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Rates of Convergence of Approximate Maximum Likelihood Estimators in the Ornstein-Uhlenbeck Process

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Abstract—Berry-Esseen bounds, with random and nonrandom normings, and large deviation probability bounds for two approximate maximum likelihood estimators of the drift parameter in the Ornstein-Uhlenbeck process are obtained when the process is observed at equally spaced dense time points. Also obtained are the rates at which these estimators converge to the maximum likelihood estimator based on continuous observation.

Keywords—Stochastic differential equation, Ornstein-Uhlenbeck process, Berry-Esseen bound, Large deviation probability, Equally spaced observations, Least squares estimator, Approximate maximum likelihood estimator.

1. INTRODUCTION AND PRELIMINARIES

Let $\{X_t,\ t \geq 0\}$ be the Ornstein-Uhlenbeck process satisfying the stochastic differential equation

$$dX_t = \theta X_t \, dt + dW_t, \qquad t \geq 0, \quad X_0 = 0, \tag{1.1}$$

where $\{W_t,\ t \geq 0\}$ is a standard Wiener process, and $\theta \in (-\infty, 0)$ be the unknown parameter to be estimated from the observations of the process $\{X_t,\ t \geq 0\}$. In the stationary case, when X_0 is normally distributed with parameters $EX_0 = 0$, $EX_0^2 = -1/(2\theta)$, the log-likelihood function based on $\{x_t,\ 0 \leq t \leq T\}$ is given by ($EX_t^2 = -1/(2\theta)$, $EW_t^2 = t$),

$$\frac{1}{2}\log\left(-\frac{\theta}{\pi}\right) - \frac{\theta^2}{2}\int_0^T X_t^2\, dt + \frac{\theta}{2}\left[X_T^2 - X_0^2\right] - \frac{\theta T}{2}. \tag{1.1a}$$

Recall that based on continuous observation of $\{X_t\}$ on $[0, T]$, the conditional log-likelihood function and the maximum likelihood estimator (MLE) are given, respectively, by

$$L_T = \theta \int_0^T X_t \, dX_t - \frac{\theta^2}{2}\int_0^T X_t^2 \, dt, \tag{1.2}$$

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and

$$\theta_T = \frac{\int_0^T X_t \, dX_t}{\int_0^T X_t^2 \, dt}. \tag{1.3}$$

For the large deviation probability bound and Berry-Esseen bounds on θ_T using different normings, see [1,2]. For the consistency and the asymptotic normality of the MLE, see [3,4]. Arató [5] and Arató *et al.* [6] obtained the exact distribution function of the MLE for different levels.

The assumption that one observes the process continuously in time is impossible to meet in actual practice. We assume that the process $\{X_t\}$ is observed at the points $0 \le t_0 < t_1 < \cdots < t_n = T$ with $\Delta t_i = t_i - t_{i-1} = T/n$, $i = 1, 2, \ldots, n$. For simplicity only, we assume equidistant time points.

The conditional least squares estimator (CLSE) based on $X(t_0), \ldots, X(t_n)$ is defined as

$$\theta_{n,T} \equiv \arg\inf_{\theta} \sum_{i=1}^{n} \frac{[X(t_i) - X(t_{i-1}) - \theta X(t_{i-1}) \Delta t_i]^2}{\Delta t_i}, \tag{1.4}$$

which is given by

$$\theta_{n,T} = \frac{\sum_{i=1}^{n} X(t_{i-1})[X(t_i) - X(t_{i-1})]}{\sum_{i=1}^{n} X^2(t_{i-1}) \Delta t_i}. \tag{1.5}$$

Note that the estimator $\theta_{n,T}$ may be viewed as an approximate maximum likelihood estimator (AMLE1) which maximizes the approximate log-likelihood given by

$$L_{n,T} = \theta \sum_{i=1}^{n} X(t_{i-1})[X(t_i) - X(t_{i-1})] - \frac{\theta^2}{2} \sum_{i=1}^{n} X^2(t_{i-1}) \Delta t_i. \tag{1.6}$$

$L_{n,T}$ is obtained by an Itô approximation of the stochastic integral and rectangular rule approximation of the ordinary integral in L_T.

Using the transformation

$$t' = \frac{t}{T}, \qquad X'_{t'} = \frac{X_{t'}}{\sqrt{T}},$$

one can see that it is enough to consider the particular case $T = 1$ where $\theta T = \kappa$ is the unknown parameter. This gives that for asymptotic results $-\kappa \to \infty$ must hold (for fixed κ and $n \to \infty$, asymptotic normality does not hold).

Le Breton [7] studied the convergence of the estimator $\theta_{n,T}$ to θ_T as $n \to \infty$ and T remains fixed. In particular, he showed that $|\theta_{n,T} - \theta_T| = O_P(h^{1/2})$ where $h = T/n$. Dorogovcev [8] and Kasonga [9], respectively, proved the weak and strong consistency of the estimator $\theta_{n,T}$ as $T \to \infty$ and $T/n \to 0$. Under the more restrictive conditions $T \to \infty$ and $T/(n^{1/2}) \to 0$, called the 'rapidly increasing experimental design' (RIED) condition, Prakasa Rao [10] proved the asymptotic normality and asymptotic efficiency of the estimator $\theta_{n,T}$.

Using the Itô formula for the stochastic integral and rectangular rule approximation for the ordinary integral in (1.2), we obtain the approximate likelihood

$$\tilde{L}_{n,T} = \frac{\theta}{2} \left(X_T^2 - T \right) - \frac{\theta^2}{2} \sum_{i=1}^{n} X^2(t_{i-1}) \Delta t_i. \tag{1.7}$$

Maximizing $\tilde{L}_{n,T}$ provides another approximate maximum likelihood estimate (AMLE2) $\tilde{\theta}_{n,T}$ given by

$$\tilde{\theta}_{n,T} = \frac{(1/2) \left(X_T^2 - T \right)}{\sum_{i=1}^{n} X^2(t_{i-1}) \Delta t_i}. \tag{1.8}$$

In this paper, we study the rates of convergence of the two estimators given in (1.5) and (1.8).

The organization of the paper is as follows. In Section 2, we obtain the Berry-Esseen bounds and bounds for large deviation probability for the estimator $\theta_{n,T}$. Using purely nonrandom and various random normings (both sample dependent and parameter dependent), we obtain different Berry-Esseen bounds for the estimator $\theta_{n,T}$. Then, we obtain the rate of convergence to zero of the probability $P\{|\theta_{n,T} - \theta| > \varepsilon\}$ for fixed $\varepsilon > 0$. Finally, we obtain probabilistic bounds on $|\theta_{n,T} - \theta_T|$. In Section 3, we derive Berry-Esseen bounds for the estimators $\tilde{\theta}_{n,T}$ using purely random and various random normings (both sample dependent and parameter dependent). Then, we find probabilistic bounds on $|\tilde{\theta}_{n,T} - \theta_T|$. Finally, in Section 4, we discuss the results and some related problems.

A bit of notation: $\Phi(\cdot)$ denotes the standard normal distribution function. C is a generic constant (which may depend on the parameter) throughout the paper. P and E denote probability and expectation under the true value of the parameter.

We shall repeatedly use the following lemma in which the proof of (a) is elementary and (b) is from Michel and Pfanzagl [11, Lemma 1].

LEMMA 1. *Let X, Y, and Z be any three random variables on a probability space $(\Omega, \mathcal{F}, P)$ with $P(Z > 0) > 0$. Then, for any $\varepsilon > 0$, we have*

(a) $\sup_{x\in\mathbb{R}} |P\{X + Y \le x\} - \Phi(x)| \le \sup_{x\in\mathbb{R}} |P\{X \le x\} - \Phi(x)| + P(|Y| > \varepsilon) + \varepsilon;$

(b) $\sup_{x\in\mathbb{R}} |P\{X/Z \le x\} - \Phi(x)| \le \sup_{x\in\mathbb{R}} |P\{X \le x\} - \Phi(x)| + P\{|Z - 1| > \varepsilon\} + \varepsilon.$

We shall also use the following lemmas in which the proof of the first one is elementary and the second one is the well-known Wick's lemma.

LEMMA 1.2. *Let Q_n, R_n, Q, and R be random variables on the same probability space $(\Omega, \mathcal{F}, P)$ with $P(R > 0) > 0$. Suppose $|Q_n - Q| = O_P(\delta_{1n})$ and $|R_n - R| = O_P(\delta_{2n})$ where $\delta_{1n}, \delta_{2n} \to 0$ as $n \to \infty$. Then,*

$$\left|\frac{Q_n}{Q} - \frac{R_n}{R}\right| = O_P(\max(\delta_{1n}, \delta_{2n})).$$

LEMMA 1.3 (Wick's lemma). *Let $(\xi_1, \xi_2, \xi_3, \xi_4)$ be a Gaussian random vector with zero mean. Then,*

$$E(\xi_1\xi_2\xi_3\xi_4) = E(\xi_1\xi_2)E(\xi_3\xi_4) + E(\xi_1\xi_3)E(\xi_2\xi_4) + E(\xi_1\xi_4)E(\xi_2\xi_3).$$

2. RATES OF CONVERGENCE OF THE AMLE1

Let us introduce the notations

$$Y_{n,T} = \sum_{i=1}^{n} X(t_{i-1})[W(t_i) - W(t_{i-1})], \qquad Y_T = \int_0^T X_t\, dW_t,$$

$$Z_{n,T} = \sum_{i=1}^{n} X(t_{i-1})[X(t_i) - X(t_{i-1})], \qquad Z_T = \int_0^T X_t\, dX_t,$$

$$I_{n,T} = \sum_{i=1}^{n} X^2(t_{i-1})(t_i - t_{i-1}), \qquad I_T = \int_0^T X_t^2\, dt,$$

$$V_{n,T} = \sum_{i=1}^{n} \int_{t_{i-1}}^{t_i} X(t_{i-1})[X_t - X(t_{i-1})]\, dt.$$

REMARK 2.1. Note that in the stationary case,

$$EI_T = E\int_0^T X_t^2\, dt = -\frac{T}{2\theta}$$

and

$$Z_t = \int_0^T X_t\,dX_t = \int_0^T X_t(\theta X_t\,dt + dW_t) = \theta\int_0^T X_t^2\,dt + \int_0^T X_t\,dW_t = \theta\int_0^T X_t^2\,dt + Y_T.$$

We shall use the following lemma in the sequel in which (a) is due to Bishwal and Bose [1] and (b) is due to Bose [12].

LEMMA 2.1.

(a) *For* $\varepsilon = \varepsilon(T) = o(1)$,

$$P\left\{\left|\frac{2\theta}{T}I_T - 1\right| > \varepsilon\right\} \le C\exp\left(\frac{T\theta}{4}\varepsilon^2\right).$$

(b)

$$\sup_{x\in\mathbb{R}}\left|P\left\{\left(\frac{2\theta}{T}\right)^{1/2} Z_T \le x\right\} - \Phi(x)\right| = O\left(T^{-1/2}\right).$$

LEMMA 2.2.

(a) $E|Y_{n,T} - Y_T|^2 = O(T^2/n)$,
(b) $E|Z_{n,T} - Z_T|^2 = O(T^2/n)$,
(c) $E|I_{n,T} - I_T|^2 = O(T^4/n^2)$.

PROOF. Let

$$g_i(t) = X(t_{i-1}) - X_t, \qquad \text{for } t_{i-1} \le t < t_i, \quad i = 1,2,\dots,n.$$

Then, since $E|X(t_{i-1}) - X_t|^{2k} \le C(t_{i-1} - t)^k$, $k = 1,2,\dots$ (see [13, p. 48]), (2.1)

$$E|Y_{n,T} - Y_T|^2 = E\left|\sum_{i=1}^n X(t_{i-1})[W(t_i) - W(t_{i-1})] - \int_0^T X_t\,dW_t\right|^2 = E\left|\int_0^T g_i(t)\,dW_t\right|^2$$

$$= \int_0^T E\left(g_i^2(t)\right)dt \le C\sum_{i=1}^n \int_{t_{i-1}}^{t_i} |t_{i-1} - t|\,dt = Cn\frac{(t_i - t_{i-1})^2}{2} = C\frac{T^2}{n}.$$

This completes the proof of (a).

Note that (b) and (c) are given in [7]. However, we give a complete proof since there are some technical errors in Lemma 6 in [7]. Using (2.1) and the second part of Remark 2.1, one gets the proof of (b) by standard calculations. (See [14], and also [2].)

We obtain

$$E|Z_{n,T} - Z_T|^2 \le 2E\left|\sum_{i=1}^n X(t_{i-1})[W(t_i) - W(t_{i-1})] - \int_0^T X_t\,dW_t\right|^2 + 2\theta^2 E\left|\sum_{i=1}^n \int_{t_{i-1}}^{t_i} X_t[X(t_{i-1}) - X_t]\,dt\right|^2 \equiv N_1 + N_2.$$

N_1 is $O(T^2/n)$ by Lemma 2.2(a). To estimate N_2, let $f_i(t) = X_t[X(t_{i-1}) - X_t]$ for $t_{i-1} \le t < t_i$, $i = 1,2,\dots,n$. Then,

$$E\left|\sum_{i=1}^n \int_{t_{i-1}}^{t_i} f_i(t)\,dt\right|^2 = \sum_{i=1}^n E\left|\int_{t_{i-1}}^{t_i} f_i(t)\,dt\right|^2 + 2\sum_{i,j=1,\, i<j}^n E\left[\int_{t_{i-1}}^{t_i} f_i(t)\,dt \int_{t_{i-1}}^{t_i} f_j(s)\,ds\right] \equiv M_1 + M_2.$$

Now, using Cauchy-Schwarts, boundedness of $E(X_t^4)$ and (2.1),

$$E\left[f_i(t)\right]^2 \le c(t - t_{i-1}). \tag{2.2}$$

Thus, again using Cauchy-Schwartz,

$$M_1 = \sum_{i=1}^{n} E\left|\int_{t_{i-1}}^{t_i} f_i(t)\,dt\right|^2 \le \frac{CT^3}{n^2}.$$

Next,

$$\begin{aligned} M_2 &= 2\sum_{i,j=1,\,i<j} E\int_{t_{i-1}}^{t_i}\int_{t_{j-1}}^{t_j} \left[f_i(t)f_j(s)\right]\,dt\,ds \\ &= 2\sum_{i,j=1,\,i<j} \int_{t_{i-1}}^{t_i}\int_{t_{j-1}}^{t_j} E\left[f_i(t)f_j(s)\right]\,dt\,ds. \end{aligned}$$

Now, by Lemma 1.3, we have

$$\begin{aligned} E\left[f_i(t)f_j(s)\right] &= E[X_t(X(t_{i-1}) - X_t)]E\left[X_s\left(X\left(t_{j-1}\right) - X_s\right)\right] \\ &\quad + E[X_tX_s]E\left[\left(X(t_{i-1}) - X_t\right)\left(X\left(t_{j-1}\right) - X_s\right)\right] \\ &\quad + E\left[X_t\left(X\left(t_{j-1}\right) - X_s\right)\right]E[X_s(X(t_{i-1}) - X_t)] \equiv A_1 + A_2 + A_3. \end{aligned} \tag{2.3}$$

Note that

$$X_t = \int_0^t e^{\theta(t-u)}\,dW_u, \qquad t \ge 0.$$

Let $a = e^\theta$. For $s \ge t$, we have

$$E(X_tX_s) = \frac{1}{2\theta}\left[a^{s+t} - a^{s-t}\right].$$

Now,

$$\begin{aligned} E(X_t - X(t_{i-1}))\left(X_s - X\left(t_{j-1}\right)\right) &= E(X_tX_s) - E\left(X_tX\left(t_{j-1}\right)\right) \\ &\quad - E(X(t_{i-1})X_s) + E\left(X(t_{i-1})X\left(t_{j-1}\right)\right) \\ &= \frac{1}{2\theta}\left(a^s - a^{t_{j-1}}\right)\left[\left(a^t - a^{t_{i-1}}\right) + \left(a^{-t_{i-1}} - a^{-t}\right)\right] \\ &= \frac{1}{2\theta}\left(s - t_{j-1}\right)a^{t^*}\left[(t - t_{i-1})a^{t^{**}} + (t - t_{i-1})a^{-t^{***}}\right] \\ &\quad \text{(where } t_{j-1} < t^* < s,\ t_{i-1} < t^{**},\ t^{***} < t\text{)} \\ &\le \frac{1}{2\theta}\left(s - t_{j-1}\right)a^t\left[(t - t_{i-1})a^{t_{i-1}} + \left(s - t_{j-1}\right)a^t(t - t_{i-1})a^{-t}\right] \\ &\le C\left(s - t_{j-1}\right)(t - t_{i-1}). \end{aligned}$$

Thus, $A_2 \le C(s - t_{j-1})(t - t_{i-1})$ since $|E(X_tX_s)|$ is bounded. Next, in the same way

$$|E[X_t(X(t_{i-1}) - X_t)]| \le \frac{1}{2|\theta|}a^t(t - t_{i-1})\left[a^{t_{i-1}} + a^{-t}\right] \le C(t - t_{i-1})$$

and

$$\left|E\left[X_s\left(X_s - X\left(t_{j-1}\right)\right)\right]\right| \le \frac{1}{2|\theta|}a^s\left(s - t_{j-1}\right)\left[a^{t_{j-1}} + a^{-s}\right] \le C\left(s - t_{j-1}\right).$$

Thus, $A_1 \le C(s-t_{j-1})(t-t_{i-1})$. Next,

$$|E\left[X_t\left(X_s - X\left(t_{j-1}\right)\right)\right]| \le \frac{1}{2|\theta|} a^t\left(1-a^{-2t}\right)\left(s-t_{j-1}\right) a^t \le \left(a^{2t}-1\right)\left(s-t_{j-1}\right) \le C\left(s-t_{j-1}\right),$$

and

$$|E[X_s(X_t - X(t_{i-1}))]| \le \frac{1}{2|\theta|} a^s (t-t_{i-1})\left[a^{t_{i-1}} + a^{-t}\right] \le C(t-t_{i-1}).$$

Thus, $A_3 \le C(s-t_{j-1})(t-t_{i-1})$. Hence, $E[f_i(t)f_j(s)] \le C(s-t_{j-1})(t-t_{i-1})$, and

$$\begin{aligned}
M_2 &= 2\sum_{i,j=1,\, i<j} \int_{t_{i-1}}^{t_i}\int_{t_{j-1}}^{t_j} E\left[f_i(t)f_j(s)\right] dt\, ds \\
&\le C \sum_{i,j=1,\, i<j} \int_{t_{i-1}}^{t_i}\int_{t_{j-1}}^{t_j} (t-t_{i-1})(s-t_{j-1})\, dt\, ds \\
&= C \sum_{i,j=1,\, i<j} (t_{i-1}-t_i)^2 (t_{j-1}-t_j)^2 \\
&= Cn^2\left(\frac{T}{n}\right)^4 = C\frac{T^4}{n^2}.
\end{aligned}$$

We next prove (c). Let $h_i(t) = X^2(t_{i-1}) - X_t^2$.

$$\begin{aligned}
E|I_{n,T} - I_t|^2 &= E\left|\sum_{i=1}^n X^2(t_{i-1})(t_i - t_{i-1}) - \int_0^T X_t^2\, dt\right|^2 = E\left|\sum_{i=1}^n \int_{t_{i-1}}^{t_i} \left[X^2(t_{i-1}) - X_t^2\right] dt\right|^2 \\
&= E\left|\sum_{i=1}^n \int_{t_{i-1}}^{t_i} h_i(t)\, dt\right|^2 = \sum_{i=1}^n E\left|\int_{t_{i-1}}^{t_i} h_i(t)\, dt\right|^2 \\
&\quad + 2\sum_{i,j=1,\, i<j} E\int_{t_{i-1}}^{t_i}\int_{t_{j-1}}^{t_j} h_i(t)h_j(s)\, dt\, ds \equiv B_1 + B_2.
\end{aligned}$$

$$\begin{aligned}
Eh_i^2(t) &= E\left[X^2(t_{i-1}) - X_t^2\right]^2 = E[X(t_{i-1}) - X_t]^2 [X(t_{i-1}) + X_t]^2 \\
&\le \left\{E[X(t_{i-1}) - X_t]^4\right\}^{1/2} \left\{E[X(t_{i-1}) + X_t]^4\right\}^{1/2} \le C(t - t_{i-1})
\end{aligned}$$

(by (2.1) and the boundedness of the second term)

$$\begin{aligned}
B_1 &= \sum_{i=1}^n E\left|\int_{t_{i-1}}^{t_i} h_i(t)\, dt\right|^2 \le \sum_{i=1}^n (t_i - t_{i-1}) \int_{t_{i-1}}^{t_i} E\left(h_i^2(t)\right) dt \\
&\le C\frac{T}{n}\sum_{i=1}^n \int_{t_{i-1}}^{t_i} (t - t_{i-1})\, dt = C\frac{T^3}{n^2}.
\end{aligned}$$

Note that

$$\begin{aligned}
E\left[h_i(t)h_j(s)\right] &= E\left(X_{t_{i-1}}^2 - X_t^2\right)\left(X_{t_{j-1}}^2 - X_s^2\right) \\
&= E\left(X_{t_{i-1}} - X_t\right)\left(X_{t_{i-1}} + X_t\right)\left(X_{t_{j-1}} - X_s\right)\left(X_{t_{j-1}} + X_s\right).
\end{aligned}$$

Now, using Lemma 1.3 and proceeding similar to the estimation of M_2, it is easy to see that

$$B_2 \le C\frac{T^4}{n^2}.$$

Combining B_1 and B_2, (c) follows. ∎

THEOREM 2.3. *Let* $a_{n,T} = \max(T^{-1/2}(\log T)^{1/2}, (T^2/n)(\log T)^{-1})$. *Then,*

$$\text{(a)} \qquad \sup_{x\in\mathbb{R}}\left|P\left\{\left(\frac{T}{-2\theta}\right)^{1/2}(\theta_{n,T}-\theta)\le x\right\}-\Phi(x)\right| = O(a_{n,T}).$$

$$\text{(b)} \qquad \sup_{x\in\mathbb{R}}\left|P\left\{I_{n,T}^{1/2}(\theta_{n,T}-\theta)\le x\right\}-\Phi(x)\right| = O(a_{n,T}).$$

$$\text{(c)} \qquad \sup_{x\in\mathbb{R}}\left|P\left\{\left(\frac{T}{|2\theta_{n,T}|}\right)^{1/2}(\theta_{n,T}-\theta)\le x\right\}-\Phi(x)\right| = O(a_{n,T}).$$

PROOF. (a) It is easy to see that

$$\theta_{n,T}-\theta = \frac{Y_{n,T}}{I_{n,T}}+\theta\frac{V_{n,T}}{I_{n,T}}. \tag{2.4}$$

Hence,

$$\begin{aligned}
&\sup_{x\in\mathbb{R}}\left|P\left\{\left(\frac{T}{-2\theta}\right)^{1/2}(\theta_{n,T}-\theta)\le x\right\}-\Phi(x)\right| \\
&= \sup_{x\in\mathbb{R}}\left|P\left\{\left(\frac{T}{-2\theta}\right)^{1/2}\frac{Y_{n,T}}{I_{n,T}}+\left(\frac{T}{-2\theta}\right)^{1/2}\theta\frac{V_{n,T}}{I_{n,T}}\le x\right\}-\Phi(x)\right| \\
&\le \sup_{x\in\mathbb{R}}\left|P\left\{\left(\frac{T}{-2\theta}\right)^{1/2}\frac{Y_{n,T}}{I_{n,T}}\le x\right\}-\Phi(x)\right| + P\left\{\left|\left(\frac{T}{-2\theta}\right)^{1/2}\frac{V_{n,T}}{I_{n,T}}\right|>\varepsilon\right\}+\varepsilon \\
&\equiv K_1+K_2+\varepsilon.
\end{aligned} \tag{2.5}$$

Note that by Lemma 1.1(b),

$$\begin{aligned}
K_1 &= \sup_{x\in\mathbb{R}}\left|P\left\{\left(\frac{T}{-2\theta}\right)^{1/2}\frac{Y_{n,T}}{I_{n,T}}\le x\right\}-\Phi(x)\right| = \sup_{x\in\mathbb{R}}\left|P\left\{\frac{(-2\theta/T)^{1/2}Y_{n,T}}{(-2\theta/T)\,I_{n,T}}\le x\right\}-\Phi(x)\right| \\
&\le \sup_{x\in\mathbb{R}}\left|P\left\{\left(\frac{-2\theta}{T}\right)^{1/2}Y_{n,T}\le x\right\}-\Phi(x)\right| + P\left\{\left(\frac{-2\theta}{T}\right)I_{n,T}-1>\varepsilon\right\}+\varepsilon \\
&\equiv J_1+J_2+\varepsilon,
\end{aligned} \tag{2.6}$$

$$\begin{aligned}
J_1 &= \sup_{x\in\mathbb{R}}\left|P\left\{\left(\frac{-2\theta}{T}\right)^{1/2}(Y_{n,T}-Y_T+Y_T)\le x\right\}-\Phi(x)\right| \\
&\le \sup_{x\in\mathbb{R}}\left|P\left\{\left(\frac{-2\theta}{T}\right)^{1/2}Y_T\le x\right\}-\Phi(x)\right| + P\left\{\left(\frac{-2\theta}{T}\right)^{1/2}|Y_{n,T}-Y_T|>\varepsilon\right\}+\varepsilon \\
&\le CT^{-1/2}+\left(\frac{-2\theta}{T}\right)\frac{E|Y_{n,T}-Y_T|^2}{\varepsilon^2}+\varepsilon \le CT^{-1/2}+C\frac{T/n}{\varepsilon^2}+\varepsilon
\end{aligned} \tag{2.7}$$

(by Lemma 2.1(b) and Lemma 2.2(a)).

$$\begin{aligned}
J_2 &= \left\{\left|\left(\frac{-2\theta}{T}\right)(I_{n,T}-I_T+I_T)-1\right|>\varepsilon\right\} \\
&\le P\left\{\left|\left(\frac{-2\theta}{T}\right)I_T-1\right|>\frac{\varepsilon}{2}\right\}+P\left\{\left(-\frac{2\theta}{T}\right)|I_{n,T}-I_T|>\frac{\varepsilon}{2}\right\} \\
&\le C\exp\left(\frac{T\theta}{16}\varepsilon^2\right)+\frac{16\theta^2}{T^2}\frac{E|I_{n,T}-I_T|^2}{\varepsilon^2} \le C\exp\left(\frac{T\theta}{16}\varepsilon^2\right)+C\frac{T^2/n^2}{\varepsilon^2}.
\end{aligned} \tag{2.8}$$

Here, the bound for the first term in (2.8) comes from Lemma 2.1(a) and that for the second term from Lemma 2.2(c). From the proof of Lemma 2.2(b), we have

$$E|V_{n,T}|^2 \le C\frac{T^4}{n^2}. \tag{2.9}$$

Next,

$$\begin{aligned}
K_2 &= P\left\{\left|\left(\frac{T}{-2\theta}\right)^{1/2}\theta\frac{V_{n,T}}{I_{n,T}}\right| > \varepsilon\right\} = P\left\{\left|\frac{(-2\theta/T)^{1/2}\,\theta V_{n,T}}{(-2\theta/T)\,I_{n,T}}\right| > \varepsilon\right\} \\
&= P\left\{\left|\left(\frac{-2\theta}{T}\right)^{1/2}\theta V_{n,T}\right| > \delta\right\} + P\left\{\left(-\frac{2\theta}{T}\right)I_{n,T} < \frac{\delta}{\varepsilon}\right\}
\end{aligned}$$

(where we choose $\delta = \varepsilon - C\varepsilon^2$)

$$\le P\left\{\left|\left(\frac{-2\theta}{T}\right)^{1/2}\theta V_{n,T}\right| > \delta\right\} + P\left\{\left|\left(-\frac{2\theta}{T}\right)I_{n,T} - 1\right| > \delta_1\right\} \tag{2.10}$$

(where $\delta_1 = (\varepsilon - \delta)/\delta = C\varepsilon$)

$$\begin{aligned}
&\le -\frac{2\theta}{T}\theta^2\frac{E|V_{n,T}|^2}{\delta^2} + C\exp\left(\frac{T\theta}{16}\delta_1^2\right) + C\frac{T^2/n^2}{\delta_1^2} \\
&\le C\frac{T^3/n^2}{\delta^2} + C\exp\left(\frac{T\theta}{16}\delta_1^2\right) + C\frac{T^2/n^2}{\delta_1^2} \qquad \text{(by (2.9) and (2.8)).}
\end{aligned}$$

Now, from (2.5)–(2.10), since $T/n^{1/2} \to 0$,

$$\begin{aligned}
\sup_{x\in\mathbb{R}}\left|P\left\{\left(\frac{T}{-2\theta}\right)^{1/2}(\theta_{n,T} - \theta) \le x\right\} - \Phi(x)\right| &\le CT^{-1/2} + C\exp\left(\frac{T\theta}{16}\varepsilon^2\right) + C\frac{T/n}{\varepsilon^2} \\
&\quad + C\frac{T^2/n^2}{\varepsilon^2} + C\frac{T^3/n^2}{\delta^2} + C\exp\left(\frac{T\theta}{16}\delta_1^2\right) + C\left(\frac{T^2/n^2}{\delta_1^2}\right) + \varepsilon.
\end{aligned} \tag{2.11}$$

Choosing $\varepsilon = T^{-1/2}(\log T)^{1/2}$, the terms of (2.11) are of the order $O(\max(T^{-1/2}(\log T)^{1/2}, (T^2/n)(\log T)^{-1}))$. This proves (a).

(b) Using the expression (2.4),

$$\begin{aligned}
\sup_{x\in\mathbb{R}}\left|P\left\{I_{n,T}^{1/2}(\theta_{n,T} - \theta) \le x\right\} - \Phi(x)\right| &= \sup_{x\in\mathbb{R}}\left|P\left\{\frac{Y_{n,T}}{I_{n,T}^{1/2}} + \theta\frac{V_{n,T}}{I_{n,T}^{1/2}} \le x\right\} - \Phi(x)\right| \\
&\le \sup_{x\in\mathbb{R}}\left|P\left\{\frac{Y_{n,T}}{I_{n,T}^{1/2}} \le x\right\} - \Phi(x)\right| + P\left\{\left|\theta\frac{V_{n,T}}{I_{n,T}^{1/2}}\right| > \varepsilon\right\} + \varepsilon \equiv H_1 + H_2 + \varepsilon.
\end{aligned} \tag{2.12}$$

Note that

$$\begin{aligned}
H_1 = \sup_{x\in\mathbb{R}}\left|P\left\{\frac{Y_{n,T} - Y_T + Y_T}{I_{n,T}^{1/2}} \le x\right\} - \Phi(x)\right| &= \sup_{x\in\mathbb{R}}\left|P\left\{\frac{Y_T}{I_{n,T}^{1/2}} \le x\right\} - \Phi(x)\right| \\
&\quad + P\left\{\frac{|Y_{n,T} - Y_T|}{I_{n,T}^{1/2}} > \varepsilon\right\} + \varepsilon \equiv L_1 + L_2 + \varepsilon.
\end{aligned} \tag{2.13}$$

Now,

$$
\begin{aligned}
L_1 &= \sup_{x\in\mathbb{R}}\left|P\left\{\frac{Y_T}{I_{n,T}^{1/2}}\le x\right\}-\Phi(x)\right| \le \sup_{x\in\mathbb{R}}\left|P\left\{\left(\frac{-2\theta}{T}\right)^{1/2}Y_T\le x\right\}-\Phi(x)\right| \\
&\quad + P\left\{\left|\left(\frac{-2\theta}{T}\right)^{1/2}I_{n,T}^{1/2}-1\right|>\varepsilon\right\}+\varepsilon \qquad \text{(by Lemma 1.1(b))} \\
&\le C_T^{-1/2}+P\left\{\left|\left(\frac{-2\theta}{T}\right)I_{n,T}-1\right|>\varepsilon\right\}+\varepsilon \qquad \text{(by Lemma 2.1(b))} \\
&\le CT^{-1/2}+C\exp\left(-\frac{T\theta}{16}\varepsilon^2\right)+C\frac{T^2/n^2}{\varepsilon^2}+\varepsilon \qquad \text{(by (2.8))}.
\end{aligned}
\tag{2.14}
$$

On the other hand,

$$
\begin{aligned}
L_2 &= P\left\{\frac{|Y_{n,T}-Y_T|}{I_{n,T}^{1/2}}>\varepsilon\right\} \le P\left\{\left(\frac{-2\theta}{T}\right)^{1/2}|Y_{n,T}-Y_T|>\delta\right\} \\
&\quad + P\left\{\left|\left(\frac{-2\theta}{T}\right)^{1/2}I_{n,T}^{1/2}-1\right|>\delta_1\right\} \quad \text{(where } \delta=\varepsilon-C\varepsilon^2 \text{ and } \delta_1=(\varepsilon-\delta)/\varepsilon>0) \\
&\le \frac{(-2\theta/T)\,E|Y_{n,T}-Y_T|^2}{\delta^2}+P\left\{\left|\left(\frac{-2\theta}{T}\right)I_{n,T}-1\right|>\delta_1\right\} \\
&\le C\frac{T/n}{\delta^2}+C\exp\left(\frac{T\theta}{16}\delta_1^2\right)+C\frac{T^2/n^2}{\delta_1^2} \qquad \text{(from Lemma 2.2(a) and (2.8))}.
\end{aligned}
\tag{2.15}
$$

Using (2.15) and (2.14) in (2.13), we obtain

$$
\begin{aligned}
H_1 &= \sup_{x\in\mathbb{R}}\left|P\left\{\frac{Y_T}{I_{n,T}^{1/2}}\le x\right\}-\Phi(x)\right| \le CT^{-1/2}+C\exp\left(\frac{T\theta}{16}\varepsilon^2\right) \\
&\quad + C\frac{T/n}{\delta^2}+C\frac{T^2/n^2}{\delta_1^2}+C\exp\left(\frac{T\theta}{16}\delta_1^2\right)+C\frac{T^2/n^2}{\varepsilon^2}+\varepsilon.
\end{aligned}
\tag{2.16}
$$

$$
\begin{aligned}
H_2 &= P\left\{\left|\theta\frac{V_{n,T}}{I_{n,T}^{1/2}}\right|>\varepsilon\right\} = P\left\{\frac{\left|(-2\theta/T)^{1/2}\,\theta V_{n,T}\right|}{\left|(-2\theta/T)^{1/2}\,I_{n,T}^{1/2}\right|}>\varepsilon\right\} \\
&\le P\left\{\left|\left(\frac{-2\theta}{T}\right)^{1/2}\theta V_{n,T}\right|>\delta\right\}+P\left\{\left|\left(\frac{-2\theta}{T}\right)^{1/2}I_{n,T}^{1/2}\right|<\frac{\delta}{\varepsilon}\right\} \le \left(-\frac{2\theta}{T}\right)\theta^2\frac{E|V_{n,T}|^2}{\delta^2} \\
&\quad + P\left\{\left|\left(-\frac{2\theta}{T}\right)I_{n,T}-1\right|>\delta_1\right\} \qquad \left(\text{where } 0<\delta<\varepsilon \text{ and } \delta_1=\frac{e-\delta}{\varepsilon}=C\varepsilon>0\right) \\
&\le C\frac{T^2/n^2}{\delta^2}+C\exp\left(\frac{T\theta}{16}\delta_1^2\right)+C\frac{T^2/n^2}{\delta_1^2} \qquad \text{(from (2.9) and (2.8))}.
\end{aligned}
\tag{2.17}
$$

Using (2.17) and (2.16) in (2.12) and choosing $\varepsilon = T^{-1/2}(\log T)^{1/2}$, the terms of (2.12) are of the order $O(\max(T^{-1/2}(\log T)^{1/2},\ (T^2/n)(\log T)^{-1}))$. This proves (b).

(c) Let

$$D_T=\{|\theta_{n,T}-\theta|\le d_T\} \quad \text{and} \quad d_T=CT^{-1/2}(\log T)^{1/2}.$$

On the set D_T, expanding $(2|\theta_{n,T}|)^{-1/2}$, we obtain

$$
\begin{aligned}
(-2\theta_{n,T})^{-1/2} &= (-2\theta)^{-1/2}\left[1-\frac{\theta-\theta_{n,T}}{\theta}\right]^{-1/2} \\
&= (-2\theta)^{-1/2}\left[1+\frac{1}{2}\left(\frac{\theta-\theta_{n,T}}{\theta}\right)+O\left(d_T^2\right)\right].
\end{aligned}
$$

Then,

$$\sup_{x\in\mathbb{R}}\left|P\left\{\left(\frac{T}{2|\theta_{n,T}|}\right)^{1/2}(\theta_{n,T}-\theta)\le x\right\}-\Phi(x)\right| \le \sup_{x\in\mathbb{R}}\left|P\left\{\left(\frac{T}{2|\theta_{n,T}|}\right)^{1/2}(\theta_{n,T}-\theta)\le x, D_T\right\}-\Phi(x)\right| + P\left(D_T^c\right) \tag{2.18}$$

$$\begin{aligned}P(D_T^c) &= P\left\{|\theta_{n,T}-\theta| > CT^{-1/2}(\log T)^{1/2}\right\}\\ &= P\left\{\left(\frac{T}{-2\theta}\right)^{1/2}|\theta_{n,T}-\theta| > C(\log T)^{1/2}(-2\theta)^{-1/2}\right\}\\ &\le C\max\left(T^{-1/2}(\log T)^{1/2}, \frac{T^2}{n}(\log T)^{-1}\right)\\ &\quad + 2\left(1-\Phi\left((\log T)^{1/2}(-2\theta)^{-1/2}\right)\right) \qquad \text{(by Theorem 2.3(a))}\\ &\le C\max\left(T^{-1/2}(\log T)^{1/2}, \frac{T^2}{n}(\log T)^{-1}\right).\end{aligned} \tag{2.19}$$

On the set D_T,

$$\left|\left(\frac{\theta_{n,T}}{\theta}\right)^{1/2}-1\right| \le CT^{-1/2}(\log T)^{1/2}.$$

Hence, upon choosing $\varepsilon = CT^{-1/2}(\log T)^{1/2}$, C large, we obtain

$$\begin{aligned}&\left|P\left\{\left(\frac{T}{-2\theta_{n,T}}\right)^{1/2}(\theta_{n,T}-\theta)\le x, D_T\right\}-\Phi(x)\right|\\ &\le \left|P\left\{\left(\frac{T}{-2\theta}\right)^{1/2}(\theta_{n,T}-\theta)\le x, D_T\right\}-\Phi(x)\right|\\ &\quad + P\left\{\left|\left(\frac{\theta_{n,T}}{\theta}\right)^{1/2}-1\right| > \varepsilon, D_T\right\}+\varepsilon \qquad \text{(by Lemma 1.1(b))}\\ &\le C\max\left(T^{-1/2}(\log T)^{1/2}, \frac{T^2}{n}(\log T)^{-1}\right) \qquad \text{(by Theorem 2.3(a))}.\end{aligned} \tag{2.20}$$

(c) follows from (2.18)–(2.20). ∎

THEOREM 2.4.

$$\sup_{x\in\mathbb{R}}\left|P\left\{I_{n,T}\left(-\frac{2\theta}{T}\right)^{1/2}(\theta_{n,T}-\theta)\le x\right\}-\Phi(x)\right| = O\left(\max\left(T^{-1/2}, \left(\frac{T}{n}\right)^{1/3}\right)\right).$$

PROOF. Let

$$A_{n,T} = Z_{n,T} - Z_T, \qquad B_{n,T} = I_{n,T} - I_T.$$

By Lemma 3.2,

$$E|A_{n,T}|^2 = O\left(\frac{T^2}{n}\right) \qquad \text{and} \qquad E|B_{n,T}|^2 = O\left(\frac{T^4}{n^2}\right). \tag{2.21}$$

From (2.6),

$$\begin{aligned}I_{n,T}\theta_{n,T} &= \sum_{i=1}^{n} X(t_{i-1})[X(t_i)-X(t_{i-1})] = \int_0^T X_t\,dX_t + A_{n,T}\\ &= \int_0^T X_t\,dW_t + \theta\int_0^T X_t^2\,dt + A_{n,T}.\end{aligned}$$

Hence, $I_{n,T}(\theta_{n,T}-\theta) = -\theta B_{n,T} + A_{n,T}$. Thus,

$$\begin{aligned}
&\sup_{x\in\mathbb{R}}\left|P\left\{I_{n,T}\left(-\frac{2\theta}{T}\right)^{1/2}(\theta_{n,T}-\theta)\le x\right\}-\Phi(x)\right| \\
&= \sup_{x\in\mathbb{R}}\left|P\left\{\left(-\frac{2\theta}{T}\right)^{1/2}[Y_T-\theta B_{n,T}+A_{n,T}]\le x\right\}-\Phi(x)\right| \\
&\le \sup_{x\in\mathbb{R}}\left|P\left\{\left(-\frac{2\theta}{T}\right)^{1/2}Y_T\le x\right\}-\Phi(x)\right| + P\left\{\left|\left(-\frac{2\theta}{T}\right)^{1/2}[-\theta B_{n,T}+A_{n,T}]\right|>\varepsilon\right\}+\varepsilon \\
&\le CT^{-1/2}+\left(-\frac{2\theta}{T}\right)\frac{E|-\theta B_{n,T}+A_{n,T}|^2}{\varepsilon^2}+\varepsilon \le CT^{-1/2}+C\frac{T^2/n}{\varepsilon^2}+\varepsilon,
\end{aligned}$$

by Lemma 2.1(b) and (2.21). Choosing $\varepsilon = (T^2/n)^{1/3}$, the rate is $O(\max(T^{-1/2},(T^2/n)^{1/3}))$. ∎

THEOREM 2.5. *For fixed* $\varepsilon > 0$,

$$P\{|\theta_{n,T}-\theta|>\varepsilon\} = O\left(\max\left(\frac{1}{T},\frac{1}{n}\right)\right).$$

PROOF. Using (2.4), we obtain

$$\begin{aligned}
P\{|\theta_{n,T}-\theta|>\varepsilon\} &= P\left\{\frac{|Y_{n,T}+\theta V_{n,T}|}{I_{n,T}}>\varepsilon\right\} \\
&\le P\left\{\frac{(-2\theta/T)\,|Y_{n,T}|+(-2\theta/T)\,|\theta V_{n,T}|}{(-2\theta/T)\,I_{n,T}}>\varepsilon\right\} \\
&\le P\left\{\left(-\frac{2\theta}{T}\right)|Y_{n,T}|+\left(-\frac{2\theta}{T}\right)|\theta V_{n,T}|>\delta\right\} \\
&\quad + P\left\{\left(-\frac{2\theta}{T}\right)I_{n,T}<\frac{\delta}{\varepsilon}\right\} \\
&\le P\left\{\left(-\frac{2\theta}{T}\right)|Y_{n,T}|>\frac{\delta}{2}\right\}+P\left\{\left(-\frac{2\theta}{T}\right)|V_{n,T}|>\frac{\delta}{2}\right\} \\
&\quad + P\left\{\left|\left(-\frac{2\theta}{T}\right)I_{n,T}<\delta_1\right\}\right| \equiv S_1+S_2+S_3
\end{aligned}\tag{2.22}$$

(where $0<\delta<\varepsilon$ and $\delta_1 = (\varepsilon-\delta)/\varepsilon = C\varepsilon > 0$).

$$\begin{aligned}
S_1 &= P\left\{\left(-\frac{2\theta}{T}\right)|Y_{n,T}|>\frac{\delta}{2}\right\}\le P\left\{\left(-\frac{2\theta}{T}\right)|Y_{n,T}-Y_T|>\frac{\delta}{4}\right\}+P\left\{\left(-\frac{2\theta}{T}\right)Y_T>\frac{\delta}{4}\right\} \\
&\le \left(-\frac{2\theta}{2}\right)^2\frac{16}{\delta^2}E|Y_{n,T}-Y_T|^2+P\left\{(-\theta)\left(\frac{X_T^2}{T}-1-\frac{2\theta}{T}I_T\right)>\frac{\delta}{4}\right\}
\end{aligned}$$

(since by Itô formula $Y_T = (1/2)\left(X_T^2-T\right)-\theta I_T$)

$$\le \frac{C}{n}+P\left\{\left|(-2\theta)\frac{I_T}{T}-1\right|>\frac{\delta}{-8\theta}\right\}+P\left\{X_T^2>\frac{T\delta}{-8\theta}\right\}.\tag{2.23}$$

Using the fact that $X_T \sim N(0,(e^{2\theta T}-1)/2\theta)$ and Lemma 2.1(a), the last two terms of (2.23) are bounded by CT^{-1}. Hence,

$$S_1 = O\left(\max\left(\frac{1}{T},\frac{1}{n}\right)\right).\tag{2.24}$$

Now, applying the Chebyshev inequality for S_2 and using (2.8) with $\varepsilon = T^{-1/2}(\log T)^{1/2}$ for S_3 in (2.22), the right side of (2.22) has the order $O(\max(1/T,1/n))$. ∎

THEOREM 2.6.
$$|\theta_{n,T}-\theta_T| = O_P\left(\frac{T^2}{n}\right)^{1/2}.$$

PROOF. Note that
$$\theta_{n,T}-\theta_T = \frac{Z_{n,T}}{I_{n,T}} - \frac{Z_T}{I_T}.$$

From Lemma 3.2, it follows that $|Z_{n,T}-Z_T| = O_P(T^2/n)^{1/2}$ and $|I_{n,T}-I_T| = O_P(T^4/n^2)^{1/2}$. The theorem follows easily from Lemma 1.2.

3. RATES OF CONVERGENCE OF THE AMLE2

THEOREM 3.1. *Let* $b_{n,T} = O(\max(T^{-1/2}(\log T)^{1/2}, (T^4/n^2)(\log T)^{-1}))$.

$$\text{(a)}\quad \sup_{x\in\mathbb{R}}\left|P\left\{\left(-\frac{T}{2\theta}\right)^{1/2}\left(\tilde{\theta}_{n,T}-\theta\right)\le x\right\}-\Phi(x)\right| = O(b_{n,T}),$$
$$\text{(b)}\quad \sup_{x\in\mathbb{R}}\left|P\left\{I_{n,T}^{1/2}\left(\tilde{\theta}_{n,T}-\theta\right)\le x\right\}-\Phi(x)\right| = O(b_{n,T}),$$
$$\text{(c)}\quad \sup_{x\in\mathbb{R}}\left|P\left\{\left(\frac{T}{2\left|\tilde{\theta}_{n,T}\right|}\right)^{1/2}\left(\tilde{\theta}_{n,T}-\theta\right)\le x\right\}-\Phi(x)\right| = O(b_{n,T}).$$

PROOF. (a) From (1.8), we have
$$I_{n,T}\tilde{\theta}_{n,T} = \frac{1}{2}\left(X_T^2-T\right) = \int_0^T X_t\,dX_t = \int_0^T X_t\,dW_t + \theta\int_0^T X_t^2\,dt = Y_T+\theta I_T.$$

Thus,
$$\begin{aligned}\left(-\frac{T}{2\theta}\right)^{1/2}\left(\tilde{\theta}_{n,T}-\theta\right) &= \frac{(-T/2\theta)^{1/2}Y_T+\theta\,(-T/2\theta)^{1/2}\,(I_T-I_{n,T})}{I_{n,T}}\\ &= \frac{(-2\theta/T)^{1/2}Y_T+(-2\theta/T)^{1/2}\,(I_T-I_{n,T})}{(-2\theta/T)\,I_{n,T}}.\end{aligned}\tag{3.1}$$

Now,
$$\begin{aligned}&\sup_{x\in\mathbb{R}}\left|P\left\{\left(-\frac{T}{2\theta}\right)^{1/2}\left(\tilde{\theta}_{n,T}-\theta\right)\le x\right\}-\Phi(x)\right|\\ &= \sup_{x\in\mathbb{R}}\left|P\left\{\frac{(-2\theta/T)^{1/2}Y_T+(-2\theta/T)^{1/2}\,(I_T-I_{n,T})}{(-2\theta/T)\,I_{n,T}}\le x\right\}-\Phi(x)\right|\\ &\le \sup_{x\in\mathbb{R}}\left|P\left\{\left(-\frac{2\theta}{T}\right)^{1/2}Y_T\le x\right\}-\Phi(x)\right| + P\left\{\left|\theta\left(-\frac{2\theta}{T}\right)^{1/2}(I_{n,T}-I_T)\right|>\varepsilon\right\}\\ &\quad + P\left\{\left|\left(-\frac{2\theta}{T}\right)I_{n,T}-1\right|>\varepsilon\right\}+2\varepsilon\\ &\le CT^{-1/2}+\theta^2\frac{(-2\theta/T)\,E|I_{n,T}-I_T|^2}{\varepsilon^2}+C\exp\left(\frac{T\theta}{4}\varepsilon^2\right)+C\frac{T^2}{n^2\varepsilon^2}+2\varepsilon\end{aligned}\tag{3.2}$$

(the bound for the third term in the r.h.s. of (3.2) is obtained from (2.8))

$$\le CT^{-1/2}+C\frac{T^2}{n^2\varepsilon^2}+C\exp\left(\frac{T\theta}{4}\varepsilon^2\right)+C\frac{T}{n^2\varepsilon^2}+\varepsilon \tag{3.3}$$

(by Lemma 2.2(a)). Choosing $\varepsilon = CT^{-1/2}(\log T)^{1/2}$, the terms in the r.h.s. of (3.3) are of the order $O(\max(T^{-1/2}(\log T)^{1/2}, (T^4/n^2)(\log T)^{-1}))$. ∎

(b) From (3.1), we have

$$I_{n,T}^{1/2}\left(\tilde{\theta}_{n,T}-\theta\right) = \frac{Y_T + \theta(I_T - I_{n,T})}{I_{n,T}^{1/2}}.$$

Now,

$$\begin{aligned}\sup_{x\in\mathbb{R}}\left|P\left\{I_{n,T}^{1/2}\left(\tilde{\theta}_{n,T}-\theta\right)\le x\right\}-\Phi(x)\right| &= \sup_{x\in\mathbb{R}}\left|P\left\{\frac{Y_T}{I_{n,T}^{1/2}}+\theta\frac{I_T-I_{n,T}}{I_{n,T}^{1/2}}\le x\right\}-\Phi(x)\right| \\ &\le \sup_{x\in\mathbb{R}}\left|P\left\{\frac{Y_T}{I_{n,T}^{1/2}}\le x\right\}-\Phi(x)\right|+P\left\{\left|\frac{\theta(I_T-I_{n,T})}{I_{n,T}^{1/2}}\right|>\varepsilon\right\}+\varepsilon \equiv U_1+U_2+\varepsilon.\end{aligned} \tag{3.4}$$

We have from (2.8)

$$U_1 \le CT^{-1/2} + C\exp\left(\frac{T\theta}{16}\varepsilon^2\right) + C\frac{T^2}{n^2\varepsilon^2} + \varepsilon. \tag{3.5}$$

Now,

$$\begin{aligned}U_2 &= P\left\{|\theta|\left|\frac{I_{n,T}-I_T}{I_{n,T}^{1/2}}\right|>\varepsilon\right\} = P\left\{|\theta|\frac{\left|(-2\theta/T)^{1/2}(I_{n,T}-I_T)\right|}{\left|(-2\theta/T)^{1/2}I_{n,T}^{1/2}\right|}>\varepsilon\right\} \\ &\le P\left\{\left|\left(-\frac{2\theta}{T}\right)^{1/2}\right||I_{n,T}-I_T|>\delta\right\}+P\left\{\left|\left(-\frac{2\theta}{T}\right)^{1/2}I_{n,T}^{1/2}-1\right|>\delta_1\right\}\end{aligned} \tag{3.6}$$

(where $\delta = \varepsilon - C\varepsilon^2$ and $\delta_1 = (\varepsilon-\delta)/\varepsilon > 0$)

$$\begin{aligned}&\le \left(-\frac{2\theta}{T}\right)\frac{E|I_{n,T}-I_T|^2}{\delta^2}+P\left\{\left|\left(-\frac{2\theta}{T}\right)I_{n,T}-1\right|>\delta_1\right\} \\ &\le C\frac{T^3}{n^2\delta^2}+C\exp\left(\frac{T\theta}{16}\delta_1^2\right)+C\frac{T^2/n^2}{\delta_1^2}.\end{aligned} \tag{3.7}$$

Here, the bound for the first term in the r.h.s. of (3.6) comes from Lemma 2.2(c) and that for the second term is obtained from J_2 in (2.8).

Now, using the bounds (3.5) and (3.7) in (3.4) with $\varepsilon = CT^{-1/2}(\log T)^{1/2}$, we obtain that the terms in (3.4) are of the order $O(\max(T^{-1/2}(\log T)^{1/2}, (T^4/n^2)(\log T)^{-1}))$. ∎

(c) Let

$$G_T = \left\{\left|\tilde{\theta}_{n,T}-\theta\right|\le d_T\right\} \qquad \text{and} \qquad d_T = CT^{-1/2}(\log T)^{1/2}.$$

On the set G_T, expanding $(2|\tilde{\theta}_{n,T}|)^{1/2}$, we obtain

$$\left(-2\tilde{\theta}_{n,T}\right)^{-1/2} = (-2\theta)^{1/2}\left[1-\frac{\theta-\tilde{\theta}_{n,T}}{\theta}\right]^{-1/2} = (-2\theta)^{1/2}\left[1+\frac{1}{2}\left(\frac{\theta-\tilde{\theta}_{n,T}}{\theta}\right)\right]+O\left(d_T^2\right).$$

Then,

$$\begin{aligned}&\sup_{x\in\mathbb{R}}\left|P\left\{\left(\frac{T}{2\left|\tilde{\theta}_{n,T}\right|}\right)^{1/2}\left(\tilde{\theta}_{n,T}-\theta\right)\le x\right\}-\Phi(x)\right| \\ &\qquad\le \sup_{x\in\mathbb{R}}\left\{P\left(\frac{T}{2\left|\tilde{\theta}_{n,T}\right|}\right)^{1/2}\left(\tilde{\theta}_{n,T}-\theta\right)\le x, G_T\right\}+P\left(G_T^c\right).\end{aligned}$$

Now,

$$P(G_T^c) = P\left\{\left|\tilde{\theta}_{n,T} - \theta\right| > CT^{-1/2}(\log T)^{1/2}\right\}$$
$$= P\left\{\left(-\frac{T}{2\theta}\right)^{1/2}\left|\tilde{\theta}_{n,T} - \theta\right| > C(\log T)^{1/2}(-2\theta)^{-1/2}\right\}$$
$$\leq C\max\left(T^{-1/2}(\log T)^{1/2}, \frac{T^3}{n^2}(\log T)^{-1}\right) + 2\left(1 - \Phi\log T^{1/2}(-2\theta)^{-1/2}\right)$$

(by Theorem 3.1(a))

$$\leq C\max\left(T^{-1/2}(\log T)^{1/2}, \frac{T^3}{n^2}(\log T)^{-1}\right).$$

On the set G_T,

$$\left|\left(\frac{\tilde{\theta}_{n,T}}{\theta}\right)^{1/2} - 1\right| \leq CT^{-1/2}(\log T)^{1/2}.$$

Hence, upon choosing $\varepsilon = CT^{-1/2}(\log T)^{1/2}, C$ large,

$$\left|P\left\{\left(\frac{T}{-2\tilde{\theta}_{n,T}}\right)^{1/2}\left(\tilde{\theta}_{n,T} - \theta\right) \leq x, G_T\right\} - \Phi(x)\right|$$
$$\leq \left|P\left\{\left(\frac{T}{-2\theta}\right)^{1/2}\left(\tilde{\theta}_{n,T} - \theta\right) \leq x, G_T\right\}\right| + P\left\{\left|\left(\frac{\theta_{n,T}}{\theta}\right)^{1/2} - 1\right| > \varepsilon, G_T\right\} + \varepsilon$$

(by Lemma 1.1(b))

$$\leq C\max\left(T^{-1/2}(\log T)^{1/2}, \frac{T^4}{n^2}(\log T)^{-1}\right)$$

(by Theorem 3.1(a)). ∎

THEOREM 3.2.

$$\sup_{x\in\mathbb{R}}\left|P\left\{I_{n,T}\left(-\frac{2\theta}{T}\right)^{1/2}\left(\tilde{\theta}_{n,T} - \theta\right) \leq x\right\} - \Phi(x)\right| = O\left(\max\left(T^{-1/2}, \left(\frac{T^3}{n^2}\right)^{1/3}\right)\right).$$

PROOF. From (3.1), we have

$$I_{n,T}\left(-\frac{2\theta}{T}\right)^{1/2}\left(\tilde{\theta}_{n,T} - \theta\right) = \left(-\frac{2\theta}{T}\right)^{1/2} Y_T + \theta\left(-\frac{2\theta}{T}\right)^{1/2}(I_T - I_{n,T}).\ldots$$

Hence, by Lemma 2.1(b) and Lemma 2.2(c),

$$\sup_{x\in\mathbb{R}}\left|P\left\{I_{n,T}\left(-\frac{2\theta}{T}\right)^{1/2}\left(\tilde{\theta}_{n,T} - \theta\right) \leq x\right\} - \Phi(x)\right|$$
$$= \sup_{x\in\mathbb{R}}\left|P\left\{\left(-\frac{2\theta}{T}\right)^{1/2} Y_T + \theta\left(-\frac{2\theta}{T}\right)^{1/2}(I_T - I_{n,T}) \leq x\right\} - \Phi(x)\right|$$
$$\leq \sup_{x\in R}\left|P\left\{\left(-\frac{2\theta}{T}\right)^{1/2} Y_T \leq x\right\} - \Phi(x)\right| + P\left\{\left|\theta\left(-\frac{2\theta}{T}\right)^{1/2}(I_T - I_{n,T})\right| > \varepsilon\right\} + \varepsilon$$
$$\leq CT^{-1/2} + C\frac{E|I_T - I_{n,T}|^2}{T\varepsilon^2} + \varepsilon \leq CT^{-1/2} + C\frac{T^3}{n^2\varepsilon^2} + \varepsilon.$$

Choosing $\varepsilon = (T^3/n^2)^{1/3}$, the theorem follows. ∎

THEOREM 3.3.

$$\left|\tilde{\theta}_{n,T} - \theta_T\right| = O_P\left(\frac{T^4}{n^2}\right)^{1/2}.$$

PROOF. We have from (2.4), $\theta_T = Z_T/I_T$. By the Itô formula, it is easy to see that

$$\tilde{\theta}_{n,T} = \frac{Z_T}{I_{n,T}}.$$

Hence, applying Lemma 1.2 with the aid of Lemma 2.2(c), the theorem follows. ∎

4. CONCLUDING REMARKS

REMARK 1. The bounds in Theorems 2.3, 2.4, 3.1, and 3.2 are uniform over compact subsets of the parameter space, but not for $\theta \leq 0$!

REMARK 2. Theorems 2.3 and 2.4 are useful for testing hypothesis about θ. They do not necessarily give confidence intervals. Theorems 2.3(b) and 2.3(c) are useful for computation of a confidence interval.

REMARK 3. To obtain bounds of the order $T^{-1/2}(\log T)^{1/2}$ in Theorem 2.3, one needs $n > T^{5/2}(\log T)^{-3/2}$. To obtain bounds of the order $T^{-1/2}(\log T)^{1/2}$ in Theorem 3.1, one needs $n > T^{9/4}(\log T)^{-3/4}$. To obtain the order $T^{-1/2}$ in Theorem 2.4, one needs $n \geq T^{10}$. To obtain bound of the order $T^{-1/2}$ in Theorem 3.2, one needs $n > T^{7/4}$. To obtain bound of the order $1/T$ in Theorem 2.5, one needs $n \geq T$.

REMARK 4. The normings in Theorems 2.3 and 3.2 are random and also contain the unknown parameter. It would be interesting to find the order $\max(T^{-1/2}, (T/n)^{1/3})$ and $\max(T^{-1/2}, (T^3/n^2)^{1/3})$, respectively, for the estimators $\theta_{n,T}$ and $\tilde{\theta}_{n,T}$ with completely random or completely nonrandom norming.

REMARK 5. Note that if we transform the Itô integral to the Rubin-Fisk-Stratonovich (RFS) integral in L_T and then apply RFS type approximation for the RFS integral and rectangle rule approximation for the ordinary in L_T, then also we obtain the approximate likelihood $\tilde{L}_{n,T}$. Using these ideas for the nonlinear stochastic differential equations, we have studied the asymptotic properties AMLEs and other estimators (except the rates) (see [14]).

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