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EXPANSION OF BAYES RISK IN THE CASE OF DOUBLE EXPONENTIAL FAMILY

By S. N. JOSHI

Indian Statistical Institute

SUMMARY. For i.i.d. random variables with a double exponential density and a smooth prior expansion of the Bayes risk upto $o(n^{-3/2})$ is obtained. Unlike the case of smooth density, the term after $O(n^{-1})$ term is of order $O(n^{-3/2})$ instead of order $O(n^{-2})$.

0. Introduction

Expansion upto $o(n^{-2})$ for the Bayes risk was obtained by Ghosh, Sinha and Joshi (1981) under certain regularity conditions which include differentiability of log likelihood function (and some of its derivatives). Under very general conditions (such as LAN condition) the limit of Bayes risk (w.r.t. bounded loss functions) has been obtained (see Strasser, 1978, Proposition 2). So a natural question to ask is whether, under such general conditions as LAN, it is possible to get an expansion of Bayes risk.

In this paper we consider a family of distributions (viz. double exponential with location parameter) which satisfy LAN condition but for which conditions of Ghosh, Sinha and Joshi (1981) do not hold and show that the Bayes risk has expansion but now the term after the n^{-1} term is not of order n^{-2} (as was the case earlier) but is of order $n^{-3/2}$. This indicates that under suitable strengthening of the LAN condition, it may be possible to get an expansion of the Bayes risk. In the following paragraph we sketch the method of proof which we followed in our special case and which is likely to succeed in the general case too.

A sequence of family of distribution is said to satisfy LAN condition if the log likelihood function

$$\Lambda(\theta_0, \theta_0+\delta n^{-1/2}) = \log L(\theta_0+\delta n^{-1/2} | x_1, \ldots, x_n) - \log L(\theta_0 | x_1, \ldots, x_n)$$

can be approximated in the following way

$$\left| \Lambda(\theta_0, \theta_0+\delta n^{-1/2}) - \frac{\delta}{\sqrt{n}} \sum_{i=1}^{n} h(\theta_0, X_i) - \delta^2 A(\theta_0)/2 \right| \xrightarrow{P_{\theta_0}} 0$$

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where $h(\theta_0, x_1)$ is normalized r.v. and $A(\theta_0) = \text{var}_{\theta_0}(h(\theta_0, X_1))$. Now if we have following type of approximation

$$P_{\theta_0}\{|\Lambda(\theta_0, \theta_0+\delta n^{-1/2})-\delta\Sigma h(\theta_0, x_i)/\sqrt{n}$$

$$-\delta^2 A(\theta_0)/2-n^{-\alpha}V_n(\theta_0)| < n^{-\beta}\} > 1-O(n^{-\gamma})$$

for some suitable α, β and γ all positive then it is likely that by plugging in both, the numerator and the denominator of

$$B_n(\theta_0) = E[n^{1/2}(\theta-\theta_0)\,|\,x_1, \ldots, x_n]$$

(w.r.t. some suitable prior for θ), the above approximation for the likelihood function one can get an approximation for $B_n(\theta_0)$ upto some suitable order (vide (2.9)). This in turn (vide (2.11)) gives the desired expansion of the Bayes risk in the present case; to extend it to the general LAN case may require non-trivial modifications. It may be observed that one of the main differences between the present investigation and Ghosh, Sinha and Joshi (1981) is that here we expand the likelihood around θ_0 rather than the m.l.e. $\hat{\theta}_n$. Technicalities regarding the delicate nature of approximation of $B_n(\theta_0)$ near the end points of the support of prior can be handled in a way similar to that of Ghosh, Sinha and Joshi (1981); see also in this regard Burnasev (1981).

1. Main result

Let $X_1, X_2, \ldots, X_n$ be i.i.d. with p.d.f.

$$f(x, \theta) = \frac{1}{2}\exp(-|x-\theta|),\ -\infty < x < \infty,\ \theta \in R.$$

Let ρ be the prior density of θ w.r.t. Lebesgue measure. Let ρ be twice continuously differentiable and let for some $\eta_j > 0$ ($j = 1$ to 4) ρ satisfy (i) to (iv) below.

(i) $\rho(\theta) > 0$ for $\theta \in (a_0, b_0)$ and $\rho(\theta) = 0$ on $[a_0, b_0]^c$ for some $-\infty < a_0 < b_0 < \infty$,

(ii) $\int \rho(\theta)[(\rho)^{(1)}(\theta)]^{2+\eta_1}\, d\theta < \infty$

where

$$(\rho)^{(i)}(\theta) = \frac{d^i}{d\theta^i} \log \rho(\theta), \qquad \ldots \quad (1.1)$$

(iii) $$\int_{D_{1(\varepsilon)}} \rho(\theta)d\theta = O(\varepsilon^{1+\eta_2}) \text{ as } \varepsilon \to 0$$

where

$$D_1(\varepsilon) = \Big[\theta : \sup_{|z| < \varepsilon} |(\rho)^{(2)}(\theta+z)| > \varepsilon^{-1+\eta_3}\Big],$$

(iv) $$\int_{(a_0,\, a_0+\varepsilon)\,\cup\,(b_0-\varepsilon,\, b)} \rho(\theta)d\theta = O(\varepsilon^{1+\eta_4}) \text{ as } \varepsilon \to 0.$$

Remark 1 : ρ satisfies (i) to (iv) if e.g. ρ satisfies (i) and for some $k > 2$ and $i = 0, 1, 2$

$$\begin{aligned} \rho^{(i)}(\theta) &= (\theta-a_0)^{k-i}(c_i+o(1)) \qquad \text{for } \theta \text{ near } a_0 \\ &= (b_0-\theta)^{k-i}(c_i'+o(1)) \qquad \text{for } \theta \text{ near } b_0 \end{aligned}$$

where c_i and c_i' are nonzero constants and

$$\rho^{(i)}(\theta) = \frac{d^i}{d\theta^i}\ \rho(\theta),\ \rho^{(0)}(\theta) = \rho(\theta).$$

Theorem : *Bayes risk, $R(\rho)$, w.r.t. squared error loss function has the following expansion*

$$R(\rho) = n^{-1}+bn^{-3/2}+o(n^{-3/2}) \qquad \dots \quad (1.2)$$

where

$$b = 24 \int \Phi(\omega)\phi^2(\omega)d\omega+4\int (1-\Phi(\omega))^2\Phi(\omega)d\omega-\frac{6}{\sqrt{\pi}}.$$

2. Notations

For a set A let $I(A)$ denote the indicator function of A and let

$$\begin{aligned} I_n(\delta) &= 1 \quad \text{if } |\delta| < (\log n)^2 \\ &= 0 \quad \text{otherwise.} \end{aligned}$$

Let

$$V_{1n}(\delta) = n^{1/4}I(\delta > 0)\{\Sigma 2(x_i-\theta_0-\delta n^{-1/2})I(0 < x_i-\theta_0 < \delta n^{-1/2})+\delta^2/2\},$$

$$V_{2n}(\delta) = n^{1/4}I(\delta < 0)\{\Sigma 2(\theta_0+\delta n^{-1/2}-x_i)I(\delta n^{-1/2} < x_i-\theta_0 < 0)+\delta^2/2\}.$$

For $i = 0$ and 1, and $j = 1$ and 2 let

$$W_j^{(i)} = \int I_n(\delta)\delta^i V_{jn}(\delta)\phi(t+\delta)d\delta$$

and

$$W_{2+j}^{(i)} = \int I_n(\delta)\delta^i (V_{jn}(\delta))^2\phi(t+\delta)d\delta,$$

where ϕ denotes the p.d.f. of standard normal variable; Φ will denote it's d.f. Let

$$t = 2n^{-1/2}\{\Sigma I(x_i-\theta_0 \leqslant 0)-n/2\},\ t_0 = \frac{1}{2}\,(n^{1/2}\,t+n)$$

and let $P(t)$ be a generic notation for polynomial in t of finite degree and with bounded (nonrandom) coefficients.

Let Y be a r.v. having p.d.f. $\frac{1}{2}\exp(-|y|)$, $-\infty < y < \infty$ and let for all $n \geqslant 1$

$$Y_{1n} = (Y-\delta n^{-1/2})\, I(0 < Y < \delta n^{-1/2}),$$
$$Y_{2n} = (\delta n^{-1/2} - Y)\, I((\delta - n^{-1}) n^{-1/2} \leqslant Y \leqslant \delta n^{-1/2}).$$

Let ρ be as in Section 1.2. Define

$$D_n^c = \Big[\theta : \sup_{|z| < n^{-1/2} \log^2 n} |(\rho)^{(2)}(\theta+z)| < n^{1/2-\eta},\ |(\rho)^{(1)}(\theta)| < n^{1/4-\eta}\Big]$$
$$\bigcap (\rho(\theta) > n^{-1}) \bigcap (a_0 + n^{-1/2}\log^3 n,\ b_0 - n^{-1/2}\log^2 n) \quad \ldots \quad (2.1)$$

then using (i) to (iv) of Section 1.2 we have for some $\eta > 0$ and $\eta' > 0$

$$\int_{D_n} \rho(\theta) d\theta = O(n^{-1/2-\eta'}). \quad \ldots \quad (2.2)$$

Hence forward η and η' will be used as generic notations for positive constants.

In the rest of the paper we fix a $\theta_0 \in D_n^c$; all probability statements and orders hold good uniformly in $\theta_0 \in D_n^c$.

Let A_n denote common part of events of (2.5) and (2.11); we have for every l and λ, both positive,

$$P_{\theta_0}(A_n) \geqslant 1 - O(n^{-\lambda}). \quad \ldots \quad (2.3)$$

Note that as $\theta_0 \in D_n^c$ we have

$$\min(\log^2 n,\ \sqrt{n}(b_0 - \theta_0),\ \sqrt{n}(\theta_0 - a_0)) = \log^2 n. \quad \ldots \quad (2.4)$$

$\underline{x}_n$ will denote $(x_1, \ldots, x_n)$.

2.2. *Some lemmas* : The following Lemma is a consequence of Lemma 3.1 of Reiss (1976).

Lemma 2.1 : *Let $\bar{\theta}_n$ be median of sample of size n from $f(x, \theta_0)$. For each $k > 0$ there exists a constant $c > 0$ such that*

$$P_{\theta_0}(n^{1/2}|\bar{\theta}_n - \theta_0| < c \log^{1/2} n) = 1 - O(n^{-k}).$$

Following lemma is a well known result (vide e.g., Serfling, 1980, page 95, Lemma A).

Lemma 2.2 :

$$P_{\theta_0}(|t| < ((s+1) \log n)^{1/2}) \geqslant 1 - O(n^{-s}) \quad \textit{for } s > 0.$$

Lemma 2.3 : *For each positive λ and l*

$$P_{\theta_0}\{\int I(\log^2 n, n^{1/2}(b_0-\theta_0))\delta\phi(t+\delta)\exp[n^{-1/4}V_{1n}(\delta)]d\delta < n^{-l},$$
$$\int I(-n^{1/2}(\theta_0-a_0), \log^2 n)\delta\phi(t+\delta)\exp[n^{-1/4}V_{2n}(\delta)]d\delta < n^{-l},$$
$$|t| < ((\lambda+4)\log n)^{1/2}\} \geqslant 1-O(n^{-\lambda}). \quad \dots \quad (2.5)$$

Proof: *Step* 1 : For each $\varepsilon_1 > 0$ and $\log^2 n < \delta < n^{1/2}(b_0-\theta_0)\log^{-1} n$

$$P_{\theta_0}[n^{-1/4}V_{1n}(\delta)+(\varepsilon_1-1)\delta^2/2 > 0] \leqslant \exp[-\delta^2(1-\varepsilon_1+O(\log^{-1}n))/4]. \quad \dots \quad (2.6)$$

To get (2.6) note that LHS of (2.6) $\leqslant \exp(\varepsilon_1\delta^2/4)E^n\exp(Y_{1n})$. Hence LHS of (2.6) $\leqslant \exp\{\varepsilon_1\delta^2/4+n\log(1-\delta^2 n^{-1}/4+O(\delta^3 n^{-3/2}))\}$.

Step 2 : For $n^{1/2}(b_0-\theta_0)\log^{-1}n \leqslant \delta \leqslant n^{1/2}(b_0-\theta_0)$ and $\varepsilon_2 > 0$ LHS of (2.6) with ε_1 replaced by ε_2

$$\leqslant \exp\{\varepsilon_2\delta^2/4-n(b_0-\theta_0)^2(\log^{-2}n)(1+O(\log^{-1}n))/4\}. \quad \dots \quad (2.7)$$

To get (2.7) note that

$$\text{LHS of } (2.6) \leqslant \exp[\varepsilon_2\delta^2/4+n\log(1+e^{-\delta n^{-1/2}}+\delta n^{-1/2}e^{-\delta n^{-1/2}})-n\log^2]$$
$$\leqslant \exp[\varepsilon_2\delta^2/4+n\log(1+\exp[-(b_0-\theta_0)\log^{-1}n]+(b_0-\theta_0)\log^{-1}n.$$
$$\exp[-(b_0-\theta_0)\log^{-1}n]-n\log^2].$$

Step 3 : For $\log^2 n < \delta < n^{1/2}(b_0-\theta_0)$

$$P_{\theta_0}\left[\inf_{(\delta-n^{-1})\leqslant\delta'<\delta} n^{-1/4}[V_{1n}(\delta)-V_{1n}(\delta')] < -\log n\right] \leqslant \exp[-\log^2 n]. \quad \dots \quad (2.8)$$

To get (2.8) note that

$$\inf_{\delta-n^{-1}<\delta'<\delta} n^{-1/4}[V_{1n}(\delta)-V_{1n}(\delta')]$$
$$= \inf_{\delta-n^{-1}\leqslant\delta'<\delta}[2\Sigma(x_i-\theta_0-\delta n^{-1/2})I(\delta' n^{-1/2} < x_i-\theta_0 \leqslant \delta n^{-1/2})$$
$$-2(\delta-\delta')n^{-1/2}\Sigma I(0 < x_i-\theta_0 \leqslant \delta' n^{-1/2})-(\delta-\delta')(\delta+\delta')/2]$$
$$\geqslant 2\Sigma(x_i-\theta_0-\delta n^{-1/2})I((\delta-n^{-1})n^{-1/2} \leqslant x_i-\theta_0 \leqslant \delta n^{-1/2})$$
$$-2n^{-1/2}-n^{1/2}(b_0-\theta_0)n^{-1}.$$

Now (2.8) can be obtained by using exponential probability inequality for the first term of the last expression.

Step 4 : Note that

$$P_{\theta_0}[\int I(\log^2 n, n^{1/2}(b_0-\theta_0))\delta\phi(t+\delta)\exp[n^{-1/4}V_{1n}(\delta)]d\delta > n^{-l},$$
$$|t| < (\lambda+4)\log n)^{1/2}] \leqslant \sum_{i\in J_n} P_i \quad \dots \quad (2.9)$$

where

$$J_n = \left\{ [\log^2 n] + \frac{j}{n} : j = 0, 1, \ldots, n([n^{1/2}(b_0 - \theta_0)] + 1 - [\log^2 n]) \right\},$$

$[g]$ = integer part of g,

and

$$P_i = P_{\theta_0}[\int I(i - n^{-1} < \delta < i) \delta \phi(t+\delta) \exp[n^{-1/4} V_{1n}(\delta)] d\delta > n^{-3/2},$$

$$|t| < ((\lambda+4)\log n)^{1/2}].$$

Now choosing ε_1 of step 1 to be 3/4, ε_2 of step 2 to be $n^{-1/4}$ in view of step 3 and Lemma 2.2 it is easy to see that

$$P_i \leqslant O(n^{-\lambda-3/2}) \text{ uniformaly in } i \in J_n.$$

Hence

$$\text{LHS of (2.9)} \leqslant O(n^{-\lambda}). \qquad \ldots \quad (2.10)$$

A statement analogues to (2.10) for the part containing $V_{2n}(\delta)$ can be proved in a similar way; this completes the proof of the lemma.

Lemma 2.4 : *For* $\lambda > 0$ *we have*

$$P_{\theta_0}\left[\sup_{0 < |\delta| < \log^2 n} | V_{in}(\delta)| < 2 \log^3 n \right] \geqslant 1 - O(n^{-\lambda}) \text{ for } i = 1, 2. \qquad \ldots \quad (2.11)$$

Proof: Step 1 : For $0 \leqslant \delta \leqslant \log^2 n$

$$P_{\theta_0}[| V_{1n}(\delta)| < \log^3 n] \geqslant 1 - \exp[-\log^2 n(1+o(1))/2]. \qquad \ldots \quad (2.12)$$

To prove (2.12) first note that

$$P_{\theta_0}[V_{1n}(\delta) > \log^3 n]$$
$$\leqslant P_{\theta_0}[2\Sigma(x_i - \theta_0 - \delta n^{-1/2}) I(0 < x_i - \theta_0 \leqslant \delta n^{-1/2}) + \delta^2/2 > n^{-1/4}\log^3 n]$$
$$\leqslant \exp[\delta^2 h/4 - n^{-1/4}(\log^3 n)h/2] E^n \exp[Y_{1n} h]$$
$$\leqslant \exp[-\log^2 n(1+o(1))/2] \text{ by choosing } h = n^{1/4}\log^{-1} n.$$

Remaining part of (2.12) can be proved in a similar way.

Step 2 : For $0 \leqslant \delta \leqslant \log^2 n$

$$P_{\theta_0}\left[\sup_{\delta - n^{-1} < \delta' < \delta} | V_{1n}(\delta) - V_{1n}(\delta')| > \log^2 n \right] \leqslant \exp[-\log^2 n/2]. \qquad \ldots \quad (2.13)$$

To prove (2.13) first note that for $n^{-1} \leqslant \delta \leqslant \log^2 n$

$$\sup_{\delta - n^{-1} < \delta' < \delta} | V_{1n}(\delta) - V_{1n}(\delta') |$$

$$= \sup_{\delta - n^{-1} < \delta' < \delta} | 2\Sigma(x_i - \theta_0 - \delta n^{-1/2}) I(\delta' n^{-1/2} < x_i - \theta_0 \leqslant \delta n^{-1/2})$$

$$-(\delta-\delta')(\delta+\delta')/2 - 2(\delta-\delta')n^{-1/2}\Sigma I(0 < x_i - \theta_0 \leqslant \delta' n^{-1/2}) | n^{1/4}$$

$$\leqslant n^{1/4}[n^{-1/2}\log^2 n + 2\Sigma(\theta_0 + \delta n^{-1/2} - x_i) I((\delta - n^{-1})n^{-1/2} < x_i - \theta_0 \leqslant \delta n^{-1/2})].$$

Hence LHS of (2.13)

$$\leqslant \exp[n^{-1/2}(\log^2 n)h/2 - n^{-1/4}(\log^2 n)h/2] E^n \exp[Y_{2n} h]$$

$$\leqslant \exp[-\log^2 n/2] \text{ by choosing } h = n^{1/4}\log n.$$

For $0 \leqslant \delta \leqslant n^{-1}$ (2.13) can be proved in a similar way.

Step 3 : Proof of (2.11) for $i = 1$ is completed by combining steps 1 and 2; for $i = 2$ (2.11) can be proved analogously, completing the proof of the lemma.

The following lemma will be used to simplify the coefficient of $n^{-3/2}$ in $R(\rho)$. The proof of the lemma is straightforward and we will give only a sketch of it.

Lemma 2.5 : *For* $i = 1, 2$

$$E_{\theta_0} E[c(W_i^{(1)})^2 | t] = 2(f_1 + f_2 + f_3) + O(n^{-\eta})$$

$$E_{\theta_0} E[t^2 (W_i^{(0)})^2 | t] = 2f_1 + O(n^{-\eta})$$

$$E_{\theta_0} E[t W_i^{(0)} W_i^{(1)} | t] = -(2f_1 + f_3) + O(n^{-\eta})$$

$$n^{1/4} E_{\theta_0} E[t W_i^{(1)} | t] = \frac{1}{2} f_4 + \frac{1}{6} f_6 + O(n^{-\eta})$$

$$E[W_i^{(j)} | t] = n^{-1/4+\eta} + P(t) \quad \text{for } j = 0,$$

$$n^{1/4} E_{\theta_0} E[t^2 W_i^{(0)} | t] = \frac{1}{6} f_4 + \frac{1}{2} f_6 + O(n^{-\eta})$$

$$E_{\theta_0} E[t W_{2+i}^{(1)} | t] = \frac{2}{3} f_5 + O(n^{-\eta})$$

$$E_{\theta_0} E[t^2 W_{2+i}^{(0)} | t] = \frac{2}{3} f_4 + O(n^{-\eta})$$

where

$$f_1 = \int \int_t^\infty [\phi(\omega)-\omega(1-\Phi(\omega))]^2 t^2 \phi(t) d\omega dt$$

$$f_2 = \int \int_t^\infty (1-\Phi(\omega))^2 \phi(t) d\omega dt$$

$$f_3 = \int \int_t^\infty 2(1-\Phi(\omega))[\omega(1-\Phi(\omega))-\phi(\omega)] t\phi(t) d\omega dt$$

$$f_4 = \int \int_t^\infty (\omega-t)^2 t^2 \phi(\omega)\phi(t) d\omega dt$$

$$f_5 = \int \int_t^\infty t(\omega-t)^4 \phi(\omega)\phi(t) d\omega dt$$

$$f_6 = \int \int_t^\infty t^2(\omega-t)^3 \phi(\omega)\phi(t) d\omega dt.$$

Sketch of the proof: First note that for g_1, g_2 measeurable functions; $g_1 : R^- \to R$, $g_2 : R^+ \to R$ conditional joint distribution of

$$\sum_{i=1}^n g_1(x_i-\theta_0)I(x_i-\theta_0 \leqslant 0)$$

and

$$\sum_{i=1}^n g_2(x_i-\theta_0)I(x_i-\theta_0 > 0)$$

given t_0 is same as the joint distributions of

$$\sum_{i=1}^{t_0} g_1(y_i) \qquad \text{and} \qquad \sum_{i=1}^{n-t_0} g_2(z_i)$$

where y_i's and z_i's are independent of each other and of $X_1, \ldots, X_n$; y_i's are i.i.d. with p.d.f. e^y, $y < 0$ and z_i's are i.i.d. with p.d.f. e^{-z}, $z > 0$. Thus we have

$$E_{\theta_0}E[(W_1^{(1)})^2 | t] = E_{\theta_0}E\left\{ \sum_{i=1}^{n-t_0} (u_{in}-E(u_{1n}))+n^{-1/4+\epsilon}P(t)\right\}^2 \quad \ldots \quad (2.14)$$

where

$$u_{in} = 2\int I_n(\delta)\delta n^{1/4}(z_i-\delta n^{-1/2})I(z_i < \delta n^{-1/2})\phi(t+\delta)d\delta.$$

R.H.S. of (2.14) can be simplified and be found to be equal to $2(f_1+f_2+f_3)+O(n^{-\epsilon})$. Other relations of Lemma 2.5 are proved in a similar way.

3. Proof of the Theorem

Note that the likelihood ratio

$$\exp[\Lambda(\theta_0, \theta_0+\delta n^{-1/2})]$$

$$=\exp\Big[\sum_{i=1}^{n}|x_i-\theta_0|-\sum_{i=1}^{n}|x_i-\theta_0-\delta n^{-1/2}|\Big]$$

$$=\exp[-(\delta+t)^2/2+t^2/2+n^{-1/4}(V_{1n}(\delta)+V_{2n}(\delta))]$$

(vide Section 2.1 for definitions of t, $V_{1n}(\delta)$ and $V_{2n}(\delta)$).

Let

$$B_n(\theta_0)=E(\sqrt{n}(\theta-\theta_0)\,|\,\underline{x}_n)\text{ and for } i=0,1$$

$$N(i)=\int \delta^i I(-\sqrt{n}(\theta_0-a_0),\sqrt{n}(b_0-\theta_0))\phi(\delta+t).$$

$$\exp[n^{-1/4}(V_{1n}(\delta)+V_{2n}(\delta))+\log\rho(\theta_0+\delta n^{-1/2})-\log\rho(\theta_0)]d\delta.$$

Hence,

$$B_n(\theta_0)=N(1)/N(0). \qquad \dots (3.1)$$

Step 1: We first approximate $B_n(\theta_0)$. Let

$$R_n^{(i)}=\int_{\sqrt{n}(a_0-\theta_0)}^{\sqrt{n}(b_0-\theta_0)} I(|\delta|\geqslant \log^2 n)\delta^i\exp[n^{-1/4}(V_{1n}(\delta)+V_{2n}(\delta))$$

$$+\log\rho(\theta_0+\delta n^{-1/2})-\log\rho(\theta_0)]\phi(t+\delta)d\delta.$$

Then in view of (2.4) and definition of $I_n(\delta)$ (vide Section 2.1) we have for $i=0,1$

$$N(i)=\int I_n(\delta)\delta^i\exp[n^{-1/4}(V_{1n}(\delta)+V_{2n}(\delta))+\log\rho(\theta_0+\delta n^{-1/2})$$

$$-\log\rho(\theta_0)]\phi(t+\delta)d\delta+R_{1n}^{(i)}$$

$$=\int I_n(\delta)\delta^i\phi(t+\delta)\{1+n^{-1/4}(V_{1n}(\delta)+V_{2n}(\delta))$$

$$+n^{-1/2}(V_{1n}(\delta)+V_{2n}(\delta))^2/2+n^{-3/4}(V_{1n}(\delta)+V_{2n}(\delta))^3e^{x_{1n}}/3!\}$$

$$\cdot\{1+n^{-1/2}\delta(\rho)^{(1)}(\theta_0)+n^{-1}((\rho)^{(1)}(\theta_0))^2\delta^2e^{x_{2n}}/2\}$$

$$\cdot\{1+n^{-1}\delta^2(\rho)^{(2)}(\theta_1)e^{x_{3n}}/2\}d\delta+R_{1n}^{(i)}$$

where x_{1n} is between 0 and $n^{-1/4}(V_{1n}(\delta)+V_{2n}(\delta))$, x_{2n} is between O and $n^{-1/2}(\rho)^{(1)}(\theta_0)$, x_{3n} is between O and $n^{-1}\delta^2(\rho)^{(2)}(\theta_1)/2$ and θ_1 is between θ_0 and $\theta_0+\delta n^{-1/2}$ (vide (1.1) for definition of $(\rho)^{(i)}(\theta)$). Now let $R_{2n}^{(i)}$ be such that (vide Section (2.1) for definitions of $W_j^{(i)}$ and $W_{2+j}^{(i)}$)

$$N(i)=\int I_n(\delta)\delta^i\phi(t+\delta)d\delta$$

$$+n^{-1/4}(W_1^{(i)}+W_2^{(i)}+\frac{1}{2}n^{-1/2}(W_3^{(i)}+W_4^{(i)})$$

$$+n^{-1/2}(\rho)^{(1)}(\theta_0)\int I_n(\delta)\delta^{i+1}\phi(t+\delta)d\delta+R_{1n}^{(i)}+R_{2n}^{(i)}).$$

Let for $i = 0, 1$

$$a_{0n}^{(i)} = \int I_n(\delta)\delta^i\phi(t+\delta)d\delta$$

$$a_{1n}^{(i)} = (W_1^{(i)}+W_2^{(i)})$$

and

$$a_{2n}^{(i)} = \frac{1}{2}(W_3^{(i)}+W_4^{(i)})+(\rho)^{(1)}(\theta_0)\int I_n(\delta)\delta^{i+1}\phi(t+\delta)d\delta.$$

Note that on A_n (vide (2.3)) we have by Lemmas 2.1 to 2.4 and (2.1) for $i = 0, 1$ and $j = 1, 2$

$$a_{0n}^{(i)} = (-1)^i t^i + O(n^{-1/2-\eta}),$$

$$a_{2n}^{(i)} = \frac{1}{2}(W_3^{(i)}+W_4^{(i)})+(\rho)^{(1)}(\theta_0)[1+(i-1)(t+1)+it^{2i}+O(n^{-2})], \quad \ldots \quad (3.2)$$

$$a_{1n}^{(i)} = W_1^{(i)}+W_2^{(i)} \text{ and } |R_{jn}^{(i)}| = O(n^{-1/2-\eta}).$$

Absorbing $O(n^{-1/2-\eta})$ terms of $a_{0n}^{(i)}$ in $R_{2n}^{(i)}$ and writting

$$c_{1n} = a_{1n}^{(1)}+ta_{1n}^{(0)},$$

$$c_{2n} = a_{2n}^{(1)}-a_{1n}^{(0)}a_{1n}^{(1)}-t(a_{1n}^{(0)})^2+ta_{2n}^{(0)},$$

$$R_{3n} = \{R_{1n}^{(1)}+R_{2n}^{(1)}-(R_{1n}^{(0)}+R_{2n}^{(0)})(-t+n^{-1/4}c_{1n}+n^{-1/2}c_{2n})$$
$$-n^{-3/4}(a_{1n}^{(0)}c_{2n}+a_{2n}^{(0)}c_{1n})-n^{-1}a_{2n}^{(0)}c_{2n}\}$$
$$\times\{1+n^{-1/4}a_{1n}^{(0)}+n^{-1/2}a_{2n}^{(0)}+R_{1n}^{(0)}+R_{2n}^{(0)}\}^{-1}$$

and

$$B_n^*(\theta_0) = -t+c_{1n}n^{-1/4}+c_{2n}n^{-1/2} \quad \ldots \quad (3.3)$$

we have, in view of (3.1) and (3.2), on A_n

$$B_n(\theta_0) = B_n^*(\theta_0)+R_{3n}.$$

As $\theta_0 \in D_n^c$ (vide (2.1)) we have in view of (3.3) on A_n

$$B_n^2(\theta_0) = t^2-2tc_{1n}n^{-1/4}+(c_{1n}^2-2tc_{2n})n^{-1/2}+O(n^{-1/2-\eta}).$$

Step 2 : We can now approximate the Bayes risk. Note that by using arguments similar to those given to prove claim 2 (Section 9) of Ghosh, Sinha and Joshi (1982) (here we use Lemma 2.1 in place of their Lemma 4.3 and (2.2) in place of their (9.3)) we have

$$\int I(D_n)E_{\theta_0}(B_n^2(\theta_0))\rho(\theta_0)d\theta_0 = O(n^{-1/2-\eta}).$$

Hence

$$R(\rho) = n^{-1}\int_{1} E_{\theta_0}(B_n^2(\theta_0))\rho(\theta_0)d\theta_0$$

$$= n^{-1}\int I(D_n^c)E_{\theta_0}(B_n^2(\theta_0))I(A_n)\rho(\theta_0)d\theta_0 + O(n^{-3/2-\eta}) \text{ (vide 2.3))}$$

$$= n^{-1}\int I(D_n^c)E_{\theta_0}[\{l^2+n^{-1/2}(-2lc_{2n}+c_{1n}^2-2lc_{1n}n^{1/4})\}$$

$$\cdot I(A_n)]\rho(\theta_0)d\theta_0 + O(n^{-3/2-\eta}).$$

Now RHS of the above expression can be simplified (by first expressing c_{i_n}'s in terms of $W_j^{(i)}$'s (vide (3.3) and (3.2)) and then using Lemma 2.5) and found to be equal to RHS of (1.2).

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