$$E_\theta[\,|\, h_i^{i(1)}(X) + (\alpha_i - 1) h_i(X)/X_i \,|\,] < \infty, \text{ for every } \theta.$$

Lemma 3.2 : (Ghosh and Parsian, 1980) : *For given functions* $\nu_i(x_i)$, $\psi(x) > 0$, $w_i(x)$, *define* $\xi_i(x_i) = \nu_i^{-1}(x_i)$ *and* $S = \sum_{j=1}^{p} d_j |\xi_j(x_j)|^\beta$ *where* d_j *and* β *are positive constants to be chosen later. If*

$$\sum_{i=1}^{p} w_i(x) \xi^2(x_i)/\psi(x) \leqslant KS \text{ for some } K, d_j \text{ and } \beta \text{ (all positive) and for all } x \in \mathcal{X}^p, \quad \ldots (3.1)$$

then

$$\phi(x) = \frac{-c\xi_i(x_i)}{S+b} \quad \ldots (3.2)$$

provides a solution to $\Delta(x) < 0$ *for all* $b > 0$ *and* $0 < c < K^{-1}(p-\beta)$ *where*

$$\Delta(x) = \psi(x) \sum_{i=1}^{p} \nu_i(x_i) \phi_i^{i(1)}(x) + \sum_{i=1}^{p} w_i(x) \phi_i^2(x) \quad \ldots (3.3)$$

Remark : Such solutions to $\Delta(x) < 0$ were first obtained by Berger (1980). The constant c can be generalized to a non-decreasing function $\tau(S)$, with $0 < \tau(S) < K^{-1}(p-\beta)$.

Lemma 3.3 : *For* $|x| < \frac{1}{2}$, $\log(1+x) \geqslant x - \frac{3x^2}{2}$.

Proof : Define $f(x) = \log(1+x) - x + \frac{3x^2}{2}$.

Then
$$f'(x) = \frac{x(3x+2)}{1+x}$$
$$\geqslant 0 \text{ for } 0 \leqslant x < \frac{1}{2}.$$
$$\leqslant 0 \text{ for } -\frac{1}{2} < x \leqslant 0.$$

Consequently, $f(x) \geqslant f(0) = 0$, for $|x| < \frac{1}{2}$.

Theorem 3.1 : *Let $X_1, \ldots, X_p$ be independent gamma variables with $E(X_i) = \frac{\alpha_i}{\theta_i}$, α_i known. Consider the loss $L(\theta^{-1}, \delta) = \sum_{i=1}^{p} \delta_i\theta_i - \sum_{i=1}^{p} \log \delta_i\theta_i - p$. Then $\left(\frac{X_1}{\alpha_1}, \ldots, \frac{X_p}{\alpha_p}\right)$ is an inadmissible estimator of $(\theta_1^{-1}, \ldots, \theta_p^{-1})$ for $p \geqslant 3$.*

Proof : Let $\delta(X)$ be a competitor to the natural estimate
$$\delta_0(X) = \left(\frac{X_1}{\alpha_1}, \ldots, \frac{X_p}{\alpha_p}\right).$$

Write $\delta_i(x) = \frac{x_i}{\alpha_i} + h_i(x)$, $1 \leqslant i \leqslant p$. We assume $h_i(x)$ are such that Lemma 3.1 holds. Then,
$$\alpha(\theta) = R(\theta, \delta) - R(\theta, \delta_0)$$
$$= \sum_{i=1}^{p} E\left\{\delta_i(X)\theta_i - \log(\delta_i(X)\theta_i) - \frac{X_i}{\alpha_i}\theta_i + \log\left(\frac{X_i}{\alpha_i}\theta_i\right)\right\}$$
$$= \sum_{i=1}^{p} E\left\{\theta_i h_i(X) - \log\frac{\alpha_i\delta_i(X)}{X_i}\right\}$$
$$= \sum_{i=1}^{p} E\left\{\theta_i h_i(X) - \log\left(1 + \frac{\alpha_i h_i(X)}{X_i}\right)\right\} \qquad \ldots \quad (3.4)$$

If the competitor $\delta(X)$ is such that $\left|\frac{\alpha_i h_i(x)}{x_i}\right| < \frac{1}{2}$ uniformly in x, for every $1 \leqslant i \leqslant p$, then by Lemma 3.3,
$$\alpha(\theta) \leqslant \sum_{i=1}^{p} E\left\{\theta_i h_i(X) - \frac{\alpha_i h_i(X)}{X_i} + \frac{3}{2}\frac{\alpha_i^2 h_i^2(X)}{X_i^2}\right\}$$
$$= \sum_{i=1}^{p} E\left\{h_i^{(1)}(X) + \frac{(\alpha_i - 1)h_i(X)}{X_i} - \frac{\alpha_i h_i(X)}{X_i} + \frac{3}{2}\frac{\alpha_i^2 h_i^2(X)}{X^2}\right\}$$
(by Lemma 3.1)
$$= \sum_{i=1}^{p} E\left\{h_i^{(1)}(X) - \frac{h_i(X)}{X_i} + \frac{3}{2}\frac{\alpha_i^2 h_i^2(X)}{X_i^2}\right\}. \qquad (3.5)$$

Now make the transformation

$$h_i(x) = x_i\phi_i(x), \quad 1 \leqslant i \leqslant p.$$

Then,
$$h_i^{(1)} = \phi_i(x)+x_i\phi_i^{(1)}(x)$$
$$= \frac{h_i(x)}{x_i}+x_i\phi_i^{(1)}(x).$$

Hence, (3.5) gives,

$$\alpha(\theta) \leqslant E\left[\sum_{i=1}^{p} x_i\phi_i^{(1)}(x)+\frac{3}{2}\sum_{i=1}^{p}\alpha_i^2\phi_i^2(x)\right]. \quad \ldots \quad (3.6)$$

Note now the differential expression within braces in (3.6) is of the form (3.3), and (3.1) of Lemma 3.2 is satisfied with $K = \frac{3}{2}$, $\beta = 2$, $d_j = \alpha_j^2$.

Therefore, for $0 < c < \frac{2}{3}(p-2)$, $b > 0$,

$$\phi_i(x) = \frac{-c \log x}{\sum_{i=1}^{p}\alpha_i^2(\log x_i)^2+b} \quad \ldots \quad (3.7)$$

is a solution to $\sum_{i=1}^{p} x_i\,\phi_i^{(1)}(x)+\frac{3}{2}\sum_{i=1}^{p}\alpha_i^2\phi_i^2(x) < 0$.

Also observe that $|\alpha_i\,\phi_i(x)|^2 = \frac{c^2\alpha_i^2(\log x_i)^2}{\left\{\sum_{j=1}^{p}\alpha_j^2(\log x_j)^2+b\right\}^2} < \frac{1}{4}$ if $b > 4c^2$.

Hence, if
$$\delta_i(x) = \frac{x_i}{\alpha_i} - \frac{cx_i \log x_i}{\sum_{j=1}^{p}\alpha_j^2(\log x_j)^2+b}, \quad 1 \leqslant i \leqslant p, \quad \ldots \quad (3.8)$$

where $0 < c < \frac{2}{3}(p-2)$, $b > 4c^2$, then $\alpha(\theta) < 0$ for all θ.

This proves the theorem.

Some remarks on theorem 3.1

(1) It is easy to check that the tail condition required on $h_i(x)$ for Lemma 3.1 to hold is satisfied by the solutions eventually obtained in Theorem 3.1.

(2) It was shown by Berger (1980) that the critical dimension of inadmissibility of the natural estimate of the gamma scale-parameters is frequently 2, rather than 3. Berger contended that the critical dimension of

inadmissibility is typically 2, and 3 dimension is required only in special situations. Brown (1980) discussed Berger's phenomenon and some of its peripheral aspects in the context of simultaneous estimation of independent normal means, and gave examples to assert that the critical dimension of inadmissibility depends on the loss, rather than the underlying coordinate distributions. Theorem 3.1 gives another example of a natural and invariant loss for which the critical dimension of inadmissibility could be 3 in the gamma distribution itself, although the peripheral aspects relating to the point of shrinkage are not illustrated by this example. Interestingly, Berger (1980) also required 3 dimension for inadmissibility only for the invariant quadratic loss. This is probably expected from Brown (1966) and Brown and Fox (1974). It follows from Brown (1966) that under the loss described in Theorem 3.1, the standard estimate is admissible if $p = 1$, although at this moment we do not know if admissibility for $p = 2$ follows readily from Brown and Fox (1974). We conjecture the standard estimate is admissible in two dimension.

(3) The improved estimate in Theorem 3.1 bears similarity to the James-Stein estimate of the multinormal mean. This is expected since on making a log transform, the problem reduces to the estimation of a location vector. One should also observe that our improved estimate is practically the same as Berger's (1980) for the other invariant loss $\sum_{i=1}^{p} (\delta_i\theta_i - 1)^2$.

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