QUANTUM MARKOV PROCESSES WITH A CHRISTENSEN–EVANS GENERATOR IN A VON NEUMANN ALGEBRA

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1. *Introduction*

Let $\mathscr{A}$ be a unital von Neumann algebra of operators on a complex separable Hilbert space $\mathscr{H}_0$, and let $\{T_t, t \geqslant 0\}$ be a uniformly continuous quantum dynamical semigroup of completely positive unital maps on $\mathscr{A}$. The infinitesimal generator $\mathscr{L}$ of $\{T_t\}$ is a bounded linear operator on the Banach space $\mathscr{A}$. For any Hilbert space $\mathscr{K}$, denote by $\mathscr{B}(\mathscr{K})$ the von Neumann algebra of all bounded operators on $\mathscr{K}$. Christensen and Evans [**3**] have shown that $\mathscr{L}$ has the form

$$\mathscr{L}(X) = R^*\pi(X)R + K_0^* X + XK_0, \quad X \in \mathscr{A}, \tag{1.1}$$

where π is a representation of $\mathscr{A}$ in $\mathscr{B}(\mathscr{K})$ for some Hilbert space $\mathscr{K}$, $R: \mathscr{H}_0 \to \mathscr{K}$ is a bounded operator satisfying the 'minimality' condition that the set $\{(RX - \pi(X)R)u, u \in \mathscr{H}_0, X \in \mathscr{A}\}$ is total in $\mathscr{K}$, and K_0 is a fixed element of $\mathscr{A}$. The unitality of $\{T_t\}$ implies that $\mathscr{L}(1) = 0$, and consequently $K_0 = iH - \frac{1}{2}R^*R$, where H is a hermitian element of $\mathscr{A}$. Thus (1.1) can be expressed as

$$\mathscr{L}(X) = i[H, X] - \tfrac{1}{2}(R^*RX + XR^*R - 2R^*\pi(X)R), \quad X \in \mathscr{A}. \tag{1.2}$$

We say that the quadruple $(\mathscr{K}, \pi, R, H)$ constitutes the set of Christensen–Evans (CE) parameters which determine the CE generator $\mathscr{L}$ of the semigroup $\{T_t\}$. It is quite possible that another set $(\mathscr{K}', \pi', R', H')$ of CE parameters may determine the same generator $\mathscr{L}$. In such a case, we say that these two sets of CE parameters are *equivalent*. In Section 2 we study this equivalence relation in some detail.

It is known from [**1, 2**] that, corresponding to the quantum dynamical semigroup $\{T_t\}$, there exists, up to unitary equivalence, a unique minimal Markov flow $(\mathscr{H}, F_t, j_t)$, $t \geqslant 0$, satisfying the following properties. (1) $\mathscr{H}$ is a Hilbert space containing $\mathscr{H}_0$ as a subspace. (2) $\{F_t\}$ is an increasing family of projections in $\mathscr{H}$, increasing to 1 (the identity projection) in $\mathscr{H}$ as $t \to \infty$, and F_0 is the projection on $\mathscr{H}_0$. (3) j_t is a $*$ homomorphism from $\mathscr{A}$ into $\mathscr{B}(\mathscr{H})$ such that $j_0(X) = XF_0$, $j_t(1) = F_t$, $F_s j_t(X) F_s = j_s(T_{t-s}(X))$ for all $s \leqslant t$, and the map $t \to j_t(X)$ is strongly continuous for each X in $\mathscr{A}$. (4) The set

$$\{j_{t_1}(X_1) j_{t_2}(X_2) \cdots j_{t_n}(X_n)u,\ u \in \mathscr{H}_0,\ t_1 > t_2 > \cdots > t_n \geqslant 0,\ n = 1, 2, \ldots,\ X_j \in \mathscr{A}\}$$

is total in $\mathscr{H}$.

If we drop condition (4) in the preceding paragraph, then we say that $(\mathscr{H}, F_t, j_t)$ is a *Markov dilation* for the semigroup $\{T_t\}$ or, equivalently, the generator $\mathscr{L}$. In [**1, 2**], the construction of the minimal dilation was achieved on the basis of a full knowledge

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of the semigroup $\{T_t\}$ and an application of the GNS principle. However, it would be desirable to construct Markov dilations starting from $\mathscr{L}$ or some parameters (like the CE parameters) determining $\mathscr{L}$. In the simplest case, when $\mathscr{A} = \mathscr{B}(\mathscr{H}_0)$, the CE generator assumes the Lindblad form [**8**]:

$$\mathscr{L}(X) = i[H, X] - \tfrac{1}{2}\sum_j (L_j^* L_j X + X L_j^* L_j - 2L_j^* X L_j),$$

where $H, L_j \in \mathscr{B}(\mathscr{H}_0)$, H is hermitian, and $\sum_j L_j^* L_j$ is a finite or strongly convergent countable sum. From the methods of quantum stochastic calculus [**6**, **9**, **11**], it is known how to construct Markov dilations of $\mathscr{L}$ by solving quantum stochastic differential equations (qsde) involving H and the L_j in its 'diffusion' coefficients [**6**, **10**, **11**]. However, even in this case, there does not seem to exist a procedure for constructing the minimal dilation starting from the parameters H, L_j. In Section 3 of this paper we start from the CE parameters in (1.2), and construct a Markov dilation for $\mathscr{L}$. The Markov process thus obtained turns out to be a Poisson imbedding of a discrete time quantum Markov chain, but looked at in an 'interaction' picture. The idea of an interaction picture of a quantum diffusion goes back to [**4**], [**5**] and [**7**].

The Markov dilation presented here depends very much on the parameters $(\mathscr{K}, \pi, R, H)$ which determine $\mathscr{L}$ through (1.2). It should be interesting to explore the connection between the dilations determined by different parametrizations for the same generator $\mathscr{L}$.

2. *An equivalence relation for the Christensen–Evans parameters*

Let $\mathscr{H}_0, \mathscr{A}, \mathscr{L}$ be as in Section 1, and let $(\mathscr{K}_j, \pi_j, R_j, H_j)$, $j = 1, 2$, be two quadruples determining the same CE generator $\mathscr{L}$ via (1.2), so that $H_j, R_j^* R_j \in \mathscr{A}$, and

$$\mathscr{L}(X) = i[H_j, X] - \tfrac{1}{2}(R_j^* R_j X + X R_j^* R_j - 2R_j^* \pi(X) R_j), \quad X \in \mathscr{A}, j = 1, 2. \tag{2.1}$$

Denote by $\mathscr{A}'$ the commutant of $\mathscr{A}$ in $\mathscr{B}(\mathscr{H}_0)$.

PROPOSITION 2.1. *There exists a unitary isomorphism $\Gamma: \mathscr{K}_1 \to \mathscr{K}_2$ such that, for all $X \in \mathscr{A}$, the following hold*:
(1) $\Gamma \pi_1(X) = \pi_2(X) \Gamma$;
(2) $(\Gamma^* R_2 - R_1) X = \pi_1(X)(\Gamma^* R_2 - R_1)$.

Proof. Let

$$\delta_j(X) = R_j X - \pi_j(X) R_j, \quad X \in \mathscr{A}, j = 1, 2. \tag{2.2}$$

By elementary algebra, we have

$$\delta_j(X)^* \delta_j(Y) = \mathscr{L}(X^* Y) - X^* \mathscr{L}(Y) - \mathscr{L}(X^*) Y, \quad X, Y \in \mathscr{A}, j = 1, 2, \tag{2.3}$$

where $\mathscr{L}$ satisfies (2.1). By the definition of the CE parameters, the set $\{\delta_j(X)u, u \in \mathscr{H}_0, X \in \mathscr{A}\}$ is total in $\mathscr{K}_j$. Hence (2.3) implies that the correspondence $\delta_1(X)u \to \delta_2(X)u$ is scalar product preserving, and there exists a unique unitary isomorphism $\Gamma: \mathscr{K}_1 \to \mathscr{K}_2$ satisfying

$$\Gamma \delta_1(X) = \delta_2(X), \quad X \in \mathscr{A}. \tag{2.4}$$

Replacing X by XY and using the relation $\delta_j(XY) = \delta_j(X) Y + \pi_j(X) \delta_j(Y)$ for all X, Y in $\mathscr{A}$, we obtain from (2.4) the relation $\Gamma \pi_1(X) \delta_1(Y) = \pi_2(X) \Gamma \delta_1(Y)$, which proves property (1) of the proposition.

Substituting for δ_1, δ_2 in (2.4) from (2.2), and using property (1), we obtain property (2).

PROPOSITION 2.2. *Let* Γ *be as in Proposition* 2.1. *Then there exist* $C \in \mathscr{A}$, $D \in \mathscr{A}'$, $Z \in \mathscr{A} \cap \mathscr{A}'$ *such that*:

(1) $R_2^* \Gamma R_1 = C + D$;

(2) $H_2 - H_1 = \frac{1}{2} i(C^* - C) + Z$.

Proof. Write $L = \Gamma^* R_2 - R_1$. From the remarks at the beginning of this section, we know that $R_j^* \pi_j(X) R_j \in \mathscr{A}$, $j = 1, 2$, for all X in $\mathscr{A}$. We have, from Proposition 2.1,

$$(\Gamma(R_1 + L))^* \pi_2(X) \Gamma(R_1 + L) = R_1^* \pi_1(X) R_1 + L^* L X + R_1^* L X + X L^* R_1,$$

so

$$L^* L X + R_1^* L X + X L^* R_1 \in \mathscr{A} \quad \text{for all } X \in \mathscr{A}. \tag{2.5}$$

From (2.1) and Proposition 2.1, we also have

$$\begin{aligned} & i[H_1, X] - \tfrac{1}{2}(R_1^* R_1 X + X R_1^* R_1 - 2 R_1^* \pi_1(X) R_1) \\ &= i[H_2, X] - \tfrac{1}{2}((R_1 + L)^* (R_1 + L) X + X (R_1 + L)^* (R_1 + L) - 2 (R_1 + L)^* \pi_1(X) (R_1 + L)), \end{aligned}$$

which simplifies to

$$i[H_1 - H_2, X] = \tfrac{1}{2}[R_1^* L - L^* R_1, X], \quad X \in \mathscr{A}.$$

Since every derivation of $\mathscr{A}$ is inner and $H_1 - H_2 \in \mathscr{A}$, it follows that

$$H_2 = H_1 + \tfrac{1}{2} i (R_1^* L - L^* R_1) + B, \tag{2.6}$$

where $B = B^* \in \mathscr{A}'$.

Substituting for L in (2.5), we conclude that $[R_2^* \Gamma R_1, X] \in \mathscr{A}$, and hence, by the same argument as above, $R_2^* \Gamma R_1$ can be expressed as

$$R_2^* \Gamma R_1 = C + D, \quad C \in \mathscr{A},\ D \in \mathscr{A}'. \tag{2.7}$$

Substituting for L in (2.6), we conclude that

$$H_2 - H_1 - \tfrac{1}{2} i \{R_1^* (\Gamma^* R_2 - R_1) - (R_2^* \Gamma - R_1^*) R_1\} \in \mathscr{A}'.$$

Now (2.7) implies that $H_2 - H_1 - \frac{1}{2} i (C^* - C) \in \mathscr{A} \cap \mathscr{A}'$, which together with (2.7) completes the proof.

THEOREM 2.3. *Two CE quadruples* $(\mathscr{K}_j, \pi_j, R_j, H_j)$, $j = 1, 2$, *determine the same CE generator* $\mathscr{L}$ *if and only if there exist a unitary isomorphism* $\Gamma : \mathscr{K}_1 \to \mathscr{K}_2$, *and elements* $C \in \mathscr{A}$, $D \in \mathscr{A}'$, $Z = Z^* \in \mathscr{A} \cap \mathscr{A}'$ *such that*:

(1) $\Gamma \pi_1(X) = \pi_2(X) \Gamma$;

(2) $(\Gamma^* R_2 - R_1) X = \pi_1(X) (\Gamma^* R_2 - R_1)$;

(3) $R_2^* \Gamma R_1 = C + D$;

(4) $H_2 - H_1 = \frac{1}{2} i (C^* - C) + Z$.

Proof. Propositions 2.1 and 2.2 imply the 'only if' part. To prove the converse, consider Γ, C, D, Z satisfying conditions (1)–(4), and the CE generators $\mathscr{L}_j$ defined by

$$\mathscr{L}_j(X) = i[H_j, X] - \tfrac{1}{2}(R_j^* R_j X + X R_j^* R_j - 2 R_j^* \pi_j(X) R_j), \quad X \in \mathscr{A},\ j = 1, 2.$$

Write $L = \Gamma^* R_2 - R_1$, so that $LX = \pi_1(X) L$ and $R_2 = \Gamma(R_1 + L)$. Then, substituting for H_2, R_2 and π_2 from (1)–(4) in $\mathscr{L}_2(X)$, we obtain

$$\begin{aligned}\mathscr{L}_2(X) &= i[H_1, X]-\tfrac{1}{2}[C^*-C, X] \\ &\quad -\tfrac{1}{2}\{(R_1+L)^*(R_1+L)X+X(R_1+L)^*(R_1+L)-2(R_1+L)^*\pi_1(X)(R_1+L)\} \\ &= \mathscr{L}_1(X)-\tfrac{1}{2}[C^*-C-R_1^*L+L^*R_1, X] \\ &= \mathscr{L}_1(X)-\tfrac{1}{2}[C^*-C-R_1^*\Gamma^*R_2+R_2^*\Gamma R_1, X] \\ &= \mathscr{L}_1(X)\end{aligned}$$

for all $X \in \mathscr{A}$.

For constructing Markov dilations, it is useful to modify the CE parametrization. To this end, we prove the following result.

THEOREM 2.4. *Let $\mathscr{L}$ be the generator of a conservative and uniformly continuous quantum dynamical semigroup on a von Neumann algebra $\mathscr{A} \subset \mathscr{B}(\mathscr{H}_0)$. Then there exist a unital completely positive map $\Psi: \mathscr{A} \to \mathscr{A}$, a positive element $K \in \mathscr{A}$, and a hermitian element $H \in \mathscr{A}$ such that*

$$\mathscr{L}(X) = i[H, X]-\tfrac{1}{2}(K^2X+XK^2-2K\Psi(X)K), \quad X \in \mathscr{A}. \tag{2.8}$$

Proof. In (1.2), put $K = (R^*R)^{1/2}$ and consider the polar decomposition $R = VK$, where V is an isometry from the closure of the range of K in $\mathscr{H}_0$ onto the closure of the range of R in $\mathscr{K}$. Denoting by P the projection on the closure of the range of K in $\mathscr{H}_0$, we see that

$$R^*\pi(X)R = KPV^*\pi(X)VPK = K\Psi_0(X)K,$$

where

$$\Psi_0(X) = PV^*\pi(X)VP.$$

Clearly, Ψ_0 is a contractive completely positive map satisfying $\Psi_0(1) = P$. Now $\mathscr{L}$ can be expressed as

$$\mathscr{L}(X) = i[H, X]-\tfrac{1}{2}(K^2X+XK^2-2K\Psi_0(X)K), \quad X \in \mathscr{A}. \tag{2.9}$$

Since $\mathscr{L}(X), H, K \in \mathscr{A}$, it follows that $K\Psi_0(X)K \in \mathscr{A}$ for all X in $\mathscr{A}$. Hence $K^m\Psi_0(X)K^n \in \mathscr{A}$ for $m, n \geqslant 1$. Thus for any two polynomials p, q such that $p(0) = q(0) = 0$, it follows that $p(K)\Psi_0(X)q(K) \in \mathscr{A}$. Hence for any two continuous functions φ, ψ on $[0, \infty)$ satisfying $\varphi(0) = \psi(0) = 0$, we have $\varphi(K)\Psi_0(X)\psi(K) \in \mathscr{A}$. Define

$$\varphi_n(x) = \begin{cases} nx & \text{if } 0 \leqslant x < 1/n, \\ 1 & \text{if } x \geqslant 1/n, \end{cases}$$

and observe that

$$w\cdot\lim_{n\to\infty} \varphi_n(K)\Psi_0(X)\varphi_n(K) = P\Psi_0(X)P = \Psi_0(X) \in \mathscr{A}.$$

Define

$$\Psi(X) = \Psi_0(X)+(1-P)X(1-P).$$

Then Ψ is a unital completely positive map from $\mathscr{A}$ into itself, and $\mathscr{L}$ assumes the form (2.8).

REMARK. Our construction of a Markov dilation for $\mathscr{L}$ in the next section depends on the discrete time quantum Markov chain defined by the unital completely positive map Ψ on $\mathscr{A}$. It should be interesting to know the exact relationship between the parameter triples (H, K, Ψ) and (H', K', Ψ') which determine the same $\mathscr{L}$ according to (2.8) in Theorem 2.4.

3. *A Markov dilation for the semigroup* $e^{t\mathscr{L}}$

We consider a CE generator $\mathscr{L}$ expressed in the form (2.8) of Theorem 2.4 in terms of the parameters H, K, Ψ. Since Ψ is a unital completely positive map on $\mathscr{A}$, it follows from [**1**, **2**] that there exists a unique (up to unitary equivalence) minimal discrete time Markov dilation $(\mathscr{H}, F_n, j_n)$, $n = 0, 1, 2, \ldots$, where $\mathscr{H}$ is a Hilbert space containing $\mathscr{H}_0$ as a subspace, $\{F_n\}$ is an increasing sequence of projections in $\mathscr{H}$, F_0 is the projection on $\mathscr{H}_0$, $s\cdot\lim_{n\to\infty} F_n = 1$,

$$F_m j_n(X) F_m = j_m(\Psi^{n-m}(X)), \quad X \in \mathscr{A},\ 0 \leqslant m \leqslant n < \infty,$$
$$j_0(X) = XF_0$$

and $\{j_n(X_n) j_{n-1}(X_{n-1}) \cdots j_0(X_0)u,\ X_i \in \mathscr{A},\ n = 0, 1, 2, \ldots,\ u \in \mathscr{H}_0\}$ is total in $\mathscr{H}$.

Our strategy for constructing the dilation for $\mathscr{L}$ will be to imbed $(\mathscr{H}, F_n, j_n)$ in a quantum version of the Poisson process and look at it in an appropriate interaction picture. To this end, we introduce the boson Fock space $\Gamma(L^2(\mathbb{R}_+))$, and consider the Poisson process $\{N(t)\}$, where $N(t)$ is a selfadjoint operator realized as the closure of $A^\dagger(t) + \Lambda(t) + A(t) + t$ on the domain of exponential vectors, $A^\dagger$, Λ, A being the creation, conservation and annihilation processes of quantum stochastic calculus. We write (forgoing rigour) $N(t) = A^\dagger(t) + \Lambda(t) + A(t) + t$, with the convention that 1 denotes the identity operator, and a scalar times the identity operator is denoted by the scalar itself. We now make the Poisson imbedding of the discrete time chain by putting $\tilde{\mathscr{H}} = \mathscr{H} \otimes \Gamma(L^2(\mathbb{R}_+))$ and defining

$$j_{N(t)}(X) := \sum_{n=0}^{\infty} j_n(X) \otimes 1_{\{n\}}(N(t)),$$

where $1_{\{n\}}$ denotes the indicator of the singleton $\{n\}$ in $\mathbb{R}$. We have used the fact that $N(t)$ has spectrum $\{0, 1, 2, \ldots\}$ for $t > 0$, and $N(0) = 0$.

Proposition 3.1. *Let* $F_{N(t)} = j_{N(t)}(1)$. *Then*:
(i) $F_{N(0)} = F_0 \otimes 1_{\Gamma(L^2(\mathbb{R}_+))}$;
(ii) $F_{N(s)} \leqslant F_{N(t)}$ *for all* $0 \leqslant s \leqslant t < \infty$;
(iii) $s\cdot\lim_{t\to\infty} F_{N(t)} = 1_{\tilde{\mathscr{H}}}$.

Proof. (i) is obvious since $N(0) = 0$. To prove (ii), we first observe that $N(t) = N(s) + N(t) - N(s)$, where $N(s)$ and $N(t) - N(s)$ are ampliations of operators in $\Gamma(L^2[0, s])$ and $\Gamma(L^2[s, t])$, respectively, in the factorization

$$\Gamma(L^2(\mathbb{R}_+)) = \Gamma(L^2[0, s]) \otimes \Gamma(L^2[s, t]) \otimes \Gamma(L^2[t, \infty)).$$

Thus

$$\begin{aligned} F_{N(t)} &= \sum_{n=0}^{\infty} F_n \otimes 1_{\{n\}}(N(t)) \\ &= \sum_{n=0}^{\infty} F_n \otimes \sum_{j=0}^{n} 1_{\{j\}}(N(s)) \otimes 1_{\{n-j\}}(N(t) - N(s)) \\ &= \sum_{j\geqslant 0,\, k\geqslant 0} F_{j+k} \otimes 1_{\{j\}}(N(s)) \otimes 1_{\{k\}}(N(t) - N(s)) \\ &\geqslant \sum_{j\geqslant 0,\, k\geqslant 0} F_j \otimes 1_{\{j\}}(N(s)) \otimes 1_{\{k\}}(N(t) - N(s)) \\ &= F_{N(s)}. \end{aligned}$$

This proves (ii). Finally,

$$F_{N(t)} = \sum_{n=0}^{\infty} F_n \otimes (1_{\{n,n+1,\dots\}}(N(t)) - 1_{\{n+1,n+2,\dots\}}(N(t)))$$
$$= \sum_{n=0}^{\infty} (F_n - F_{n-1}) \otimes (1 - 1_{\{0,1,2,\dots,n-1\}}(N(t))).$$

By the isomorphism [**11**] between $\Gamma(L^2(\mathbb{R}_+))$ and the L^2 space with respect to the probability measure of the Poisson process of unit intensity, and the fact that $N(t)$ viewed as a Poisson random variable tends to ∞ with probability 1 as $t \to \infty$, it follows that

$$s\cdot\lim_{t\to\infty} F_{N(t)} = \sum_{n=0}^{\infty} (F_n - F_{n-1}) \otimes 1_{\Gamma(L^2(\mathbb{R}_+))} = 1_{\tilde{\mathscr{H}}}.$$

In the von Neumann algebra $\mathscr{B}(\tilde{\mathscr{H}})$, we consider the Fock vacuum conditional expectation $\mathbb{E}_{t]}$ which is defined as follows. For any $X \in \mathscr{B}(\tilde{\mathscr{H}})$, consider the operator X_t on $\mathscr{H} \otimes \Gamma(L^2[0,t])$ defined by $\langle \varphi, X_t \psi \rangle = \langle \varphi \otimes \Omega_{[t}, X\psi \otimes \Omega_{[t} \rangle$, where $\Omega_{[t}$ is the Fock vacuum vector in $\Gamma(L^2[t,\infty))$, and put $\mathbb{E}_{t]} X = X_t \otimes 1_{[t}$, where $1_{[t}$ is the identity operator in $\Gamma(L^2[t,\infty))$.

PROPOSITION 3.2. *Let $F_{N(t)}, j_{N(t)}$ be as in Proposition* 3.1. *Then*

$$\mathbb{E}_{s]} F_{N(s)} j_{N(t)}(X) F_{N(s)} = j_{N(s)}(S_{t-s}(X)), \quad 0 \leqslant s \leqslant t < \infty,\ X \in \mathscr{A},$$

where

$$S_t(X) = e^{t(\Psi - \mathrm{id})}(X), \quad X \in \mathscr{A},$$

id *being the identity map on $\mathscr{A}$.*

Proof. We have, from properties of the Poisson process $\{N(t)\}$,

$$F_{N(s)} j_{N(t)}(X) F_{N(s)} = \sum_{n \geqslant 0} F_n \otimes 1_{\{n\}}(N(s)) \sum_{n \geqslant 0} j_n(X) \otimes 1_{\{n\}}(N(t)) \sum_{n \geqslant 0} F_n \otimes 1_{\{n\}}(N(s))$$
$$= \sum_{k,n \geqslant 0} F_k j_n(X) F_k \otimes 1_{\{k\}}(N(s)) 1_{\{n\}}(N(t))$$
$$= \sum_{n \geqslant k \geqslant 0} F_k j_n(X) F_k \otimes 1_{\{k\}}(N(s)) 1_{\{n-k\}}(N(t) - N(s))$$
$$= \sum_{k \geqslant 0,\, n-k \geqslant 0} j_k(\Psi^{n-k}(X)) 1_{\{k\}}(N(s)) 1_{\{n-k\}}(N(t) - N(s)).$$

Now, applying $\mathbb{E}_{s]}$ on both sides,

$$\mathbb{E}_{s]} F_{N(s)} j_{N(t)}(X) F_{N(s)} = \sum_{k \geqslant 0,\, \ell \geqslant 0} j_k(\Psi^{\ell}(X)) 1_{\{k\}}(N(s)) e^{-(t-s)} \frac{(t-s)^{\ell}}{\ell\,!}$$
$$= j_{N(s)}(e^{(t-s)(\Psi - \mathrm{id})}(X)).$$

COROLLARY 3.3. *Let*

$$\tilde{j}_t(X) = j_{N(t)}(X) \otimes |\Omega_{[t} >< \Omega_{[t}|,$$
$$\tilde{F}_t = \tilde{j}_t(1) = F_{N(t)} \otimes |\Omega_{[t} >< \Omega_{[t}|.$$

Then $(\tilde{\mathscr{H}}, \tilde{F}_t, \tilde{j}_t)$, $t \geqslant 0$, is a Markov dilation for the conservative quantum dynamical semigroup $\{e^{t(\Psi - \mathrm{id})}\}$, $t \geqslant 0$.

Proof. Immediate.

PROPOSITION 3.4. *Let H, K be hermitian elements in $\mathscr{A}$. Then the quantum stochastic differential equation*

$$dW(t) = \{j_{N(t)}(H)(dA^{\dagger} - dA) + j_{N(t)}(-iK - \tfrac{1}{2}H^2)dt\}W(t) \tag{3.1}$$

with $W(0) = 1$ admits a unique isometric solution $W(t)$.

Proof. The proof is along the same lines as in Section 4 of [**4**]. Write $W_0(t) \equiv 1$, and define iteratively

$$W_n(t) = 1 + \int_0^t \{j_{N(s)}(H)(dA^{\dagger} - dA) + j_{N(s)}(-iK - \tfrac{1}{2}H^2)ds\}W_{n-1}(s).$$

By the inequality (ii) of Proposition 27.1, page 222 of [**11**], we conclude that

$$\sum_n \|(W_n(t) - W_{n-1}(t))fe(u)\| < \infty$$

for all $f \in \mathscr{H}$ and exponential vectors $e(u)$ in $\Gamma(L^2(\mathbb{R}_+))$. This implies the convergence of $W_n(t)fe(u)$ in $\tilde{\mathscr{H}}$ as $n \to \infty$. Denoting this limit by $W(t)fe(u)$, we obtain a solution of (3.1). A routine application of quantum Ito's formula implies the isometric property of $W(t)$. Uniqueness follows from the fact that any solution of (3.1) with initial value 0 is identically 0.

PROPOSITION 3.5. *Let*

$$j_{N(t)+k}(X) = \sum_{n=0}^{\infty} j_{n+k}(X) \otimes 1_{\{n\}}(N(t)), \quad k \geqslant 0.$$

Then

$$dj_{N(t)+k}(X) = (j_{N(t)+k+1}(X) - j_{N(t)+k}(X))dN(t).$$

Proof. We have

$$\begin{aligned} dj_{N(t)+k}(X) &= \left\{\sum_{n=0}^{\infty} j_{n+k}(X) \otimes (1_{\{n\}}(N(t)+1) - 1_{\{n\}}(N(t)))\right\}dN(t) \\ &= \left\{\sum_{n=1}^{\infty} j_{n+k}(X) \otimes 1_{\{n-1\}}(N(t)) - j_{N(t)+k}(X)\right\}dN(t) \\ &= (j_{N(t)+k+1}(X) - j_{N(t)+k}(X))dN(t). \end{aligned}$$

PROPOSITION 3.6. *The isometric process $\{W(t)\}$ of Proposition 3.4 is unitary.*

Proof. Let $X(t) = 1 - W(t)W(t)^*$. Then $\{X(t)\}$ is a projection-valued Fock adapted process with initial value 0. The proposition will be proved if we show that $dX(t) = 0$. By a routine application of quantum Ito's formula and some algebra, we obtain

$$\begin{aligned} dX(t) = {} & [j_{N(t)}(H), X(t)](dA^{\dagger} - dA)(t) \\ & - [\{j_{N(t)}(iK), X(t)] + \tfrac{1}{2}[j_{N(t)}(H), [j_{N(t)}(H), X(t)]]\}dt. \end{aligned} \tag{3.2}$$

Define $P_n(t) = 1_{\{n\}}(N(t))$, and observe that

$$dP_0(t) = -P_0(t)dN(t),$$
$$dP_n(t) = (P_{n-1}(t) - P_n(t))dN(t) \quad \text{if } n \geqslant 1.$$

This, together with (3.2), quantum Ito's formula and some tedious algebra, implies

$$\begin{aligned} dP_n XP_n(t) = {} & (P_{n-1}XP_{n-1} - P_n XP_n)(t)dN(t) \\ & + P_{n-1}(t)[j_{N(t)}(H), X(t)]P_n(t)dA^\dagger(t) + P_n(t)[X(t), j_{N(t)}(H)]P_{n-1}(t)dA(t) \\ & + \{P_{n-1}(t)[j_{N(t)}(H), X(t)]P_n(t) + P_n(t)[X(t), j_{N(t)}(H)]P_{n-1}(t) \\ & - P_n(t)([j_{N(t)}(iK), X(t)] + \tfrac{1}{2}[j_{N(t)}(H), [j_{N(t)}(H), X(t)]])P_n(t)\}dt. \end{aligned} \tag{3.3}$$

Note that operators and their ampliations to tensor products have been denoted by the same symbols. Since $P_k(t)$ and $j_{N(t)}(B)$ commute with each other, and $P_k(t)j_{N(t)}(B) = j_k(B)P_k(t) = P_k(t)j_k(B)$ for any B in $\mathscr{A}$, (3.3) can be expressed as

$$\begin{aligned} dP_n XP_n = {} & (P_{n-1}XP_{n-1} - P_n XP_n)dN \\ & + (j_{n-1}(H)P_{n-1}XP_n - P_{n-1}XP_n j_n(H))dA^\dagger \\ & + (P_n XP_{n-1}j_{n-1}(H) - j_n(H)P_n XP_{n-1})dA \\ & + \{j_{n-1}(H)P_{n-1}XP_n - P_{n-1}XP_n j_n(H) + P_n XP_{n-1}j_{n-1}(H) \\ & \quad - j_n(H)P_n XP_{n-1} + [j_n(-iK), P_n XP_n] \\ & \quad + \tfrac{1}{2}[j_n(H), [j_n(H), P_n XP_n]]\}dt. \end{aligned} \tag{3.4}$$

Putting $n = 0$, we obtain

$$dP_0 XP_0 = -P_0 XP_0 dN + \{[j_0(-iK), P_0 XP_0] - \tfrac{1}{2}[j_0(H), [j_0(H), P_0 XP_0]]\}dt.$$

This is a constant operator coefficient quantum stochastic differential equation (qsde) for $P_0 XP_0$ with initial value 0. Hence $(P_0 XP_0)(t) = 0$. Since $X(t)$ and $P_0(t)$ are projections, we conclude that $P_0(t)X(t) = X(t)P_0(t) = 0$. Let us now make the induction hypothesis that $P_{n-1}(t)X(t) = X(t)P_{n-1}(t) = 0$. Then (3.4) becomes

$$dP_n XP_n = -P_n XP_n dN + \{[j_n(-iK), P_n XP_n] + \tfrac{1}{2}[j_n(H), [j_n(H), P_n XP_n]]\}dt,$$

which is once again a constant operator coefficient qsde for $P_n XP_n$ with initial value 0. Hence $(P_n XP_n)(t) = 0$, which implies that $P_n(t)X(t) = X(t)P_n(t) = 0$. Thus $X(t)P_n(t) = 0$ for every $n \geqslant 0$. Since $\sum_{n \geqslant 0} P_n(t) = 1$, we conclude that $X(t) \equiv 0$.

PROPOSITION 3.7. *Let $\{W(t)\}$ be the unique unitary solution of the equation* (3.1) *in Proposition* 3.4. *Then, for any $X \in \mathscr{A}$,*

$$\begin{aligned} & dW(t)^* j_{N(t)}(X)W(t) \\ & = W(t)^*\{(j_{N(t)+1}(X) - j_{N(t)}(X))dN(t) + (j_{N(t)+1}(X)j_{N(t)}(H) - j_{N(t)}(HX))dA^\dagger(t) \\ & \quad + (j_{N(t)}(H)j_{N(t)+1}(X) - j_{N(t)}(XH))dA(t) \\ & \quad + (j_{N(t)}(H\Psi(X)H - \tfrac{1}{2}(H^2X + XH^2) - HX - XH + i[K, X]) \\ & \qquad + j_{N(t)+1}(X)j_{N(t)}(H) + j_{N(t)}(H)j_{N(t)+1}(X))dt\}W(t). \end{aligned} \tag{3.5}$$

Proof. This is immediate from Proposition 3.5 for the case $k = 0$, equation (3.1), quantum Ito's formula, and the fact that

$$\begin{aligned} j_{N(t)}(H)j_{N(t)+1}(X)j_{N(t)}(H) &= j_{N(t)}(H)F_{N(t)}j_{N(t)+1}(X)F_{N(t)}j_{N(t)}(H) \\ &= j_{N(t)}(H\Psi(X)H). \end{aligned}$$

PROPOSITION 3.8. *Let $W(t)$ be as in Proposition* 3.7. *Then*

$$F_{N(t)}\,W(t) = W(t)F_{N(t)}.$$

Proof. Put $X=1$ in Proposition 3.7. Since $\Psi(1)=1$ and $F_{N(t)+1} \geqslant F_{N(t)}$, we have, from (3.5),

$$W(t)^* F_{N(t)}\,W(t) = F_0 + \int_0^t W(s)^*(F_{N(s)+1} - F_{N(s)})W(s)\,dN(s). \tag{3.6}$$

On the other hand, the differential equation for W implies

$$\begin{aligned} W(t) &= 1 + \int_0^t \{j_{N(s)}(H)\,(dA^\dagger - dA)\,(s) + j_{N(s)}(-iK - \tfrac{1}{2}H^2)ds\}W(s) \\ &= 1 + F_{N(t)}\int_0^t \{j_{N(s)}(H)\,(dA^\dagger - dA)\,(s) + j_{N(s)}(-iK - \tfrac{1}{2}H^2)ds\}W(s) \\ &= 1 + F_{N(t)}(W(t)-1), \end{aligned}$$

or $W(t) = 1 - F_{N(t)} + F_{N(t)}\,W(t)$. Substituting this in the right-hand side of (3.6), we have

$$\begin{aligned} W(t)^* F_{N(t)}\,W(t) &= F_0 + \int_0^t (F_{N(s)+1} - F_{N(s)})\,dN(s) \\ &= F_{N(t)}, \end{aligned}$$

by Proposition 3.5.

PROPOSITION 3.9. *Let $\{W(t)\}$ be as in Proposition* 3.7. *Then*

$$F_{N(s)}\,\mathbb{E}_{s]}(W(t)^* j_{N(t)}(X)W(t))F_{N(s)} = W(s)^* j_{N(s)}(e^{(t-s)\mathscr{M}}(X))W(s)$$

for all $X \in \mathscr{A}$, $0 \leqslant s \leqslant t < \infty$, where

$$\mathscr{M}(X) = i[K, X] - \tfrac{1}{2}((H+1)^2 X + X(H+1)^2 - 2(H+1)\Psi(X)\,(H+1)).$$

Proof. From Proposition 3.7 and basic quantum stochastic calculus, we have

$$\begin{aligned} &\mathbb{E}_{s]}\,W(t)^* j_{N(t)}(X)W(t) \\ &\quad = W(s)^* j_{N(s)}(X)W(s) \\ &\qquad + \int_s^t \mathbb{E}_{s]}\,W(\tau)^*\{j_{N(\tau)}(H\Psi(X)H - \tfrac{1}{2}(H^2X + XH^2) - HX - XH + i[K,X]) \\ &\qquad + j_{N(\tau)+1}(X)j_{N(\tau)}(H) + j_{N(\tau)}(H)j_{N(\tau)+1}(X) + j_{N(\tau+1)}(X) - j_{N(\tau)}(X)\}W(\tau)\,d\tau. \end{aligned}$$

Pre- and post-multiplying by $F_{N(s)}$ on both sides, noting that $F_{N(s)} = F_{N(s)}\,F_{N(\tau)}$ for $\tau \geqslant s$, and using Proposition 3.8, we obtain

$$\begin{aligned} &F_{N(s)}\{\mathbb{E}_{s]}\,W(t)^* j_{N(t)}\,(X)W(t)\}F_{N(s)} \\ &\quad = W(s)^* j_{N(s)}(X)W(s) + \int_s^t F_{N(s)}\,\mathbb{E}_{s]}\,W(\tau)^* j_{N(\tau)}(H\Psi(X)H - \tfrac{1}{2}(H^2X + XH^2) \\ &\qquad - HX - XH + i[K,X] + \Psi(X)H + H\Psi(X) + \Psi(X) - X)W(\tau)F_{N(s)}\,d\tau \\ &\quad = W(s)^* j_{N(s)}(X)W(s) + \int_s^t F_{N(s)}\{\mathbb{E}_{s]}\,W(\tau)^* j_{N(\tau)}(\mathscr{M}(X))W(\tau)\}F_{N(s)}\,ds. \end{aligned}$$

Now the result follows from general principles of ordinary differential equations.

THEOREM 3.10. *Let $\mathscr{L}$ be the Christensen–Evans generator of a uniformly continuous semigroup of unital completely positive maps on a unital von Neumann algebra $\mathscr{A} \subset \mathscr{B}(\mathscr{H}_0)$ given by*

$$\mathscr{L}(X) = i[K, X] - \tfrac{1}{2}(H^2X + XH^2 - 2H\Psi(X)H), \quad X \in \mathscr{A},$$

where H and K are hermitian elements in $\mathscr{A}$, $H \geqslant 0$, and Ψ is a unital completely positive map on $\mathscr{A}$. Let $(\mathscr{H}, F_n, j_n)$, $n \geqslant 0$, be a Markov dilation of the discrete semigroup $\{\Psi^n\}$, $n \geqslant 0$. Let $\tilde{\mathscr{H}} = \mathscr{H} \otimes \Gamma(L^2(\mathbb{R}_+))$, $N(t) = A^\dagger(t) + \Lambda(t) + A(t) + t$,

$$\tilde{F}(t) = F_{N(t)}(1_{t]} \otimes |\Omega_{[t} > < \Omega_{[t]}|),$$

where $1_{t]}$ is the identity operator in $\mathscr{H} \otimes \Gamma(L^2[0, t])$ and $\Omega_{[t}$ is the Fock vacuum vector in $\Gamma(L^2[t, \infty))$, and

$$\tilde{j}_t(X) = W(t)^* j_{N(t)}(X) W(t)\,(1_{t]} \otimes |\Omega_{[t} > < \Omega_{[t]}|),$$

where $\{W(t)\}$ is the unique unitary solution of the qsde

$$dW(t) = \{j_{N(t)}(H-1)\,(dA^\dagger - dA)\,(t) - j_{N(t)}(iK + \tfrac{1}{2}(H-1)^2)dt\} W(t)$$

with $W(0) = 1$. Then $(\tilde{\mathscr{H}}, \tilde{F}(t), \tilde{j}_t)$, $t \geqslant 0$, is a Markov dilation of the semigroup $\{e^{t\mathscr{L}}\}$, $t \geqslant 0$.

Proof. This is immediate from Proposition 3.9.

REMARK. It is curious that a shift of H by -1 is required in the equation for W in order to construct the Poisson imbedding in the interaction picture for obtaining the dilating homomorphisms $\tilde{j}_t$. It is also to be noted that we have dealt with the case when no 'structure maps' in the sense of Evans and Hudson may be available for writing a flow equation for the required dilation.

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