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Solution of a System of Generalized Abel Integral Equations Using Fractional Calculus

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Abstract—A method is presented for obtaining useful closed form solution of a system of generalized Abel integral equations by using the ideas of fractional integral operators and their applications. This system appears in solving certain mixed boundary value problems arising in the classical theory of elasticity.

Keywords—Abel integral equation, System of generalized Abel integral equations, Fractional calculus.

1. INTRODUCTION

Certain mixed boundary value problems arising in the classical theory of elasticity lead to the problem of solving systems of generalized Abel integral equations. Many workers (cf. [1–3], and the references cited therein) have analysed such systems of generalized Abel integral equations for their solutions by reducing them to equivalent systems of coupled Riemann boundary value problems (see [4]).

In the present paper, we have obtained the solution, in closed form, of the system of generalized Abel integral equations as given by

$$a\int_0^x \frac{\phi_1(t)\,dt}{(x-t)^\mu} + b\int_x^1 \frac{\phi_2(t)\,dt}{(t-x)^\mu} = f_1(x), \qquad x \in (0,1), \tag{1}$$

$$c\int_x^1 \frac{\phi_1(t)\,dt}{(t-x)^\mu} + d\int_0^x \frac{\phi_2(t)\,dt}{(x-t)^\mu} = f_2(x), \qquad x \in (0,1), \tag{2}$$

where a, b, c, and d are suitable quadruples of real constants with $0 < \mu < 1$. We have used the ideas of fractional integral operators and their applications, as developed recently by Chakrabarti and George [5], while obtaining the closed form solution of a single generalized Abel integral equation. The advantage of the present method over the existing ones, is that, it avoids the details of the complex function theory needed to analyse Riemann boundary value problems, normally used to solve problems of this type.

2. DERIVATION OF THE SOLUTION OF THE SYSTEM (1) AND (2)

To start with, we cast, following Chakrabarti and George's method [1], equation (1) in the equivalent form as given by

$$\phi_1(x) + \frac{b}{a} L(\phi_2)(x) = h_1(x), \tag{3}$$

where the operator L is defined by [5]

$$L(\psi)(x) = \frac{\sin \pi\nu}{\pi} x^{-\nu} \int_0^1 \frac{y^\nu \psi(y)\, dy}{y - x} + \psi(x) \cos \pi\nu, \tag{4}$$

with $\mu + \nu = 1$ and

$$h_1(x) = \frac{1}{a} \frac{1}{\Gamma(\nu)\Gamma(1-\nu)} \frac{d}{dx} \int_0^x \frac{f_1(t)\, dt}{(x-t)^\nu}. \tag{5}$$

In a similar manner, equation (2) can also be cast into the form

$$\phi_2(x) + \frac{c}{d} L(\phi_1)(x) = h_2(x), \tag{6}$$

where

$$h_2(x) = \frac{1}{d} \frac{1}{\Gamma(\nu)\Gamma(1-\nu)} \frac{d}{dx} \int_0^x \frac{f_2(t)\, dt}{(x-t)^\nu}. \tag{7}$$

Defining

$$\Phi_1 = \alpha\phi_1 + \beta\phi_2 \quad \text{and} \quad \Phi_2 = \alpha\phi_1 - \beta\phi_2, \tag{8}$$

where $\alpha^2/\beta^2 = ac/bd$ and introducing the parameter λ such that

$$\lambda^2 = \frac{bc}{ad}, \tag{9}$$

the system of coupled equations (3) and (6) can be decoupled in the form as given by

$$\Phi_1(x) + \lambda L(\Phi_1)(x) = H_1(x), \tag{10}$$

and

$$\Phi_2(x) - \lambda L(\Phi_2)(x) = H_2(x), \tag{11}$$

where

$$H_1(x) = \alpha h_1(x) + \beta h_2(x) \quad \text{and} \quad H_2(x) = \alpha h_1(x) - \beta h_2(x). \tag{12}$$

Using the expression (4) for the operator L, in equation (10), we obtain

$$\lambda_1 \psi_1(x) = \overline{H}_1(x) + \int_0^1 \frac{\psi_1(y)\, dy}{y - x}, \qquad 0 < x < 1, \tag{13}$$

where

$$x^\nu \Phi_1(x) = -\psi_1(x), \quad \frac{\pi}{\lambda \sin \pi\nu} + \pi \cot \pi\nu = -\lambda_1, \tag{14}$$

and

$$\overline{H}_1(x) = \frac{\pi x^\nu}{\lambda \sin \pi\nu} H_1(x). \tag{15}$$

Equation (13) is a Cauchy-type singular integral equation of second kind and its solution is well known [6]. Utilizing this known solution, along with relations (8) and (14), the solution of the integral equation (10) is found to be given by

$$\alpha\phi_1(x) + \beta\phi_2(x) = \begin{cases} -\dfrac{\sin^2 \pi\gamma_1}{\pi^2 x^{1-\mu-\gamma_1}} \dfrac{d}{dx} \displaystyle\int_x^1 \dfrac{ds}{(s-x)^{\gamma_1}} \int_0^s \dfrac{t^{-\gamma_1} \overline{H}_1(t)\, dt}{(s-t)^{1-\gamma_1}} + \dfrac{C_1 x^{\gamma_1+\mu-2}}{(1-x)^{\gamma_1}}, & \lambda_1 < 0 \\ -\dfrac{x^{\mu-1} \sin^2 \pi\gamma_1}{\pi^2 (1-x)^{-\gamma_1}} \dfrac{d}{dx} \displaystyle\int_0^x \dfrac{ds}{(x-s)^{\gamma_1}} \int_s^1 \dfrac{(1-t)^{-\gamma_1} \overline{H}_1(t)\, dt}{(t-s)^{1-\gamma_1}} + \dfrac{C_1 x^{\mu-\gamma_1-1}}{(1-x)^{1-\gamma_1}}, & \lambda_1 > 0, \end{cases} \tag{16}$$

where γ_1 is defined by $|\lambda_1| = \pi \cot \pi\gamma_1$ with $0 < \gamma_1 < 1/2$,

$$\lambda_1 = -\left[\frac{\pi}{\lambda \sin \pi\mu} - \pi \cot \pi\mu\right] \quad \text{and} \quad \overline{H}_1(t) = \frac{t^{1-\mu}}{\lambda a d}\frac{d}{dt}\int_0^t \frac{(\alpha d f_1(x) + a\beta f_2(x))\,dx}{(t-x)^{1-\mu}}. \tag{17}$$

In a similar way, the solution of the integral equation (11) can also be obtained as

$$\alpha\phi_1(x) - \beta\phi_2(x) = \begin{cases} \dfrac{\sin^2 \pi\gamma_2}{\pi^2 x^{1-\mu-\gamma_2}}\dfrac{d}{dx}\int_x^1 \dfrac{ds}{(s-x)^{\gamma_2}}\int_0^s \dfrac{t^{-\gamma_2}\overline{H}_2(t)\,dt}{(s-t)^{1-\gamma_2}} + \dfrac{C_2 x^{\gamma_2+\mu-2}}{(1-x)^{\gamma_2}}, & \lambda_2 < 0 \\ \dfrac{x^{\mu-1}\sin^2 \pi\gamma_2}{\pi^2(1-x)^{-\gamma_2}}\dfrac{d}{dx}\int_0^x \dfrac{ds}{(x-s)^{\gamma_2}}\int_s^1 \dfrac{(1-t)^{-\gamma_2}\overline{H}_2(t)\,dt}{(t-s)^{1-\gamma_2}} + \dfrac{C_2 x^{\mu-\gamma_2-1}}{(1-x)^{1-\gamma_2}}, & \lambda_2 > 0, \end{cases} \tag{18}$$

where γ_2 is defined by $|\lambda_2| = \pi \cot \pi\gamma_2$ with $0 < \gamma_2 < 1/2$,

$$\lambda_2 = \left[\frac{\pi}{\lambda \sin \pi\mu} + \pi \cot \pi\mu\right] \quad \text{and} \quad \overline{H}_2(t) = \frac{t^{1-\mu}}{\lambda a d}\frac{d}{dt}\int_0^t \frac{(\alpha d f_1(x) - a\beta f_2(x))\,dx}{(t-x)^{1-\mu}}. \tag{19}$$

Ultimately, the original unknown functions ϕ_1 and ϕ_2 can be fully recovered by solving the system of algebraic equations (16) and (18).

3. A SPECIAL CASE

It is interesting to observe that for the very special case $a = b = c = d = 1$, we can choose $\alpha = \beta = 1$ and $\lambda = 1$. For this special case if we also select $\mu = 1/2$ and f_1, f_2 as known constants, we deduce that

$$\lambda_1 = -\pi, \quad \lambda_2 = \pi, \quad \gamma_1 = \gamma_2 = \frac{1}{4}, \quad \overline{H}_1(t) = f_1 + f_2, \quad \text{and} \quad \overline{H}_2(t) = f_1 - f_2.$$

The first equation of (16) and the second equation of (18) then become useful and we derive the solutions of system (1) and (2), in the form

$$\phi_1(x) = \frac{(f_1+f_2)}{2\sqrt{2}\pi}\{x(1-x)\}^{-1/4} + \frac{(f_1-f_2)}{2\sqrt{2}\pi}x^{-3/4}(1-x)^{1/4} + \frac{1}{2}\left\{C_1 x^{-5/4}(1-x)^{-1/4} + C_2 x^{-3/4}(1-x)^{-3/4}\right\}, \tag{20}$$

and

$$\phi_2(x) = \frac{(f_1+f_2)}{2\sqrt{2}\pi}\{x(1-x)\}^{-1/4} - \frac{(f_1-f_2)}{2\sqrt{2}\pi}x^{-3/4}(1-x)^{1/4} + \frac{1}{2}\left\{C_1 x^{-5/4}(1-x)^{-1/4} - C_2 x^{-3/4}(1-x)^{-3/4}\right\}, \tag{21}$$

in the case when $a = b = c = d = 1$, $\mu = 1/2$, and f_1, f_2 as known constants.

It can be easily observed that the integral equation

$$\int_0^1 \frac{\Phi(t)\,dt}{|x-t|^{1/2}} = f_1 + f_2,$$

is the result of adding the two equations (1) and (2), in the special case, as mentioned above, in which $\Phi = \phi_1 + \phi_2$, and we derive, by using solutions (20) and (21), that

$$\Phi(x) = \frac{(f_1+f_2)}{2\sqrt{2}\pi}\{x(1-x)\}^{-1/4} + C_1 x^{-5/4}(1-x)^{-1/4},$$

which agrees completely with Porter and Stirling [6], if the arbitrary constant C_1 is chosen to be zero.

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