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A NEW APPROACH TO EFFICIENT CHANNEL ASSIGNMENT FOR HEXAGONAL CELLULAR NETWORKS

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ABSTRACT

Given a hexagonal cellular network with specific demand vector and frequency separation constraints, we introduce the concept of a *critical block* of the network, that leads us to an efficient channel assignment scheme for the whole network. A novel idea of partitioning the critical block into several smaller sub-networks with homogeneous demands has been introduced which provides an elegant way of assigning frequencies to the critical block. This idea of partitioning is then extended for assigning frequencies to the rest of the network. The proposed algorithm provides an optimal assignment for all well-known benchmark instances including the most difficult two. It is shown to be superior to the existing frequency assignment algorithms, reported so far, in terms of both bandwidth requirement and computation time.

Keywords: Channel assignment problem, lower bound, bandwidth, frequency separation constraints, k-band buffering.

1. Introduction

In a mobile cellular network, each cell of the network is assigned a set of channels to provide services to the individual calls of the cell. The *Channel Assignment Problem* (CAP) deals with the task of assigning frequency channels to the cells satisfying some frequency separation constraints to avoid channel interference and using as small bandwidth as possible. We are considering here the static model of the channel assignment problem, where the number of calls to each cell is known *a priori*. For a network, the available radio spectrum is divided into non-overlapping frequency bands. We assume that the frequency bands are of equal length and are numbered as $0, 1, 2, \cdots$ from the lower end. Each such frequency band is termed as a channel. In this context, the terms *channel assignment* and *frequency assignment* will be used interchangeably in our discussions. The highest numbered channel required in an assignment problem is termed as the required bandwidth. Three types of interference are generally taken into consideration in the form of constraints: i) *co-channel constraint*, due to which the same channel is not allowed to be assigned to certain pairs of cells simultaneously, ii) *adjacent channel constraint*, for which adjacent channels are not allowed to be assigned in certain pairs of cells simultaneously, and iii) *co-site constraint*, which implies that any pair of channels assigned to the same cell must be separated by a certain number [24].

In its most general form, the channel assignment problem is equivalent to the generalized graph-coloring which is a well-known NP-complete problem [8]. The cellular network is often modeled as a graph and the channel assignment problem has been formulated as a graph coloring problem by several authors [10, 22, 26]. In all these studies the graph used to model the cellular network ignores the geometry of the network. Some authors [9, 15, 18, 20, 21] have, however, considered the geometry of the network and solved the channel assignment problem optimally in some cases. In [20], Sen, Roxborough and Medidi presented three channel assignment algorithms taking the hexagonal cell structure into account. The first considered only co-channel constraint and the remaining two considered both the co-channel and adjacent channel constraints. Approximate algorithms using neural networks [5, 12, 14, 23], simulated annealing [4, 17] and genetic algorithms [2, 16, 19], have also been proposed to solve this problem.

In [2, 5, 11, 12, 13, 19, 22, 24, 25, 26] authors have used their assignment algorithms on eight well-known benchmark instances for the given channel demands on hexagonal cells. It is quite easy to derive the optimal solution for the six benchmark instances other than problems 2 and 6, because in all these six cases the required number of channels is primarily limited by the co-channel interference constraint. Most difficult is, however, to get the optimal solution for the other two benchmark instances - problems 2 and 6 [1, 2]. For instance, the optimal assignment for problem 6 needs 253 channels, whereas the assignment algorithm given in [19] requires 165 hours on an unloaded HP Apollo 9000/700 workstation, to produce only a non-optimal solution with 268 channels. Later, however, the authors in [2] proposed an algorithm which provided an optimal solution for both the problems 2 and 6 with a running time of 8 and 10 minutes respectively, on the same workstation. Recently, an algorithm for CAP called FESR (Frequency Exhaustive Strategy with Rearrangement) has been proposed in [25] which produces only non-optimal solutions to the benchmark problems 2 and 6. A Randomized Saturation Degree (RSD) heuristic reported in [1] also produces non-optimal solutions for both the problems 2 and 6. However, combining the RSD heuristic with a Local Search (LS) algorithm, the authors in [1] were able to find an optimal solution for problem 2 but not for problem 6. Most recently, the heuristic algorithm in [3] also produces non-optimal results for problems 2 and 6 both.

In this paper, we first introduce the notion of a *critical block* of cellular network of hexagonal structure with a *2-band buffering*, where the interference does not extend beyond two cells. Next, we present an algorithm for finding the critical block of the cellular network, followed by the introduction of a novel idea of partitioning the critical block into several smaller sub-networks with homogeneous demands, using integer programming. This partition makes the frequency assignment to the critical block very simple. After assigning frequencies to the cells of the critical block, we extend the partitioning technique further to consider the assignment for the remaining cells of the network.

The proposed algorithm provides an optimal assignment for all eight well-known benchmark instances including the most difficult two, i.e., problems 2 and 6. Using our proposed assignment algorithm, we need, on an average, only a few seconds for channel assignment of all the six benchmark instances other than problems 2 and 6, on an unloaded Sun Ultra 60 workstation. For the benchmark problems 2 and 6, however, our algorithm needs only 60 seconds and 72 seconds of running time, respectively on the same workstation. These running times may be contrasted with 8 minutes and 10 minutes, respectively on an unloaded HP Apollo 9000/700 workstation, as reported in [2].

The rest of the paper is organized as follows. Section 2 describes the basic model and the preliminaries. Section 3 presents the algorithm for assigning channels to a given *distance-2 clique*. In section 4, the notion of a critical block is introduced. Section 5 describes the algorithm for assigning frequency channels to the entire cellular network. Simulation results and its comparison with other well-known CAP algorithms are discussed in section 6. Concluding remarks are included in section 7.

2. Preliminaries

Here, we first present the general model for Channel Assignment Problem (CAP) for any arbitrary cellular network. Next, considering the regular geometry of the cellular network, we describe a notational framework for the concepts developed later.

2.1. General Model of CAP

We use here the same model as described in [10, 21, 22], which consists of the following components:

1. The number of distinct cells, say n, with cell numbers as $0, 1, ..., n-1$.
2. A demand vector $W = (w_i)$ $(0 \le i \le n-1)$ where w_i represents the number of channels required for cell i.
3. A frequency separation matrix $C = (c_{ij})$ where c_{ij} represents the frequency separation requirement between a call in cell i and a call in cell j $(0 \le i, j \le n-1)$.
4. A frequency assignment matrix $\phi = (\phi_{ij})$, where ϕ_{ij} represents the frequency assigned to call j in cell i $(0 \le i \le n-1, 0 \le j \le w_i - 1)$. The assigned frequencies ϕ_{ij}'s are assumed to be evenly spaced, and can be represented by integers ≥ 0.
5. A set of frequency separation constraints specified by the frequency separation matrix : $|\phi_{ik} - \phi_{jl}| \ge c_{ij}$ for all i, j, k, l (except when both $i = j$ and $k = l$).

The goal of the channel assignment problem is to assign frequencies to the cells satisfying the frequency separation constraints, as specified by the component 5 above, in such a manner that the required system bandwidth becomes optimal.

Call j in cell i is represented as a node (ij) of a graph and the nodes (ij) and (kl) are connected by an edge with weight c_{ik}, if $c_{ik} > 0$. We call this graph as a Channel Assignment Problem (CAP) graph following the terminology in [21]. In our model, we assume that the channels are assigned to the nodes of the CAP graph in a specific order and a node will be assigned the channel corresponding to the smallest integer that will satisfy the frequency separation constraints with all the previously assigned nodes. Suppose there are m nodes in the CAP graph. Therefore, the nodes can be ordered in $m!$ ways and hence for sufficiently large m, it is impractical to find the best ordering by an exhaustive search due to exponentially increasing computation time. Usually, algorithms are developed so as to find the optimal or at least a near-optimal solution to the channel assignment problem within a reasonable amount of computation time.

2.2. Cellular Graph and Distance-2 Clique

The above model represents the CAP in its most general form. However, the regular geometry of hexagonal cellular network enables us to reformulate the problem. Here follow some definitions for the cellular network having a regular geometry.

Definition 1 *The cellular graph is a graph where each cell of the cellular network is represented by a node and two nodes have an edge between them if the corresponding cells are adjacent to each other (i.e., when the two cell boundaries share a common segment) [20].*

Definition 2 *The cellular network is said to belong to a k-band buffering system if it is assumed that the interference does not extend beyond k cells from the call originating cell [20].*

We assume that the calls in the same cell should be separated by at least s_0 and the calls in the cells those are distance i apart should be separated by at least s_i, $1 \leq i \leq k$, for avoiding channel interference.

Definition 3 *Suppose $G = (V, E)$ is a cellular graph. A subgraph $G' = (V', E')$ of the graph $G = (V, E)$ is defined to be a distance-k clique, if every pair of nodes in G' is connected in G by a path of length at most k [20].*

In all our later discussions, we assume that the cellular graph is hexagonal in nature with a 2-band buffering restriction.

Definition 4 *A distance-2 clique with 7 nodes is defined as a complete distance-2 clique of the hexagonal cellular network, and the node at distance-1 from all other remaining nodes is termed as the central node of that distance-2 clique. Nodes other than the central node are termed as the peripheral nodes.*

Example 1 *Fig. 1(a) shows a complete distance-2 clique of a hexagonal cellular structure, where node 4 is its central node, and all other nodes are peripheral nodes.*

Definition 5 *In any distance-2 clique, joining the node pairs at distance-2 by dashed edges, we generate the graph defined here as the cellular clique Q_2. The*

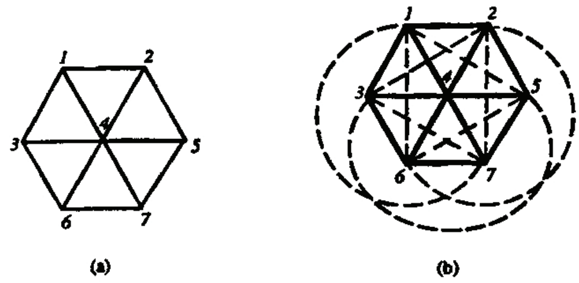

Fig. 1. (a) A complete distance-2 clique G (b) A complete cellular clique Q_2.

cellular clique generated from a complete distance-2 clique is defined as a complete cellular clique.

Example 2 *Fig. 1(b) shows the complete cellular clique Q_2 corresponding to the complete distance-2 clique G shown in Fig. 1(a).*

2.3. Clique-Classes

Fig. 1(b) shows that any cellular clique consists of two types of edge sets - i) set E_1 for connecting the node pairs at distance-1 (shown by solid lines), and ii) set E_2 for connecting the node pairs at distance-2 (shown by dashed lines). We represent it as $Q_2(V, E_1 \cup E_2)$. With reference to Q_2, we extend the usual definitions of an *induced subgraph*, and *graph isomorphism* in the following way:

Definition 6 *Given any graph $G = (V, E_1 \cup E_2)$, E_1, E_2 being the sets of two types of edges (solid and dashed), for any V' ($V' \subseteq V$), $G' = (V', E_1' \cup E_2')$ is the induced subgraph of G if and only if $E_1' \subseteq E_1$ contains all the solid edges existing in G between two nodes $v_i, v_j \in V'$, and $E_2' \subseteq E_2$ contains all the dashed edges of G existing between two nodes $v_p, v_q \in V'$.*

Definition 7 *Two graphs $G = (V, E_1 \cup E_2)$ and $G' = (V', E_1' \cup E_2')$ are said to be isomorphic to each other, if there is a one-to-one correspondence between their vertices and between their edges of respective types, such that the incidence relationship is preserved for both.*

Example 3 *Fig. 2 shows three induced subgraphs of cellular clique Q_2 shown in Fig. 1(b), where Figs. 2(a) and 2(b) are isomorphic to each other, but Fig. 2(c) is not isomorphic to any of them.*

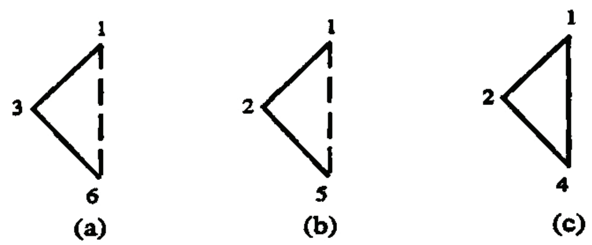

Fig. 2. Three induced subgraphs of Q_2.

Let us denote the set of nodes in the complete cellular clique Q_2 of Fig. 1(b) by $V = \{1, 2, 3, 4, 5, 6, 7\}$, and let $S^{(i)}$ be the set of all possible subsets of i nodes taken from V. That is, $S^{(i)} = \{V^{(i)} : V^{(i)} \subseteq V$ and $|V^{(i)}| = i, 1 \leq i \leq 7\}$. Clearly $|S^{(i)}| =$

${}^7C_i$, $1 \le i \le 7$. Suppose, $V' \in S^{(i)}$. A conflict-free assignment of w frequencies to each of the nodes in V' will be termed as a *homogeneous assignment* with weight w and will be denoted by $A_w(V')$. The minimum bandwidth required for $A_w(V')$ actually depends on the connection pattern of the nodes in the subgraph of Q_2 induced by the nodes in V'.

Definition 8 *All subsets of nodes $V^{(i)} \in S^{(i)}, 1 \le i \le 7$, are classified in r_i disjoint clique-classes, say, $C^{(i)}(1), C^{(i)}(2), \cdots, C^{(i)}(r_i)$ so that two node sets $V_1^{(i)}$ and $V_2^{(i)}$ belong to the same clique-class if and only if the subgraphs of Q_2 induced by $V_1^{(i)}$ and $V_2^{(i)}$ are isomorphic to each other.*

Example 4 *: Consider $V' = \{1,3,6\}$, $V'' = \{1,2,5\}$, and $V''' = \{1,2,4\}$ belonging to $S^{(3)}$ of the complete cellular clique Q_2. The corresponding induced subgraphs are shown in Fig. 2. As already mentioned, the graphs of Figs. 2(a) and 2(b) are isomorphic to each other, but Fig. 2(c) is not. Therefore, V' and V'' belong to the same clique-class, while V''' belongs to a different clique-class.*

The member sets in different clique-classes corresponding to each $S^{(i)}$, $1 \le i \le 7$, for the complete cellular clique of Fig. 1(b) are shown in *Table 1*.

With reference to *Table 1*, the u^{th} subset of nodes of the class $C^{(i)}(j)$, is denoted as $V_u^{(i)}(j)$ $(1 \le i \le 7, 1 \le j \le r_i, 1 \le u \le |C^i(j)|)$.

Remark 1 *Given any distance-2 clique G of the hexagonal cellular network, even if it is not complete, Table 1 can be used to identify the corresponding clique classes just by deleting the nodes which are not present in G.*

2.4. Class-Bandwidth and Increments

As the subgraphs of the complete cellular clique Q_2 induced by the elements in each $C^{(i)}(j)$ $(1 \le j \le r_i, 1 \le i \le 7)$ are all isomorphic to each other, the minimum bandwidth required for the assignment $A_1(V')$ is same for all $V' \in C^{(i)}(j)$.

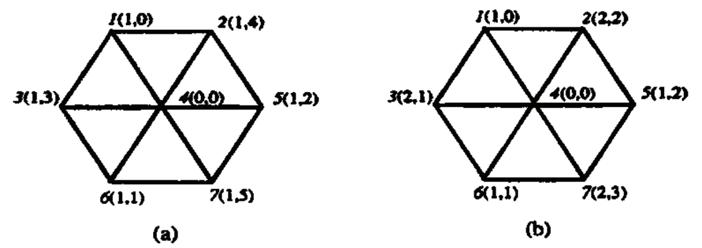

Fig. 3. Different frequency assignments to a complete distance-2 clique for (a) $s_2 \le s_1 \le 2s_2$ (b) $s_1 \ge 2s_2$.

Definition 9 *Given any $V' \in C^{(i)}(j)$, the minimum bandwidth required for the assignment $A_1(V')$ is defined as the class bandwidth for the class $C^{(i)}(j)$, and is denoted by $P^{(i)}(j)$ when $s_2 \le s_1 \le 2s_2$, and $P'^{(i)}(j)$ when $s_1 \ge 2s_2$.*

In [6], it has been shown that *the minimum bandwidth* required for assigning channels to a complete distance-2 clique of a hexagonal cellular network having homogeneous demand of only one channel and 2-band buffering, with frequency separation $s_1 \ge s_2$, is $(s_1 + 5s_2)$ when $s_2 \le s_1 \le 2s_2$, and $(2s_1 + 3s_2)$ when $s_1 \ge 2s_2$. The corresponding assignments are shown in Figs. 3(a) and 3(b) respectively, where the label (α, β) associated with a node indicates that a frequency $(\alpha s_1 + \beta s_2)$ is assigned to that node.

Table 1. Class bandwidths and increments of different clique-classes for a complete cellular clique.

$V^{(i)}$	$C^{(i)}(j)$	$V_u^{(i)}(j)$	$P^{(i)}(j)$	$P'^{(i)}(j)$	$I^{(i)}(j)$	$J^{(i)}(j)$	$s_0 = 5$, $s_1 = 2$, $s_2 = 1$
$V^{(1)}$	$C^{(1)}(1)$	[{1}, {2}, {3}, {4}, {5}, {6}, {7}]	0	0	s_0	s_0	5
$V^{(2)}$	$C^{(2)}(1)$	[{1, 2}, {1, 3}, {1, 4}, {2, 4}, {2, 5}, {3, 4}, {3, 6}{4, 5}, {4, 6}, {4, 7}, {5, 7}, {6, 7}]	s_1	s_1	$max(s_0, 2s_1)$	$max(s_0, 2s_1)$	5
	$C^{(2)}(2)$	[{1, 5}, {1, 6}, {1, 7}, {2, 3}, {2, 6}, {2, 7}, {3, 5}, {3, 7}, {5, 6}]	s_2	s_2	$max(s_0, 2s_2)$	$max(s_0, 2s_2)$	5
$V^{(3)}$	$C^{(3)}(1)$	[{1, 4, 5}, {1, 4, 6}, {1, 4, 7}, {2, 4, 3}, {2, 4, 6}, {2, 4, 7}, {3, 4, 5}, {3, 4, 7}, {5, 4, 6}, {1, 2, 5}, {2, 5, 7}, {5, 7, 6}, {7, 6, 3}, {6, 3, 1}, {3, 1, 2}]	$s_1 + s_2$	$s_1 + s_2$	$max(s_0, 2s_1 + s_2)$	$max(s_0, 2s_1 + s_2)$	5
	$C^{(3)}(2)$	[{1, 7, 2}, {1, 7, 5}, {1, 7, 3}, {1, 7, 6}, {2, 6, 1}, {2, 6, 3}, {2, 6, 5}, {2, 6, 7}, {3, 5, 1}, {3, 5, 2}, {3, 5, 6}, {3, 5, 7}]	$2s_2$	s_1	$max(s_0, s_1 + 2s_2)$	$max(s_0, 2s_1)$	5
	$C^{(3)}(3)$	[{1, 2, 4}, {2, 5, 4}, {4, 5, 7}, {4, 7, 6}, {4, 6, 3}, {4, 3, 1}]	$2s_1$	$2s_1$	$max(s_0, 3s_1)$	$max(s_0, 3s_1)$	6
	$C^{(3)}(4)$	[{1, 5, 6}, {2, 3, 7}]	$2s_2$	$2s_2$	$max(s_0, 3s_2)$	$max(s_0, 3s_2)$	5
$V^{(4)}$	$C^{(4)}(1)$	[{1, 2, 4, 6}, {1, 2, 4, 7}, {2, 4, 5, 3}, {2, 4, 5, 6}, {4, 5, 7, 1}, {4, 5, 7, 3}, {4, 6, 7, 2}, {4, 6, 7, 1}, {3, 4, 6, 2}, {3, 4, 6, 5}, {1, 3, 4, 5}, {1, 3, 4, 7}]	$s_1 + 2s_2$	$2s_1$	$max(s_0, 2s_1 + 2s_2)$	$max(s_0, 3s_1)$	6
	$C^{(4)}(2)$	[{1, 2, 4, 5}, {1, 2, 4, 3}, {2, 4, 5, 7}, {4, 5, 7, 6}, {4, 6, 7, 3}, {3, 4, 6, 1}]	$2s_1 + s_2$	$2s_1 + s_2$	$max(s_0, 3s_1 + s_2)$	$max(s_0, 3s_1 + s_2)$	7
	$C^{(4)}(3)$	[{1, 2, 5, 7}, {1, 3, 6, 7}, {6, 3, 1, 2}, {6, 7, 5, 2}, {3, 1, 2, 5}, {3, 6, 7, 5}]	$3s_2$	$s_1 + s_2$	$max(s_0, s_1 + 3s_2)$	$max(s_0, 2s_1 + s_2)$	5
	$C^{(4)}(4)$	[{1, 2, 6, 7}, {3, 6, 2, 5}, {1, 3, 5, 7}]	$3s_2$	$s_1 + s_2$	$max(s_0, 4s_2)$	$max(s_0, 2s_1)$	5
	$C^{(4)}(5)$	[{1, 5, 7, 6}, {2, 3, 6, 7}, {5, 1, 3, 6}, {7, 3, 1, 2}, {6, 1, 2, 5}, {3, 2, 5, 7}]	$3s_2$	$s_1 + s_2$	$max(s_0, s_1 + 3s_2)$	$max(s_0, 2s_1 + s_2)$	5
	$C^{(4)}(6)$	[{1, 4, 5, 6}, {2, 4, 3, 7}]	$s_1 + 2s_2$	$s_1 + 2s_2$	$max(s_0, 2s_1 + 2s_2)$	$max(s_0, 2s_1 + 2s_2)$	6
$V^{(5)}$	$C^{(5)}(1)$	[{1, 2, 4, 6, 7}, {2, 4, 5, 3, 6}, {1, 3, 4, 5, 7}]	$s_1 + 3s_2$	$2s_1 + s_2$	$max(s_0, 2s_1 + 3s_2)$	$max(s_0, 3s_1 + s_2)$	7
	$C^{(5)}(2)$	[{3, 4, 5, 6, 7}, {3, 4, 5, 1, 2}, {1, 4, 7, 3, 6}, {1, 4, 7, 2, 5}, {2, 4, 6, 1, 3}, {2, 4, 6, 5, 7}]	$s_1 + 3s_2$	$2s_1 + s_2$	$max(s_0, 2s_1 + 3s_2)$	$max(s_0, 3s_1 + s_2)$	7
	$C^{(5)}(3)$	[{1, 3, 4, 6, 5}, {3, 6, 7, 4, 2}, {6, 7, 5, 4, 1}, {2, 5, 7, 4, 3}, {1, 2, 5, 4, 6}, {3, 1, 2, 4, 7}]	$s_1 + 3s_2$	$2s_1 + s_2$	$max(s_0, 2s_1 + 3s_2)$	$max(s_0, 3s_1 + s_2)$	7
	$C^{(5)}(4)$	[{1, 2, 3, 5, 6}, {1, 2, 3, 6, 7}, {1, 3, 6, 7, 5}, {3, 6, 7, 5, 2}, {6, 7, 5, 2, 1}, {7, 5, 2, 1, 3}]	$4s_2$	$s_1 + 2s_2$	$max(s_0, 5s_2)$	$max(s_0, 2s_1 + s_2)$	5
$V^{(6)}$	$C^{(6)}(1)$	[{1, 2, 3, 5, 6, 7}]	$5s_2$	$s_1 + 3s_2$	$max(s_0, 6s_2)$	$max(s_0, 2s_1 + 2s_2)$	6
	$C^{(6)}(2)$	[{1, 2, 3, 4, 5, 6}, {1, 2, 3, 4, 5, 7}, {1, 2, 3, 4, 6, 7}, {1, 2, 4, 5, 6, 7}, {1, 3, 4, 5, 6, 7}, {2, 3, 4, 5, 6, 7}]	$s_1 + 4s_2$	$2s_1 + 2s_2$	$max(s_0, 2s_1 + 4s_2)$	$max(s_0, 3s_1 + 2s_2)$	8
$V^{(7)}$	$C^{(7)}(1)$	[{1, 2, 3, 4, 5, 6, 7}]	$s_1 + 5s_2$	$2s_1 + 3s_2$	$max(s_0, 2s_1 + 5s_2)$	$max(s_0, 3s_1 + 3s_2)$	9

Note that, the class bandwidth for every class can be found by suitably rearranging the assignments of Figs. 3(a) and 3(b) for the cases when $s_2 \leq s_1 \leq 2s_2$ and $s_1 \geq 2s_2$, respectively. All the class bandwidths of the respective classes are shown in *Table 1*.

Example 5 *For $s_2 \leq s_1 \leq 2s_2$, the class bandwidth $P^{(5)}(4)$ for the clique-class $C^{(5)}(4)$ is $4s_2$. Two possible assignments for $A_1(V')$ where $V' = \{1,2,3,5,6\} \in C^{(5)}(4)$ are shown in Figs. 4(a) and 4(b). These assignments are obtained by suitably rearranging the assignments shown in Fig. 3(a). Similarly, for $s_1 \geq 2s_2$, two assignments of the same node subset are shown in Figs. 5(a) and 5(b). The corresponding class bandwidth is $P'^{(5)}(4) = s_1+2s_2$. These assignments are obtained by suitably rearranging the assignments shown in Fig. 3(b).*

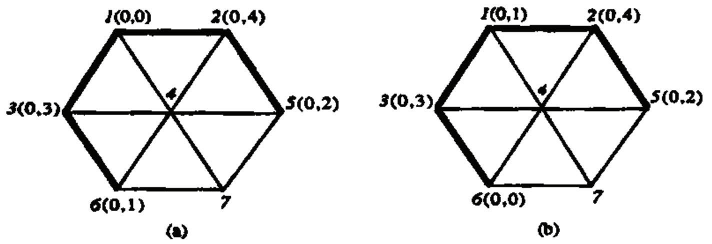

Fig. 4. Two different assignments $A_1(V')$ with $V' = \{1,2,3,5,6\}$ for $s_2 \leq s_1 \leq 2s_2$.

Let us now consider an assignment $A_1(V')$ where $V' \in C^{(i)}(j)$. In the first round, a single channel is assigned to each node, using bandwidth equal to the respective class bandwidth $P^{(i)}(j)$. In such an assignment, say u and v are two nodes which are assigned the minimum and maximum frequencies 0 and $P^{(i)}(j)$ respectively. In case of multiple channel demand, our idea is to assign the channels in the second round, starting from the node u again and following the same order as it was in the first round and so on. In that case, *increment*, i.e., the minimum frequency by which we can start again at node u without conflicting the already assigned frequencies, depends on: i) the distance between nodes u and v and ii) the relative values of s_0, s_1 and s_2, and it can be defined formally as follows:

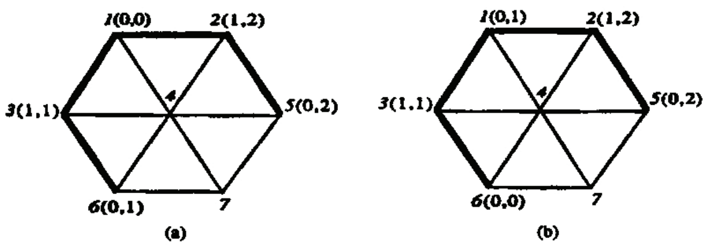

Fig. 5. Two different assignments $A_1(V')$ with $V' = \{1,2,3,5,6\}$ for $s_1 \geq 2s_2$.

Definition 10 *For a given subset of nodes $V' \in C^{(i)}(j)$, let u and v be two nodes which are assigned the minimum (zero) and maximum ($P^{(i)}(j)$) frequencies, respectively in an assignment $A_1(V')$. Then an assignment $A_1^*(V')$ is defined as the optimal partition assignment, if it requires a bandwidth equal to the class bandwidth $P^{(i)}(j)$, with u and v being farthest apart. If $I^{(i)}(j)$ is the minimum frequency that can be assigned next to the node u for $A_2(V')$ without any conflict to the already assigned frequencies in $A_1^*(V')$, then we define $I^{(i)}(j)$ as the increment for the clique-class $C^{(i)}(j)$.*

Example 6 *Figs. 4(a) and 4(b) show two different assignments for $A_1(V')$, where $V' = \{1,2,3,5,6\} \in C^{(5)}(4)$. Both require the minimum bandwidth $P^{(5)}(4)=4s_2$. But the distance between nodes u(minimum frequency) and v(maximum frequency) are 1 and 2 in Figs. 4(a) and (b) respectively. Thus, for the next round frequency* *assignment in Fig. 4(a), the minimum frequency by which we can start at node 1 (i.e., node u) is* $max(s_0, s_1 + 4s_2)$*; while in Fig. 4(b), the minimum by which we can start at node 6 (i.e., node u) is* $max(s_0, 5s_2)$.

Therefore, the assignment of Fig. 4(b) is the *optimal partition assignment* and $max(s_0, 5s_2)$ is the *increment* of class $C^{(5)}(4)$.

Similarly, for $s_1 \geq 2s_2$, where the class bandwidth is $P'^{(i)}(j)$, the *increment* $J^{(i)}(j)$ can be defined accordingly. Figs. 5(a) and 5(b) show two different assignments for $s_1 \geq 2s_2$ with the same bandwidth $P'^{(i)}(j) = s_1 + 2s_2$. But the assignment of Fig. 5(b) is the *optimal partition assignment* with corresponding increment $J^{(i)}(j) = max(s_0, 2s_1 + s_2)$.

Remark 2 *As all the members of a particular class* $C^{(i)}(j)$ *are isomorphic to each other, the increment for all the members of a particular class is same. We compute all* $I^{(i)}(j)$ *and* $J^{(i)}(j)$, $(1 \leq j \leq r_i, 1 \leq i \leq 7)$, *for the complete distance-2 clique and present these values in Table 1.*

From now onwards, we will mention the results for $s_1 \leq 2s_2$ with increment $I^{(i)}(j)$ only. Similar results would also be true for $s_1 \geq 2s_2$ with increment $J^{(i)}(j)$.

After getting the optimal partition assignment $A_1^*(V')$ where $V' \in C^{(i)}(j)$, and the increment $I^{(i)}(j)$ for class $C^{(i)}(j)$, the multiple weight assignment can be obtained by the following result.

Lemma 1 *For the multiple weight assignment* $A_w(V')$, $w \geq 1$, *the node u can be successively assigned the frequencies* $0, I^{(i)}(j), 2I^{(i)}(j), \ldots, (w-1)I^{(i)}(j)$. *Similarly, each of the remaining nodes in* V' *can be assigned w frequencies with successive gaps of* $I^{(i)}(j)$, *giving rise to a total bandwidth requirement of* $(P^{(i)}(j) + (w-1)I^{(i)}(j))$.

Proof : See [7].

Next we present some lower bounds on the bandwidth requirement of a distance-2 clique.

2.5. Bandwidth-Bounds for Distance-2 Clique

Before describing the algorithm to assign channels to the distance-2 clique, it is necessary to know the lower bounds on the minimum number of frequencies needed for its assignment to check the optimality of the solutions achieved. For a complete distance-2 clique G, the minimum bandwidth ($B_{min}^{(h)}$) required to satisfy a homogeneous demand w, has been derived in [6]. Let us now consider the complete distance-2 clique G with non-homogeneous demand vector $W = (w_i)$ (w_i being the channel requirement for cell i) where $w = max(w_i)$, $i = 1, 2, \cdots, 7$. It is evident that the minimum bandwidth ($B_{min}^{(h)}$) required to satisfy the homogeneous demand w, is an *upper bound* on the minimum bandwidth requirement B_{min} of G with the demand vector $W = (w_i)$. It is also evident that a *lower bound* on B_{min} for G is $(w-1)s_0$. However, this lower bound is not always tight for all values of s_0, s_1, s_2, and W. We find a tighter *lower bound* on bandwidth for the general case in the following way:

Lemma 2 *A lower bound on* B_{min} *for G with demand vector* $W = (w_i)$, *where* $w = max(w_i)$, $i = 1, 2, \cdots, 7$, *is :*

1. $max((w-1)s_0, (\sum_{i=1}^{7} w_i - 1)s_2 + (s_0 - s_2)(w_4 - 2) + 2(s_1 - s_2))$ *for* $s_1 \leq s_0 \leq (2s_1 - s_2)$, *and*
2. $max((w-1)s_0, (\sum_{i=1}^{7} w_i - 1)s_2 + 2(s_1 - s_2)(w_4 - 2) + 2(s_1 - s_2))$ *for* $s_0 \geq (2s_1 - s_2)$.

Proof : See [7].

It does not, however, necessarily mean that there always exists a conflict-free assignment of G with the lower bound given above.

3. Frequency Assignment of a Distance-2 Clique

We present an algorithm for assigning frequencies to the nodes of a given distance-2 clique G_2 with a demand vector $W = (w_i)$, $1 \le i \le 7$ and the frequency separation constraints s_0, s_1, and s_2. The assignment is done in two steps. First, we break-up the total demand of G_2 which is non-homogenous in general, in terms of homogeneous demands on different subgraphs of G_2. Each such subgraph of G_2 is also a distance-2 clique. This process will be termed as *partitioning of demand into homogeneous subsets* and will be done through an integer programming formulation. After this partitioning of demand is done, the actual assignment of frequencies with homogeneous demands taken together on the appropriate subgraphs of the given distance-2 clique, is done by another algorithm. Finally, all these homogeneous assignments on the appropriate subgraphs of the given distance-2 clique together constitute the non-homogeneous assignments of G_2.

We now present the integer programming (IP) formulation for the partitioning of demand into homogeneous subsets.

3.1. Integer Programming (IP) Formulation

Given any distance-2 clique G_2 with its demand vector $W = (w_p)$ and frequency separation constraints (s_0, s_1, and s_2), we apply IP technique to find the homogeneous weight $X_u^{(i)}(j)$ for the subset $V_u^{(i)}(j)$, for all i, j, u, such that they together satisfy the total demand given by W, and at the same time keeps the required bandwidth minimum. We formulate the problem in the following way:

Minimize $[(\sum_{1\le i\le 7, 1\le j\le r_i, 1\le u\le |C^{(i)}(j)|} I^{(i)}(j) X_u^{(i)}(j))]$
subject to the following constraints :

1. $\sum_{1\le i\le 7, 1\le j\le r_i, 1\le u\le |C^{(i)}(j)|, p\in V_u^{(i)}(j)} X_u^{(i)}(j) = w_p,\ p = 1, 2, \cdots, 7.$
2. All $X_u^{(i)}(j)$'s are integers.

The non-zero $X_u^{(i)}(j)$'s obtained from the solution to the above integer programming problem consists of the partitions of G_2 into several sub-sets with homogeneous demands.

Let the solution set consist of k non-zero $X_u^{(i)}(j)$'s denoted as $\{X_{u_1}^{(i_1)}(j_1), X_{u_2}^{(i_2)}(j_2), \dots, X_{u_k}^{(i_k)}(j_k)\}$ where $1 \le i_m \le 7$, $1 \le j_m \le r_{i_m}$, $1 \le m \le k$, and $1 \le u_m \le |C^{(i_m)}(j_m)|$. For ease of notation we refer to the value of $X_{u_m}^{(i_m)}(j_m)$ by α_m, $1 \le m \le k$, and the complete solution set will be denoted by $\alpha = \{\alpha_1, \alpha_2, \dots, \alpha_k\}$. Let the corresponding set of partitions be $\mathcal{S} = \{V_{u_1}^{(i_1)}(j_1), V_{u_2}^{(i_2)}(j_2), \dots, V_{u_k}^{(i_k)}(j_k)\}$.

Example 7 *Consider the distance-2 clique G_2 shown in Fig. 6(a) where the label $[\alpha]$ associated with a node indicates the demand of that node. Let the frequency separation constraints be $s_0 = 5$, $s_1 = 2$, and $s_2 = 1$. The increments for different classes are taken from the last column of Table 1. The solution to the IP formulation for this problem is: $X_2^{(5)}(1) = 7$, $X_1^{(5)}(4) = 5$, $X_{11}^{(4)}(1) = 3$, $X_{10}^{(4)}(1) = 13$, $X_7^{(3)}(1) = 12$, $X_9^{(3)}(1) = 5$. From Table 1, $\mathcal{S} = \{ V_2^{(5)}(1), V_1^{(5)}(4), V_{11}^{(4)}(1), V_{10}^{(4)}(1), V_7^{(3)}(1),$*

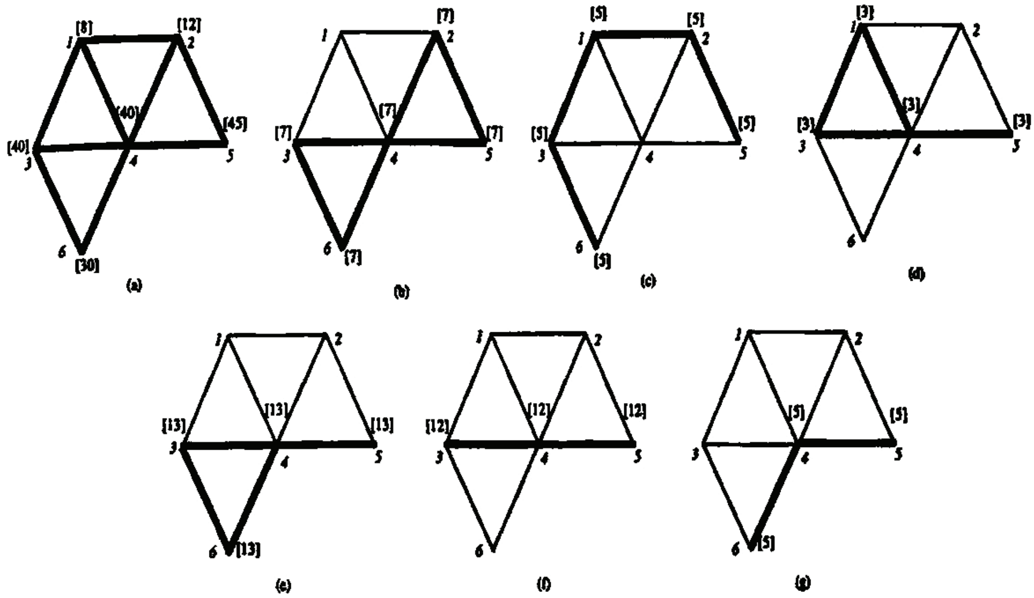

Fig. 6. A distance-2 clique G_2 and its partitions.

$V_9^{(3)}(1)\}$, *with the corresponding subgraphs as shown in Figs. 6(b)-(g). We call these subgraphs of Figs. 6(b)-(g) as partitions* $P_1, P_2, \ldots, P_6$ *respectively.*

After the partitioning of demands, the assignment of frequencies to different nodes is to be done according to an optimal ordering of partitions, as described in the following subsection.

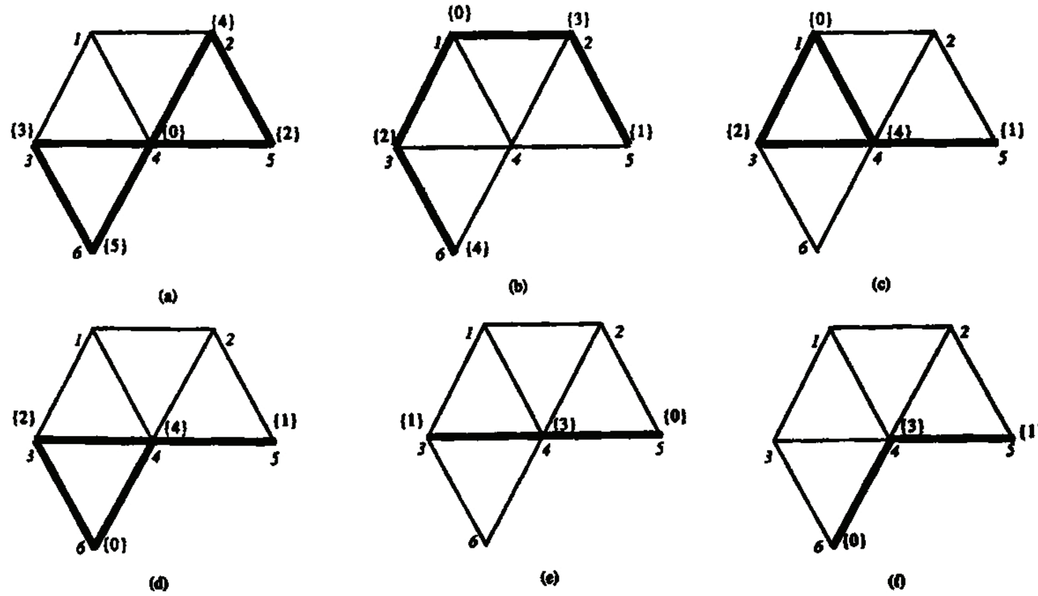

Fig. 7. Optimal partition assignment for G_2.

3.2. Ordering of Partitions

The actual assignment of frequencies to different nodes, following the above partitioning of demands is a bit tricky. We first restrict to the case where assignment for each partition is done exactly once. We would then generalize this for multiple assignments on each partition using *Lemma 1*. For example, the assignment of a single frequency to each of the partitions P_1, P_2, P_3, P_4, P_5, and P_6 separately corresponding to the Example 7, is shown in Figs. 7(a)-(f). Note that each of these assignments in these partitions is done requiring the corresponding class bandwidths of the respective classes. Within a particular partition P_i, $1 \leq i \leq 6$, the assignment of frequencies to the different nodes is, however, not unique. Figs. 7(a)-(f) show only one possible such assignment. Now, to combine these assignments so as to meet the required total demand on each node, we have to assign frequencies to all these partitions in a certain order.

For single assignment to each partition, we observe that assigning frequencies to these partitions in a different order would result in a different bandwidth requirement. For example, if we first assign the frequencies to the nodes of P_1, then the assignment according to the partition P_2 would necessitate a frequency of 6 on node 1, which would finish with a frequency 10 on node 6. Continuing this way for the partitions P_3, P_4, P_5, and P_6 and in this order we would see that the maximum frequency assigned is 31 on node 4 (Fig. 8). On the other hand, the assignments in the order $P_3, P_4, P_6, P_5, P_2, P_1$ would lead to a maximum frequency requirement of 34. Note that, using each of these partitions once in this whole assignment process, we actually assign 2, 2, 5, 5, 6, and 4 channels on nodes 1, 2, 3, 4, 5, and 6 of G_2 respectively. We represent this by another clique G'_2 with demand vector P:(2, 2, 5, 5, 6, 4) as shown in Fig. 9(a). The corresponding assignment is shown in Fig. 9(b).

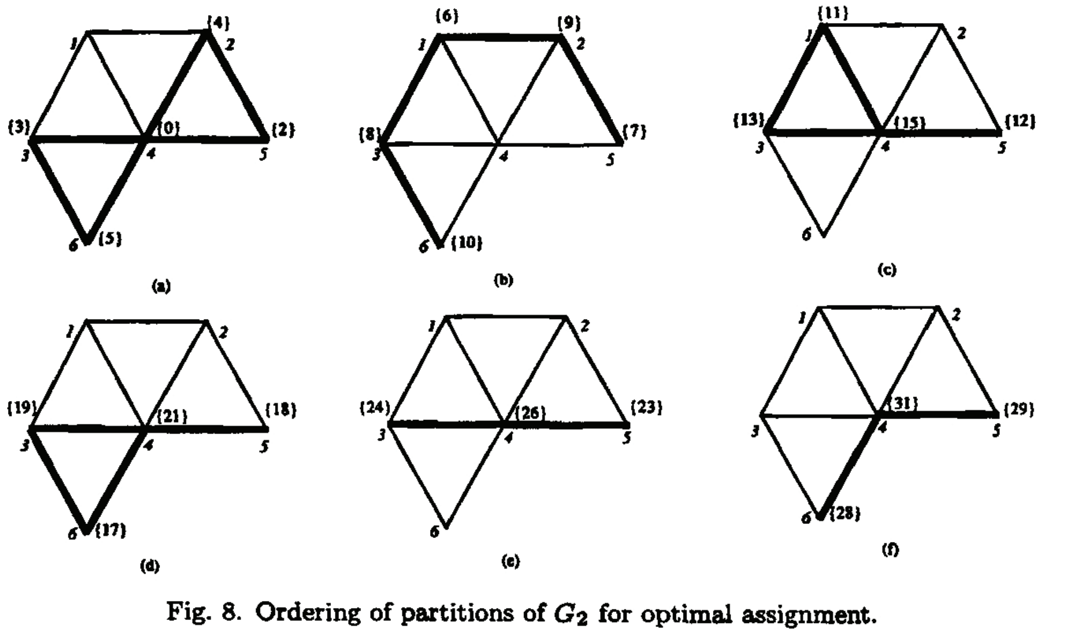

Fig. 8. Ordering of partitions of G_2 for optimal assignment.

We check that the assigned numbers of channels on all the nodes of G'_2 sum to 24. Since $s_0 = 5$, $s_1 = 2$ and $s_2 = 1$, the lower bound on bandwidth required for the assignment of G'_2 can be found from *Lemma 2* as 31 ($= max(5 \times 5, 23 \times 1 + 2 \times 1 \times 3 + 2 \times 1)$), which is the same as obtained in Fig. 9(b). However, this minimum bandwidth may not always be achievable unless we can assign both the minimum and maximum frequencies to the central node 4.

Next let us consider the general case where each partition may be assigned multiple frequencies. Before describing the algorithm, we illustrate the basic ideas with an example. With reference to Example 7, the actual weights for the partitions

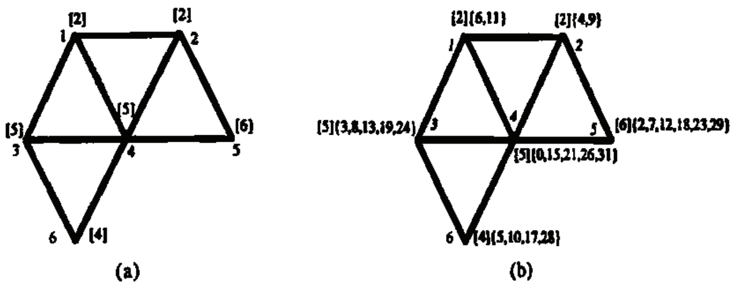

Fig. 9. (a) G_2' with demand vector P (b) Optimal solution for G_2'.

$P_1, P_2, ..., P_6$ of Figs. 6(b)-(f) are 7, 5, 3, 13, 12, and 5 respectively. The assignment shown in Fig. 10(a) is obtained from the assignment of Fig. 8(a) by assigning 7 frequencies to all the nodes of P_1 with successive gaps of 7, since the increment for the class to which the partition P_1 belongs, is equal to $max(s_0, 2s_1 + 3s_2) =$ 7. Similarly, the assignments $A_5(V_1^{(5)}(4))$, $A_3(V_{11}^{(4)}(1))$, $A_{13}(V_{10}^4(1))$, $A_{12}(V_7^3(1))$, and $A_5(V_9^3(1))$ shown in Figs. 10(b)-(f) are obtained from the assignments in Figs. 8(b)-(f) respectively, by changing the starting frequency channel accordingly and maintaining the same ordering $P_1 \to P_2 \to P_3 \to P_4 \to P_5 \to P_6$. All the assignments of Figs. 10(a)-(f) together constitute the assignment for the given distance-2 clique G_2.

It is to be noted that the total demand on all nodes of G_2 of Fig. 6(a) is 175. Since $s_0 = 5$, $s_1 = 2$ and $s_2 = 1$, the lower bound on bandwidth required for the assignment of G_2 can be found from *Lemma 2* as 252 $(= max(44 \times 5, 174 \times 1 + 2 \times 1 \times 38 + 2 \times 1))$. Therefore, the assignment of Figs. 10(a)-(f) for the clique G_2 is optimal.

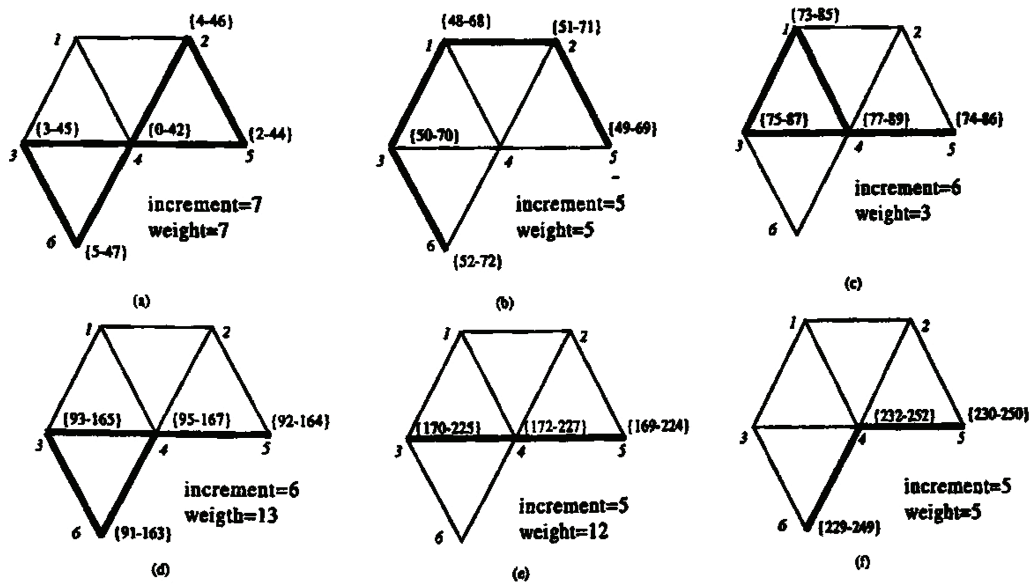

Fig. 10. Optimal assignment of G_2.

3.3. Assignment Algorithm

Once the partitioning of the distance-2 clique G_2 has been done, the following algorithm is used to assign the channels to G_2.

Algorithm Assign_Distance-2_Clique

Input : $\mathcal{S} = \{V_{u_1}^{(i_1)}(j_1), V_{u_2}^{(i_2)}(j_2), ..., V_{u_k}^{(i_k)}(j_k)\}$, $\alpha = \{\alpha_1, \alpha_2, ..., \alpha_k\}$, and the frequency separation constraints s_0, s_1, and s_2.

Output : A conflict free assignment of G_2 and the required bandwidth B^*.

Step 1 : For each $V_{u_m}^{(i_m)}(j_m)$, $1 \leq m \leq k$, find the set of optimal partition assignments $\{A_1^*(V_{u_m}^{(i_m)}(j_m))\}$. Set $D_m \leftarrow \{A_1^*(V_{u_m}^{(i_m)}(j_m))\}$.

Step 2 : Select an assignment from each D_m, $1 \leq m \leq k$, and assuming an arbitrary ordering of partitions combine all these assignments by changing the starting frequency channel accordingly to avoid conflict. Find the minimum bandwidth B_{min} required from all possible selections of A_1^* and all possible ordering of the partitions.

Step 3 : For each m $(1 \leq m \leq k)$, use the value of B_{min} in step 2 to perform the assignment $A_{\alpha_m}(V_{u_m}^{(i_m)}(j_m))$, by giving α_m frequencies to all the nodes in $V_{u_m}^{(i_m)}(j_m)$ with successive gaps of $I^{(i_m)}(j_m)$. Combine all these assignments by changing the starting frequency channel accordingly to avoid conflict, giving rise to the complete assignment for the given distance-2 clique. Return the bandwidth B^* for this assignment, and terminate.

In step 2, to find B_{min}, we may apply exhaustive search to guarantee an optimal solution, since in general, the search space will be very limited, or we may apply heuristics, e.g., GA or other techniques to find an optimal, or near-optimal solution. However, in our simulation procedure, we applied the elitist model of GA presented in [6]. To check the optimality, we present a lower bound on B_{min} in the following remark which may determine the termination criterion for the heuristic.

Remark 3 *Given a distance-2 clique G_2, and the set of partition $\mathcal{S} = \{V_{u_1}^{(i_1)}(j_1), V_{u_2}^{(i_2)}(j_2), ..., V_{u_k}^{(i_k)}(j_k)\}$, let a node i $(1 \leq i \leq 7)$ appear in a_i different partitions of $\mathcal{S}$. Clearly $a_i \leq k$ for all i, $1 \leq i \leq 7$. If we consider a distance-2 clique, say G_2', with demand vector $a = (a_i)$, $1 \leq i \leq 7$, and find out the theoretical lower bound on bandwidth required for G_2' using Lemma 2, then it would also be a lower bound on B_{min}.*

4. Critical Block and its Assignment

Let there be n nodes in the cellular graph of k-band buffering with a demand vector $W = (w_i)$, $1 \leq i \leq n$. Let us consider all possible distance-k cliques of the cellular graph, say $G_1, G_2, ..., G_m$. Let B_j be the *minimum bandwidth* required to assign frequencies to the nodes of the distance-k clique G_j $(1 \leq j \leq m)$ with respect to the given demand vector W and the frequency separation constraints s_i's, $0 \leq i \leq k$.

Definition 11 *Given a cellular graph G with a demand vector W, and the set of all possible distance-k cliques $\{G_j\}$, each with minimum bandwidth requirement B_j, the critical block CB_k is that distance-k clique, whose minimum bandwidth requirement is the maximum of all B_j's.*

Note that the critical block in a cellular graph may or may not be unique depending on the demand vector W and s_i's, $0 \leq i \leq k$. Since, we consider only a 2-band buffering system, we would consider the critical blocks with $k = 2$ only. Here follow the algorithms for identification and frequency assignment of the critical block CB_2 of a network.

Algorithm Find_Assign_Critical_Block

Input : The cellular graph G with demand vector $W = (w_i)$, and frequency separation constraints s_0, s_1 and s_2.

Output : The critical block CB_2 with a conflict free assignment to it.

Step 1 :

For *each node i of the cellular graph* do

begin

Consider the distance-2 clique centered around node i, say D_i.

$N_i \leftarrow$ the set of nodes of D_i.

end

Example 8 *In the benchmark network of Fig. 11, each node has a label of the form* $[x]$, *where* x *is the demand of that node. The values of* s_0, s_1 *and* s_2 *are given as 5, 2, and 1, respectively. In this graph,* $N_0 = \{0,1,6,7\}$, $N_7 = \{0,1,6,7,8,14,15\}$. *Since* $N_0 \subset N_7$, *we will not consider the clique centered around node* 0 *in step 2 below.*

Step 2 :

For *each* D_p *whose node set* N_p *is not a subset of any other* N_q, $p \neq q$ do

begin

$A_p \leftarrow$ the lower bound of D_p obtained by *Lemma 2.*

$B_p \leftarrow$ the upper bound of D_p [6].

end

$max\_lower \leftarrow max\{A_p\}$.

Example 9 *For Fig. 11,* $A_{19} = (95-1) \times 1 + 2 \times 1 \times 18 + 2 \times 1 = 132$ *and* $B_{19} = (25-1) \times 9 + 6 \times 1 = 222$ *[6]. Considering all other* A_p*'s, we see that* A_{10} $((175-1) \times 1 + 2 \times 1 \times 38 + 2 \times 1 = 252)$ *is maximum, i.e., max_lower=252. Since* $B_{19} < max\_lower$, *the clique* D_{19} *is not considered in step 3 below.*

Step 3 :

For *each clique* D_p *whose* B_p *value is* $\geq$ *max_lower* do

begin

Do IP formulation for the clique D_p as described in the previous section and solve it.

$F_p \leftarrow$ the value of the objective function.

Save the values S and α.

end

$F_{max} \leftarrow max\{F_p\}$.

Example 10 *For Fig. 11,* $F_{max} = 255$ $(= 7\times7+5\times5+3\times6+13\times6+12\times5+5\times5)$, *corresponding to the clique centered around node* 10.

Step 4 :

For *each clique whose* F_p *value* $=F_{max}$ do

begin

Consider the partition set S and their corresponding weights α for that clique.

Assign frequency channels to that clique by the algorithm *Assign_Distance-2_Clique* described in section 3.3.

$E_i \leftarrow$ the highest frequency assigned by *Assign_distance-2_clique*

end

$E_{max} \leftarrow max\{E_i\}$.

Step 5 :
Return the distance-2 clique whose $E_i = E_{max}$, along with its assignment, and terminate.

Example 11 *For Fig. 11, the only critical block is the distance-2 clique centered around node* 10, *i.e., consisting of the nodes {3, 4, 9, 10, 11, 17}, which is isomorphic to* G_2 *shown in Fig. 6(a). Hence, it has the same assignment as shown in Fig. 10, with 0-252 channels.*

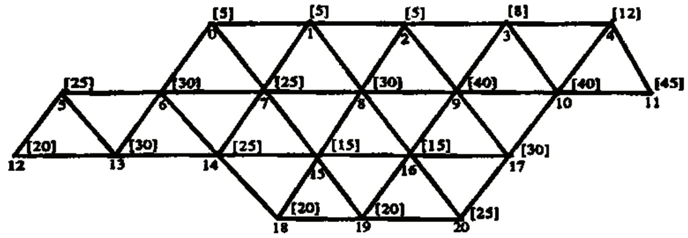

Fig. 11. The benchmark cellular network.

5. Assignment of the Cellular Network

First let us assume that for the given network, demand vector, and frequency separation constraints, there exists only one critical block. Given any cellular network and its demand vector and frequency separation constraints, we identify the critical block CB_2, find the partitions, say (P_1, P_2, $\cdots$ P_k), of the critical block and assign the critical block according to an optimal ordering of partitions, by the technique described in the previous section. Next we extend the partitions P_i, $1 \leq i \leq k$, of CB_2 over the whole network, to find a complete assignment. The exact procedure is described in the following subsections.

5.1. *Partitioning Around Critical Block*

Let all the partitions of the critical block CB_2 be $S = \{P_1, P_2, \cdots, P_k\}$. Suppose $\alpha = \{\alpha_1, \alpha_2, \cdots, \alpha_k\}$ be their corresponding non-zero weights. Now, we consider each partition P_i of CB_2 and try to extend it over the distance-2 cliques around CB_2 and repeat the procedure for the whole network. Initially, the critical block CB_2 with known partitions is termed as the *Partitioned Block PB*, i.e., $PB = CB_2$. In each iteration, PB covers more and more nodes of the cellular graph until it is exhausted. Let us denote the node set of PB as V_{PB}.

Definition 12 *A distance-2 clique* G_2 *is adjacent to a partitioned block PB, if and only if, the center of* G_2 *is a peripheral node of PB.*

Example 12 *Fig. 12 shows a partitioned block PB (marked by solid bold line) formed by the set of nodes {3,4,9,10,11,17}. The set of peripheral nodes is {3, 4, 9, 11, 17}. Then, the distance-2 clique* G_2 *(marked by dashed line) centered around node* 9 *is adjacent to* PB.

Now, for the partition P_i with node set V_{P_i} of PB, let us consider an adjacent distance-2 clique G^i_j with node set V^i_j. In G^i_j, the index i refers to the partition P_i of CB_2 and j refers to the central node of G^i_j which lies on the boundary of PB. Let $A \leftarrow V^i_j \cap V_{P_i}$, $B \leftarrow (V_{PB} \cap V^i_j) \setminus A$. We will consider only those partitions of G^i_j which includes the set A but excludes the set B, and the increment for each partition is less than or equal to that of P_i. Let us denote it as the set AP of partitions *adjacent* to PB. Now, for extension of the partition P_i to cover G^i_j, we will consider the union of P_i with each partition of AP. We refer to this set as the *Candidate Partition Set* for P_i, termed as CP_i. Now, the partitioned block is $PB = PB \cup G^i_j$. Next we are to repeat the procedure for another adjacent distance-2 clique of PB, unless PB covers the whole network.

Example 13 *Fig. 12 shows a subgraph G induced by the node set {2, 3, 4, 8, 9, 10, 11, 16, 17} of the cellular graph shown in Fig. 11. The only critical block CB_2 of G is the distance-2 clique centered around cell* 10, *i.e., consisting of the set of nodes D_{10}={3,4,9,10,11,17}. Initially $PB = CB_2 = D_{10}$. Let us now consider any partition of CB_2, say $P_5 = \{9,10,11\}$. We initially set $CP = P_5$. Consider the adjacent distance-2 clique G^5_9 centered around node* 9, *i.e., consisting of the nodes $D_9 = \{2,3,8,9,10,16,17\}$. Then $D_9 \cap CP = \{2,3,8,9,10,16,17\} \cap \{9,10,11\} = \{9,10\}$ and $(D_9 \cap PB) \setminus \{9,10\} = \{3,17\}$. Let us now consider the set $AP = \{\{9,10\},\{9,10,2\},\{9,10,8\},\{9,10,16\}\}$ of all possible subsets of $D_9 = \{2,3,8,9,10,16,17\}$ which includes $\{9,10\}$ but excludes $\{3,17\}$ and the increment (from Table 1) is less than or equal to that of P_5. Now, for the extension of P_5 to cover D_9, the candidate partition set for P_5 becomes CP_5 = {{9,10,11}, {9,10,11,2}, {9,10,11,8}, {9,10,11,16}}.*

The algorithm is formally described below.

Algorithm Candidate_Partition

Input : The critical block CB_2; set of partitions $S = \{P_1, P_2, \cdots, P_k\}$.
Output : CP_m's $(1 \le m \le k)$, i.e., candidate partition sets for P_i's to be considered for the entire cellular network.

```
For i = 1 to k do
 begin
   PB ← CB_2
   CP ← {V_{P_i}}
   Y = φ
   For adjacent distance-2 clique G^i_j of PB until PB covers the whole network
     begin
       For g = 1 to |CP| do
       /*|CP| is the cardinality of set CP and CP(1), CP(2), ..., CP(|CP|)
       are the elements of CP*/
         begin
           A ← V^i_j ∩ CP(g)
           B ← (V_PB ∩ V^i_j) \ A.
           AP ← {S : S ⊆ V^i_j and S includes the set A but excludes the set B
           and the increment for S is less or equal to that of P_i}
           Y = Y ∪ {CP(g) ∪ AP(1), CP(g) ∪ AP(2), ..., CP(g) ∪ AP(|AP|)}
           /*|AP| is the cardinality of set AP and AP(1), AP(2), ..., AP(|AP|)
           are the elements of AP*/
         end
       CP ← Y
       PB ← PB ∪ G^i_j
     end
   CP_i ← CP
 end
```

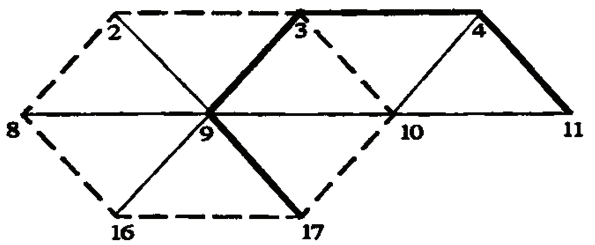

Fig. 12. Adjacent distance-2 clique (dashed line) of partitioned block PB (bold line).

5.2. IP formulation for the whole network

We now know all k *candidate partition sets* for the corresponding k partitions of the critical block. Suppose $y_m(t)$ be the weight associated with the t^{th} partition of CP_m where $1 \leq t \leq |CP_m|$ and $1 \leq m \leq k$. Now, the sum of weights associated with all partitions of CP_m must be equal to the weight of the corresponding partition P_m of the critical block giving rise to k equations of the following Integer Programming (IP). Also, the sum of the weights of all partitions within which the node i belongs to, must be equal to the demand for node i, giving rise to n other equations of the IP, as described below.

Minimize$[\sum_{1\leq m\leq k, 1\leq t\leq |CP_m|} I^{(i_m)}(j_m) y_m(t)]$
subject to the constraints

1. $\sum_{1\leq t\leq |CP_m|} y_m(t) = \alpha_m$, $m = 1, 2, ..., k$
2. $\sum_{1\leq m\leq k, 1\leq t\leq |CP_m|, p\in CP_m(t)} y_m(t) = w_p$, $p = 0, 1, ..., n-1$.
3. $y_m(t)'s$ are integers.

Suppose the non-zero solution to the above IP constitutes the final partition set FP_m=$\{FP_m(t_1)$, $FP_m(t_2)$, ..., $FP_m(t_{\gamma_m})\}$ for P_m with their nonzero weights $y_m = \{y_m(t_1), y_m(t_2), ..., y_m(t_{\gamma_m})\}$, $1 \leq m \leq k$, $1 \leq t_1, t_2, \cdots, t_{\gamma_m} \leq |CP_m|$.

Example 14 *Consider the cellular graph shown in Fig. 11. As already mentioned, the partitions of the critical block CB_2 are given by S=$\{\{4,9,10,11,17\}$, $\{3,4,9,10,11,17\}$, $\{3,9,10,11\}$, $\{9,10,11,17\}$, $\{9,10,11\}$, $\{10,11,17\}\}$ with weights $\alpha = \{7,5,3,13,12,5\}$. Based on these, we get the partitions for the whole network as shown in Table 2. The rows FP_1, FP_2, FP_3, FP_4, FP_5, and FP_6 in Table 2, show the final partitions for the whole network corresponding to the partitions P_1, P_2, P_3, P_4, P_5 and P_6, respectively of the critical block CB_2. For example, the final partition set FP_1 for the partition P_1 of CB_2 has only two partitions shown in Table 2 as rows a(i) and a(ii) with weights 6 and 1 respectively. The sum of the weights 6 and 1 is 7 which is the weight of the partition P_1 of the critical block. This is true for all other partitions of the critical block. Also, any node, say node 20, appears in the partitions corresponding to the rows b, c, d(i), d(ii), d(iv), e(i), e(ii) and e(iv) in Table 2 with respective weights 5,3,3,2,2,5,3 and 2. The sum of these weights is 25 which is equal to the total demand (as mentioned in the last row of Table 2) for the node 20.*

5.3. Assignment algorithm

At this point we know the partitions of the whole network, and their corresponding weights. We would follow a technique similar to that presented in the algorithm *Assign_distance-2_Clique* of section 3.3. We recall that the assignment of the critical block was performed partition by partition with an optimal sequence of the partitions. Here, we maintain the same ordering of partitions existing in CB_2. Corresponding to a particular partition of CB_2, now we have a final partition set for the whole network. An optimal sequence of these partitions for assigning a given final partition set are to be determined again by a heuristic search (e.g., using GA). Next, assignment of the whole network is done partition by partition following the same technique as was followed for assignment of CB_2.

Table 2. Homogeneous partitions of the cellular graph of Fig. 11.

Nodes → $FP_i's$ ↓		0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
FP_1	$a(i)$					6	6	6	6	6	6	6	6		6	6	6	6	6			
	$a(ii)$					1	1	1	1		1	1	1		1	1	1	1	1			
FP_2	b		5		5	5			5		5		5	5	5		5		5			5
FP_3	c				3		3	3	3	3	3	3	3								3	3
FP_4	$d(i)$						3	3	3		3	3	3	3		3			3	3		3
	$d(ii)$						2	2	2		2	2	2	2		2			2	2	2	2
	$d(iii)$							1		1	1	1	1		1	1	1		1		1	
	$d(iv)$							2		2	2	2	2		2	2	2		2			2
	$d(v)$						3			3	3	3	3		3	3			3		3	
	$d(vi)$							2		2	2	2	2		2				2		2	
FP_5	$e(i)$							5	5	5	5	5	5	5	5					5	5	5
	$e(ii)$						3	3			3	3	3	3				3		3		3
	$e(iii)$						2	2		2	2	2	2	2						2	2	
	$e(iv)$						2			2	2	2	2			2					2	2
FP_6	$f(i)$	4		4						4		4	4		4	4		4	4	4		
	$f(ii)$	1		1								1	1		1	1		1	1	1		
Total		5	5	5	8	12	25	30	25	30	40	40	45	20	30	25	15	15	30	20	20	25

Algorithm Assign_Cellular_Network

Input : $FP_m=\{FP_m(t_1),\ FP_m(t_2),\ ...,\ FP_m(t_{\gamma_m})\}$ with their nonzero weights $y_m=\{y_m(t_1),\ y_m(t_2),\ ...,\ y_m(t_{\gamma_m})\}$, $1 \le m \le k$, and the frequency separation constraints s_0, s_1, and s_2.

Output : A conflict free assignment of the network and the required bandwidth B^*

Step 1 : For each m and l, find the set of optimal partition assignments $D_m(t_l) \leftarrow \{A_1^*(FP_m(t_l))\}$, $1 \le m \le k$, $1 \le l \le \gamma_m$.

Remark 4 *Note that partition P_m of CB_2 has been extended to FP_m (see Algorithm Candidate_Partition in section 5.1) such that the assignment $A_1^*(FP_m(t_l))$ ($1 \le m \le k$, $1 \le l \le \gamma_m$) can be obtained, keeping the increment unchanged.*

Step 2 : For each m and l, select an assignment from $D_m(t_l)$, $1 \le m \le k$, $1 \le l \le \gamma_m$. For each m, $1 \le m \le k$, assume an arbitrary ordering of the partitions $FP_m(t_1), FP_m(t_2), ..., FP_m(t_{\gamma_m})$ of P_m. Combine all these assignments by changing the starting frequency channel accordingly to avoid conflict and maintaining the same ordering of P_m's obtained for the critical block. Find the minimum bandwidth

B_{min} required from all possible selections of A_1^* and all possible ordering of the partitions $FP_m(t_1), FP_m(t_2), ..., FP_m(t_{\gamma_m})$ of P_m, $1 \le m \le k$.

Step 3 : For each m ($1 \le m \le k$) and l ($1 \le l \le \gamma_m$), the assignment $A_{y_m(t_l)}(FP_m(t_l))$ can be done by assigning $y_m(t_l)$ frequencies to all the nodes in $FP_m(t_l)$ with a successive gaps of $I^{(i_m)}(j_m)$. Combine all these assignments by changing the starting frequency channel accordingly to avoid conflict, giving rise to the complete assignment for the given cellular network. Return the bandwidth requirement B^*, and terminate.

In step 2, to find B_{min}, we may apply exhaustive search to guarantee an optimal solution, or we may apply heuristics, e.g., GA or other techniques to find an optimal, or near-optimal solution. However, in our simulation procedure, we applied the elitist model of GA presented in [6]. Here also, we apply the idea of *Remark 3* to obtain a theoretical lower bound on the bandwidth B_{min}, for checking the optimality of the sequence of elements in a *final partition set*. Given the cellular network, and the set of partitions $FP_m = \{FP_m(t_1), FP_m(t_2), ..., FP_m(t_{\gamma_m})\}$, let a node i ($0 \le i \le n-1$) appear in p_i different partitions of all FP_m's, $1 \le m \le k$. Clearly $p_i \le \sum_1^k \gamma_m$ for all i, $0 \le i \le n-1$. If we consider a cellular network, with demand vector $P = (p_i)$, $0 \le i \le n-1$, and find out the theoretical lower bound on bandwidth required for that network, it would also be a lower bound on B_{min}.

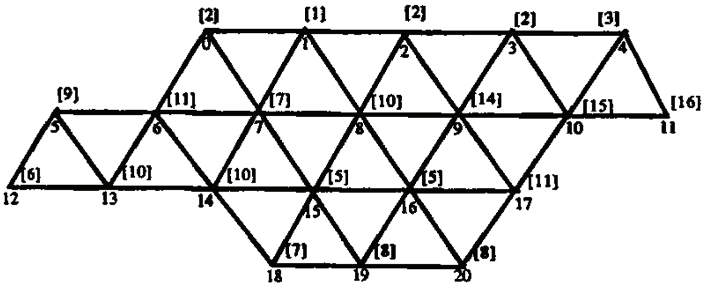

Fig. 13. The cellular network with demand vector $P = (p_i)$, $0 \le i \le 20$.

Example 15 *Fig. 13 shows the cellular network with demand vector $P = (p_i)$, ($0 \le i \le 20$). The label [α] associated with a node i indicates that node i appears in α different partitions of Table 2. For example, node 20 appears in eight different partitions corresponding to the rows b, c, d(i), d(ii), d(iv), e(i), e(ii) and e(iv) in Table 2. Thus, node 20 has a label [8] in Fig. 13. By Lemma 2, the distance-2 clique centered around node* 10 *requires at least 89* ($0-88$) *channels. So the lower bound on bandwidth for the assignment of Fig. 13 is 89. The single channel assignment of each partition of the final partition set with an optimal sequence obtained by the above algorithm is shown in Table 3, where each entry shows the channel assigned to a node corresponding to a final partition. We then assign the required number of multiple channels to each node corresponding to every final partition FP_i as specified in Table 2, using the increment I for the class of P_i and changing the starting frequencies accordingly to avoid conflict. The complete assignment requiring* 253 *channels* ($0-252$)*, has been shown in Table A.1 in the Appendix.*

Remark 5 *In the above example, it has been found that the whole network is assigned channels using the same bandwidth with* ($0-252$) *channels, as it was required for assigning the critical block only. But, it may not always be the case. It is also to be noted that if there exist more than one critical block in a network, having centers* *within distance-2, it may require a higher bandwidth.*

Example 16 *Fig. 14 shows a cellular network having homogeneous demand 20, with frequency separation constraints* $s_0 = 5$*,* $s_1 = 2$*, and* $s_2 = 1$ *respectively. Here each complete distance-2 clique is a critical block. A critical block needs at least* $0 - 177$ *channels for its assignment (from Lemma 2). An assignment requiring channels* $0 - 177$ *is possible only if both frequencies* 0 *and* 177 *are assigned to the central node of that critical block. But there are many other critical blocks having centers within distance-2. Hence, the frequencies* 0 *and* 177 *can not be assigned to the central node of those critical blocks. As a result, to assign other critical blocks we require frequencies from* 0 *to* 179*. By following our algorithm, the assignment for the whole network, demanding channels* $0 - 179$ *is shown in Fig. 14. The label* $[a - b]$ *associated with a node of Fig. 14 means that the frequency channels* $a, a + 9, a + 18, \cdots, b - 9, b$ *has been assigned to that node.*

Table 3. Optimal channel assignment for the cellular graph of Fig. 13.

Nodes → FP_i's ↓		0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
FP_1	$a(i)$					4	1	5	2	6	3	0	2		3	0	4	1	5			
	$a(ii)$					11	8	12	9		10	7	9		10	7	11	8	12			
FP_2	b		17		13	16			14		15		14	13	15		16		17			14
FP_3	c				18		22	19	21	23	20	22	19								19	21
FP_4	$d(i)$						27	24	28		26	28	25	25		26			24	29		27
	$d(ii)$						33	30	34		32	34	31	31		32			30	35	31	33
	$d(iii)$							36		41	38	40	37		38	40	37		36		39	
	$d(iv)$							42		47	44	46	43		44	46	43		42			45
	$d(v)$						48			53	50	52	49		50	52			48		51	
	$d(vi)$							54		59	56	58	55		56				54		57	
FP_5	$e(i)$							60	62	64	61	63	60	61	63					61	63	60
	$e(ii)$						68	65			66	68	65	66				69		66		67
	$e(iii)$						73	70		74	71	73	70	71						71	73	
	$e(iv)$						78			79	76	78	75			77					78	75
FP_6	$f(i)$	80		81						84		83	81		81	83		82	80	80		
	$f(ii)$	85		86								88	86		86	88		87	85	85		

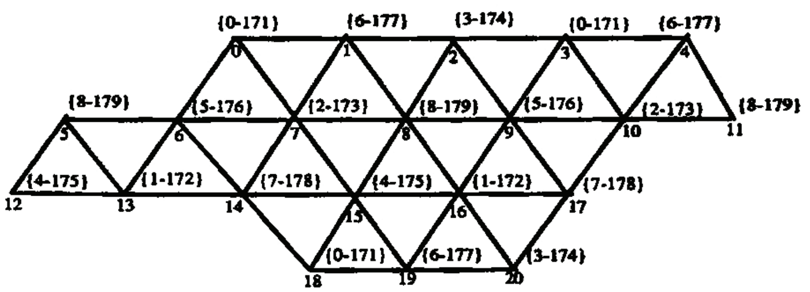

Fig. 14. Assignment of cellular network with homogeneous demand 20 for each node.

6. Simulation Results

We employ the eight CAP benchmarks widely used in the literature to compare the performance of our technique with earlier works [2, 5, 11, 12, 13, 19, 22, 24, 25, 26]. The cellular layout of the 21-cell system has already been presented in Figure 11. With this system, different CAP problems have been formulated, assuming any of the two different demand vectors D_1 and D_2, shown in *Table 4*. The i^{th} column of *Table 4* indicates the channel demand of cell i corresponding to D_1 or D_2. *Table 5* shows the specification of these eight problems (problems 1 through 8) in terms of the specific values of s_0, s_1 and s_2 for a 2-band buffering system, and the corresponding demand vector.

Problems 2 and 6 are the most difficult ones. Note that, the instance we considered in Example 8 is the benchmark problem 6. We have described the step by step solution to this problem in the previous sections and presented the complete result with 253 frequency channels in the *Appendix*.

Table 6 shows the number of frequency channels which are needed by different algorithms in order to derive a conflict-free frequency assignment for the problems described by demand vector(D_1 or D_2), and the frequency separation constraints (s_0, s_1, and s_2). The first row (Proposed approach) of *Table 6* gives the results of our proposed technique. The row *Lower Bound* corresponds to the lower bound for each of the problems as obtained by *Lemma 2*.

Table 4. Two different demand vectors for benchmark problems.

D_1	8	25	8	8	8	15	18	52	77	28	13	15	31	15	36	57	28	8	10	13	8
D_2	5	5	5	8	12	25	30	25	30	40	40	45	20	30	25	15	15	30	20	20	25

Table 5. The specification of eight benchmark problems.

Problem number		1	2	3	4	5	6	7	8
Frequency separation constraints	s_0	5	5	7	7	5	5	7	7
	s_1	1	2	1	2	1	2	1	2
	s_2	1	1	1	1	1	1	1	1
Demand vector		D_1	D_1	D_1	D_1	D_2	D_2	D_2	D_2

A comparison of the lower bounds, and the number of frequency channels required by our algorithm reveals that we find optimal solution for all eight benchmark instances. Most of the other algorithms (except the algorithm presented in [2]) determined such an optimal frequency assignment only for six of these eight problems. The average running time required in [2] for the optimal solution of problem 2 and 6 were about 8 and 10 minutes respectively, on an unloaded HP Apollo 9000/700 workstation. In contrast to this, using our proposed assignment algorithm, we need, on an average, only a few seconds for channel assignment of all the six benchmark instances other than problems 2 and 6, on an unloaded Sun Ultra 60 workstation. For the benchmark problems 2 and 6, however, our algorithm needs only 60 seconds and 72 seconds of running time, respectively on the same workstation. For comparison purposes, it may be noted that the Sun Ultra 60 workstation used by us has SPECint95 and SPECfp95 values as 13.2 and 18.4 respectively, while those for HP 9000/series 700 model 712/100 system are 3.76 and 4.03, respectively [27, 28].

7. Conclusion

We have first introduced the notion of a *critical block* of a cellular network of hexagonal structure having *2-band buffering* with respect to a given demand vector. Then, we present an algorithm (using integer programming) for finding the critical block of the cellular network, followed by the introduction of a novel idea of partitioning the critical block into several smaller sub-networks with homogeneous demands. This partition makes the frequency assignment to the critical block very simple. After the frequency assignment of the critical block, this partitioning technique is further extended to the rest of the network.

The proposed technique is able to achieve the optimum solution for all the eight well-known benchmark instances, with the minimum number of frequency channels. The results obtained by the application to the benchmark instances reveal that this novel strategy clearly outperforms the already existing algorithms in terms of both running time and required bandwidth.

Though we consider here the case of *2-band buffering*, the above results can also be extended to the cases of *k-band buffering*, in general. For *k-band buffering*, we are to consider complete *distance-k clique* to classify the different *clique-classes* and corresponding *class bandwidths* and *increments*. However, the integer programming formulation would then involve a large number of variables, and the solution may require longer time.

Table 6. Performance Comparisons between the existing CAP algorithms and our approach.

Problem	1	2	3	4	5	6	7	8
LowerBound	381	427	533	533	221	253	309	309
Proposed approach	381	427	533	533	221	253	309	309
(2001)[3]	381	463	533	533	221	273	309	309
(2001)[1]	381	427	533	533	221	254	309	309
(2000)[25]	381	433	533	533	–	260	–	309
(1998)[2]	381	427	533	533	221	253	309	309
(1998)[19]	–	–	–	–	221	268	–	309
(1997)[12]	381	–	533	533	221	–	309	309
(1997)[24]	381	436	533	533	–	268	–	309
(1996)[11]	381	–	533	533	–	–	–	–
(1996)[26]	381	433	533	533	221	263	309	309
(1994)[13]	381	464	533	536	–	293	–	310
(1992)[5]	381	–	533	533	221	–	309	309
(1989)[22]	381	447	533	533	–	270	–	310

References

1. Roberto Battiti, Alan Bertossi, and Daniela Cavallaro, "A Randomized Saturation Degree Heuristic for Channel Assignment in Cellular Radio Networks," *IEEE Transaction on Vehicular Technology,* vol. 50, No. 2, pp. 364-374, Mar. 2001.
2. Dirk Beckmann and Ulrich Killat, "A New Strategy for the Application of Genetic Algorithms to the Channel-Assignment Problem," *IEEE Transaction on Vehicular Technology,* vol. 48, no. 4, pp. 1261-1269, July 1999.
3. Goutam Chakraborty, "An Efficient Heuristic Algorithm for Channel Assignment Problem in Cellular Radio Networks," *IEEE Trans. Vehicular Technology,* vol. 50, No. 6, pp. 1528-1539, Nov. 2001.
4. M. Duque-Anton, D. Kunz and B. Ruber, "Channel Assignment for Cellular Radio Using Simulated Annealing," *IEEE Trans. on Vehicular Technology,* vol. 42, no. 1, pp. 14-21, Feb. 1993.

5. N. Funabiki and Y. Takefuji, "A Neural Network Parallel Algorithm for Channel Assignment in Cellular Radio Network," *IEEE Transaction on Vehicular Technology,* vol. 41, pp. 430-437, Nov. 1992.
6. S. C. Ghosh, B. P. Sinha, and N. Das, "Channel assignment using genetic algorithm based on geometric symmetry," *IEEE Transaction on Vehicular Technology,* to appear.
7. S. C. Ghosh, B. P. Sinha, and N. Das, "Efficient Schemes for Channel Assignment in Hexagonal Cellular Networks," *Technical Report CCSD/ACMU/01-2003*, Indian Statistical Institute, Calcutta, India.
8. W. K. Hale, "Frequency Assignment: Theory and Application," *Proc. IEEE,* vol.68, pp. 1497-1514, 1980.
9. S. Khanna and K. Kumaran, "On Wireless Spectrum Estimation and Generalized Graph Coloring," *Proc. of IEEE INFOCOM '98,* April, 1998.
10. S. Kim and S. Kim, "A Two-Phase Algorithm for Frequency Assignment in Cellular Mobile Systems," *IEEE Transaction on Vehicular Technology,* vol. 43, no. 3, pp. 542-548, Aug. 1994.
11. J. S. Kim, S. H. Park, P. W. Dowd, and N. M. Nasrabadi, "Channel Assignment in Cellular Radio using Genetic Algorithm," *Wireless Personal Commn.,* vol. 3, no. 3, pp. 273-286, Aug. 1996.
12. J. S. Kim, S. H. Park, P. W. Dowd, and N. M. Nasrabadi, "Cellular Radio Channel Assignment using a Modified Hopfield Network," *IEEE Transaction on Vehicular Technology,* vol. 46, pp. 957-967, Nov. 1997.
13. T.-M. Ko, "A Frequency Selective Insertion Strategy for Fixed Channel Assignment," *in Proc. 5th IEEE Int. Symp., Personal, Indoor and Mobile Radio Commn.,* pp. 311-314, Sept. 1994.
14. D. Kunz, "Channel Assignment for Cellular Radio using Neural Networks," *IEEE Transactions on Vehicular Technology,* vol. 40, no. 1, pp. 188-193, Feb 1991.
15. R. A. Leese, "A Unified Approach to the Assignment of Radio Channels on a Rectangular Hexagonal Grid," *IEEE Transactions on Vehicular Technology,* vol. 46, no. 4, pp. 968-980, Nov 1997.
16. W. K. Lio and George G. Coghill, "Channel Assignment Through Evolutionary Optimization," *IEEE Transaction on Vehicular Technology,* vol. 45, no. 1, pp. 91-96, Feb. 1996.
17. R. Mathar and J. Mattfeldt, "Channel Assignment in Cellular Radio Networks," *IEEE Transactions on Vehicular Technology,* vol. 42, no. 4, pp. 647-656, Nov. 1993.
18. L. Narayanan and S. Shende, "Static Frequency Assignment in Cellular Networks," *Proc. of the 4th International Colloquium on Structural Information and Communication Complexity,* 1997.
19. Chiu Y. Ngo and Victor O. K. Li, "Fixed Channel Assignment in Cellular Radio Networks Using a Modified Genetic Algorithm," *IEEE Transaction on Vehicular Technology,* vol. 47, no. 1, pp. 163-172, Feb. 1998.
20. A. Sen, T. Roxborough and S. Medidi, "Upper and Lower Bounds of a Class of Channel Assignment Problems in Cellular Networks," *Proc. of IEEE INFOCOM'98,* April, 1998.
21. A. Sen, T. Roxborough and B. P. Sinha, "On an Optimal Algorithm for Channel Assignment in Cellular Network," *Proc. of IEEE International Conference on Communications,* Vancouver, Canada, June 6-10, 1999, pp. 1147-1151.

22. K. N. Sivarajan, R. J. McEliece and J. W. Ketchum, "Channel Assignment in Cellular Radio, "*Proc. 39th IEEE Vehicular Technology Conf.*, pp. 846-850, May 1989.
23. K. Smith and M. Palaniswami, "Static and dynamic channel assignment using neural network," *IEEE J. Select. Areas Commun.*, vol. 15, pp. 238-249, Feb. 1997.
24. C. W. Sung, and W. S. Wong, "Sequential Packing Algorithm for Channel Assignment under Co-Channel and Adjacent-Channel Interference Constraint," *IEEE Transaction on Vehicular Technology*, vol. 46, pp. 676-686, Aug. 1997.
25. Dong-Wan Tcha, June-Hyuk Kwon, Taek-Jin Choi, and Se-Hyun Oh, "Perturbation-Minimizing Frequency Assignment in a Changing TDMA/FDMA Cellular Environment," *IEEE Transaction on Vehicular Technology*, vol. 49, no. 2, pp. 390-396, March 2000.
26. W. Wang and C. K. Rushforth, "An Adaptive Local-Search Algorithm for the Channel-Assignment Problem," *IEEE Transactions on Vehicular Technology*, vol. 45, no. 3, pp. 459-466, Aug. 1996.
27. http://www.specbench.org/osg/cpu95/results/cint95.html.
28. http://www.specbench.org/osg/cpu95/results/cfp95.html.

Appendix A:

Table A.1. Derived Channel Assignment for Problem 6.

0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
229	52	230	48	4	1	5	2	6	3	0	2	48	3	0	4	1	5	96	74	49
234	57	235	53	11	8	12	9	13	10	7	9	53	10	7	11	8	12	102	80	54
239	62	240	58	18	15	19	16	20	17	14	16	58	17	14	18	15	19	108	86	59
244	67	245	63	25	22	26	23	27	24	21	23	63	24	21	25	22	26	114	110	64
249	72	250	68	32	29	33	30	34	31	28	30	68	31	28	32	29	33	120	116	69
			73	39	36	40	37	41	38	35	37	92	38	35	39	36	40	170	124	76
			79	46	43	47	44	78	45	42	44	98	45	42	46	43	47	175	142	82
			85	51	77	74	49	84	50	77	49	104	50	93	51	198	52	180	148	88
				56	83	80	54	90	55	83	54	110	55	99	56	203	57	185	154	94
				61	89	86	59	126	60	89	59	116	60	105	61	208	62	190	160	100
				66	94	91	64	132	65	95	64	170	65	111	66	231	67	195	166	106
				71	100	97	69	138	70	101	69	175	70	117	71	236	72	200	172	112
					106	103	76	144	75	107	74	180	123	125	122	241	91	205	177	118
					112	109	82	150	81	113	80	185	129	131	128	246	97	210	182	130
					118	115	88	156	87	119	86	190	135	137	134	251	103	215	187	136
					139	121	95	162	93	125	92	195	141	143			109	229	192	169
					145	127	101	168	99	131	98	200	147	149			115	234	212	174
					151	133	107	173	105	137	104	205	153	155			121	239	217	179
					197	157	113	178	111	143	110	210	159	221			127	244	222	184
					202	163	119	183	117	149	116	215	165	226			133	249	227	189
					207	169	171	188	123	155	122		172	232			139			196
					212	174	176	193	129	161	128		177	237			145			201
					217	179	181	213	135	167	134		182	242			151			206
					222	184	186	218	141	172	140		187	247			157			219
					227	189	191	223	147	177	146		192	252			163			224
						194		228	153	182	152		230				229			
						199		233	159	187	158		235				234			
						204		238	165	192	164		240				239			
						209		243	170	197	169		245				244			
						214		248	175	202	174		250				249			
									180	207	179									
									185	212	184									
									190	217	189									
									195	222	194									
									200	227	199									
									205	232	204									
									210	237	209									
									215	242	214									
									220	247	219									
									225	252	224									
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