SOME DISCRETE DISTRIBUTIONS RELATED TO EXTENDED PASCAL TRIANGLES

K. Balasubramanian
Indian Statistical Institute, New Delhi, India 110 016

Román Viveros
Department of Mathematics & Statistics, McMaster University
Hamilton, Ontario, Canada L8S 4K1

N. Balakrishnan
Department of Mathematics & Statistics, McMaster University
Hamilton, Ontario, Canada L8S 4K1

(*Submitted March 1994*)

1. EXPERIMENTAL SETTING

Consider a die with m faces marked $\{0, 1, 2, \ldots, m-1\}$. Assume that the turn-up side probabilities are in geometric progression as follows:

Face (i)	0	1	2	...	$m-1$
Probability (p_i)	q^{m-1}	pq^{m-2}	p^2q^{m-3}	...	p^{m-1}

(1)

The necessary and sufficient restrictions on p and q are

$$q^{m-1} + pq^{m-2} + p^2q^{m-3} + \cdots + p^{m-1} = 1, \quad 0 \le p \le 1, \; 0 \le q \le 1. \tag{2}$$

Note that the first restriction is equivalent to $q^m - p^m = q - p$.

The die just described becomes an ordinary coin when $m = 2$. In this case $p + q = 1$. Selecting $p = q = m^{-1/(m-1)}$ will result in a fair die, i.e., each face will have probability m^{-1} of turning-up when the die is rolled. Also, from (2), when $0 \le p < m^{-1/(m-1)}$ one must have $m^{-1/(m-1)} < q \le 1$, and vice versa.

For a given p, the function $f(q) = q^m - q - p^m + p$ has derivative $f'(q) = mq^{m-1} - 1$. Thus, $f(q)$ is strictly decreasing for $0 \le q \le m^{-1/(m-1)}$ and strictly increasing for $m^{-1/(m-1)} \le q \le 1$. This fact in conjunction with the remarks in the previous paragraph assure that, for a given p $(0 \le p \le 1)$, there is a unique q satisfying (2). The value of q, which is the root of a polynomial of degree $m-1$, cannot be given explicitly in general. However, $q = 1 - p$ for $m = 2$, and $q = (-p + \sqrt{4 - 3p^2})/2$ for $m = 3$.

Alternative parametrizations to (1) that may yield other useful interpretations are also possible. For instance, if $p \le q$, then defining $\theta = p/q$ one can easily see that (1) is equivalent to $p_i = (1-\theta)\theta^i/(1-\theta^m), 0 \le i \le m-1$. In this case, rolling the die is equivalent to generating a value of a geometric random variable constrained to the range $\{0, 1, 2, \ldots, m-1\}$ with $1-\theta$ and θ being the success and failure probabilities, respectively.

2. THE EXTENDED BINOMIAL DISTRIBUTION AND PROPERTIES

The focus of this article is the random variable

$$X_n^{(m)} = \text{total score in } n \text{ rolls of the } m\text{-sided die with face probabilities as described in (1)-(2).} \tag{3}$$

It is clear that $X_n^{(m)}$ has the familiar binomial distribution with index n and success probability p when $m=2$. For this reason, the distribution of $X_n^{(m)}$ will be called the *extended binomial distribution of order m*, index n and parameter p, and will be denoted by $\text{EB}(m, n, p)$.

Note that $X_n^{(m)}$ is simply the convolution of n i.i.d. random variables corresponding to the scores of n rolls of the die. Therefore, the probability generating function (PGF) of $X_n^{(m)}$ can be written as

$$G(t) = E(t^{X_n^{(m)}}) = \left[\frac{q^m - p^m t^m}{q - pt}\right]^n. \tag{4}$$

Expanding $G(t)$ in powers of t yields an expression for the probability mass function (PMF) of $X_n^{(m)}$ as

$$\Pr(X_n^{(m)} = r; p) = C_m(n, r) p^r q^{(m-1)n-r}, \quad 0 \le r \le (m-1)n, \tag{5}$$

where $C_m(n, r)$ is the coefficient of t^r in $[(1-t^m)/(1-t)]^n$. Note the similarity between (5) and the ordinary binomial distribution.

The coefficients $C_m(n, r)$, which can be traced back to the classic work of Abraham De Moivre [6], were studied in detail by Freund [10], who discussed their role in occupancy theory. In particular, $C_m(n, r)$ can be interpreted as "the number of ways of putting n indistinguishable objects into r numbered boxes with each box containing at most $m-1$ objects." Thus,

$$C_2(n, r) = \binom{n}{r}, \quad 0 \le r \le n.$$

In the spirit of Bollinger [3] and [4], we will refer to the numbers $C_m(n, r), 0 \le r \le (m-1)n$, as the *extended binomial coefficients of order m*.

From a mathematical point of view, many theoretical properties of $C_m(n, r)$ have been established. For details, see [4] and [5] and the references therein. From a probabilistic point of view, in addition to the applications to occupancy problems discussed in [10] and those presented in this article, $C_m(n, r)$ plays an important role in describing the distribution of discrete waiting time random variables based on run criteria. For instance, see [3] and [2].

A convenient way of computing $C_m(n, r)$ is by means of the recursion

$$C_m(n, r) = \sum_{\ell=0}^{m-1} C_m(n-1, r-\ell). \tag{6}$$

For the case $m = 2$, this recursion reduces to the well-known identity

$$\binom{n}{r} = \binom{n-1}{r} + \binom{n-1}{r-1}.$$

In a manner similar to the calculation of the familiar Pascal triangle, (6) can be used to compute a table the n^{th} row of which will contain all the extended binomial coefficients of order m. These arrangements have been called *extended Pascal triangles*, see [4].

Alternatively, $C_m(n, r)$ can be calculated by means of the explicit formula

$$C_m(n,r)=\sum_{\alpha=0}^{\alpha_1}(-1)^\alpha\binom{n}{\alpha}\binom{r+n-\alpha m-1}{n-1}, \tag{7}$$

where $\alpha_1=\min\{n,\text{ integer part in } r/m\}$. For a proof of (6)-(7), see [5] and [2].

The classical hypergeometric identity also extends to arbitrary m. Namely,

$$C_m(n_1+n_2,r)=\sum_a C_m(n_1,a)C_m(n_2,r-a). \tag{8}$$

Relationship (8) will be called the *extended hypergeometric identity of order m* in this article.

It is a minor exercise to show that the property of symmetry for ordinary binomial coefficients also holds for m-binomial coefficients. That is,

$$C_m(n,r)=C_m(n,(m-1)n-r),\quad 0\le r\le (m-1)n. \tag{9}$$

As a result,

$$\Pr(X_n^{(m)}=r;\,p)=\Pr(X_n^{(m)}=(m-1)n-r;\,q),\quad 0\le r\le(m-1)n, \tag{10}$$

where p and q satisfy (2). Note that the distribution of $X_n^{(m)}$ is symmetric when $p=m^{-1/(m-1)}$ since $q=p$ in this case.

The PMF of $X_n^{(m)}$ given in (5) can also be computed recursively as follows. Write the PGF (4) as

$$(q-pt)^n G(t)=(q^m-p^m t^m)^n. \tag{11}$$

Then expand each factor in (11) using the binomial theorem and (5), and equate the coefficients of t^r from both sides to get

$$\sum_{j=0}^{\min\{n,r\}}(-1)^j\binom{n}{j}p^j q^{n-j}\Pr(X_n^{(m)}=r-j;\,p)=q^n\alpha_r, \tag{12}$$

for $r=0,1,2,\ldots$, where

$$\alpha_r=\begin{cases}0 & \text{if } b_r\ne 0,\\ (-1)^{a_r}\binom{n}{a_r}p^r q^{(m-1)n-r} & \text{if } b_r=0,\end{cases}$$

and $r=a_r m+b_r$ with $0\le b_r\le m-1$. From (12), one immediately obtains the recursion

$$\Pr(X_n^{(m)}=r;\,p)=\alpha_r-\sum_{j=1}^{\min\{r,n\}}(-1)^j\binom{n}{j}\left(\frac{p}{q}\right)^j\Pr(X_n^{(m)}=r-j;\,p),\quad 1\le r\le(m-1)n. \tag{13}$$

As an illustration of the variety of shapes exhibited by the distribution of $X_n^{(m)}$, (5) was calculated numerically for $m=4, n=10$, and several values of (p,q) using the foregoing methods. The corresponding bar plots are depicted in Figure 1. Note that the distribution of $X_n^{(m)}$ is positively skewed for $p<m^{-1/(m-1)}=0.63$ and negatively skewed for $p>m^{-1/(m-1)}$. This result holds generally for arbitrary m.

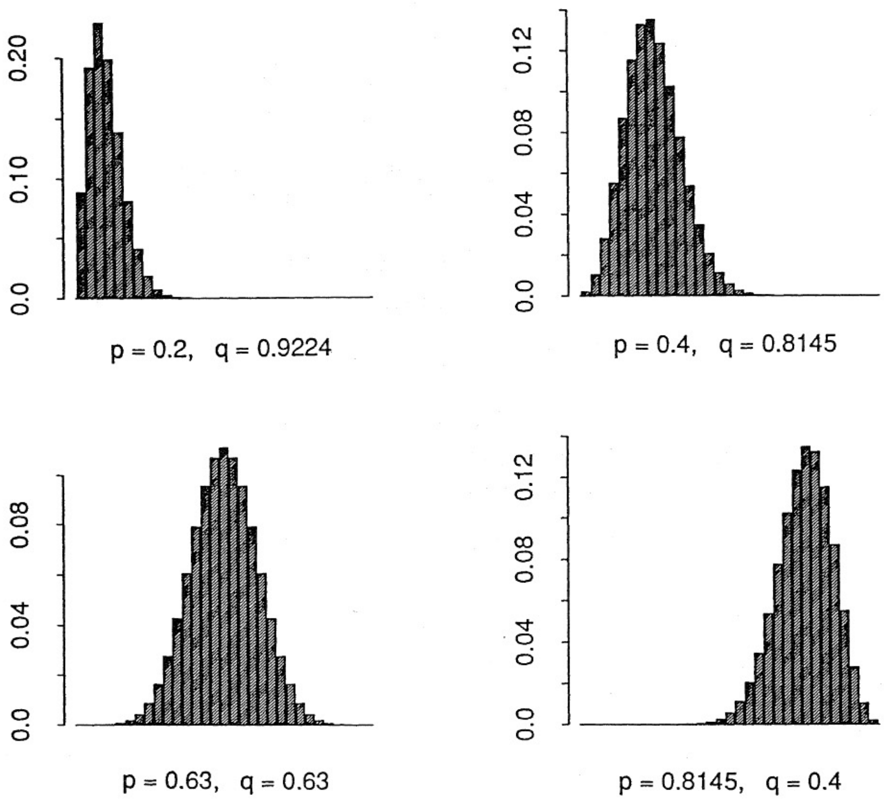

FIGURE 1. Shapes of the 4-Binomial Distribution with Index $n = 10$ and Several Values of (p, q)

Because $X_n^{(m)}$ arises as a convolution, it must have the reproductive property. Specifically, if $Y_1, Y_2, \ldots, Y_k$ are independent with $Y_i \sim \text{EB}(m, n_i, p)$, then $\sum_{i=1}^k Y_i \sim \text{EB}(m, \sum_{i=1}^k n_i, p)$.

Defining $\theta = p / q$, one can write the PMF of $X_n^{(m)}$ given in (5) as

$$\Pr(X_n^{(m)} = r; p) = \frac{C_m(n, r)\theta^r}{g(\theta)}, \quad 0 \le r \le (m-1)n, \tag{14}$$

where $0 \le \theta < \infty$ and

$$g(\theta) = \left(\frac{1-\theta^m}{1-\theta}\right)^n. \tag{15}$$

In the form (14)-(15), one can readily see that $X_n^{(m)}$ has a power series distribution. Thus, any results on this general family of distributions will apply to the distribution of $X_n^{(m)}$ as well. Note from (15) that $g(\theta) = (1+\theta)^n$ for $m = 2$ and $g(\theta) = (1-\theta)^{-n}$ for $m = \infty$ when $0 \le \theta < 1$, characterizing the binomial and negative binomial distributions, respectively. Using a standard argument, one can get the Poisson distribution by keeping m fixed, and letting $n \to \infty$ and $\theta \to 0$ in such a way that $n\theta / (1+\theta) \to \lambda$. By means of the central limit theorem, a normal approximation is also guaranteed. Specifically, $(X_n^{(m)} - \mu)/\sigma \approx N(0,1)$ for n large, where μ and σ^2 are as given in (16) below.

From (4) or otherwise, the mean and variance of $X_n^{(m)}$ are readily shown to be

$$\mu = E(X_n^{(m)}) = np\frac{1-mp^{m-1}}{q-p}, \quad \sigma^2 = Var(X_n^{(m)}) = npq\frac{1-m^2(pq)^{m-1}}{(q-p)^2}. \tag{16}$$

In comparing μ and σ^2, one can readily see that $\mu < \sigma^2$ when and only when

$$mp^{m-2}(mq^m - q + 1) < 1. \tag{17}$$

For any $3 \le m < \infty$, the left-hand side of (17) approaches 0 as $p \to 0$ and m as $p \to 1$. Hence, both $\mu < \sigma^2$ and $\mu \ge \sigma^2$ are always possible. When $m = 2$, (16) gives $\mu = np$ and $\sigma^2 = npq$ with $p + q = 1$. This case corresponds to the binomial model for which $\mu \ge \sigma^2$ for all $0 \le p \le 1$. When $p \le q$ one can easily show that $\mu = n\theta/(1-\theta)$, $\sigma^2 = n\theta^3/(1-\theta)^2$ when $m = \infty$ where $\theta = p/q$. Therefore, $\mu \le \sigma^2$ for all $0 \le \theta \le 1$ in this case, a well-known property for the negative binomial distribution.

Applying the results in [7, pp. 109-11, Th. 4.2], one can readily show that the turn-up face probability distribution (1) is strongly unimodal. Because the family of discrete strongly unimodal distributions is closed under convolution, it follows that the distribution of $X_n^{(m)}$ is strongly unimodal (i.e., log-concave). In particular, the distribution of $X_n^{(m)}$ is unimodal in the usual sense, i.e., there exists a point M such that

$$\Pr(X_n^{(m)} = r;\, p) \gtreqless \Pr(X_n^{(m)} = r-1;\, p) \text{ according as } r \lesseqgtr M.$$

A consequence of the log-concavity of the distribution of $X_n^{(m)}$ is the inequality

$$[C_m(n,r)]^2 \ge C_m(n,r-1)C_m(n,r+1), \quad 1 \le r \le (m-1)n-1,$$

which simply shows the log-concavity of the extended binomial coefficients $C_m(n,r)$, $0 \le r \le (m-1)n$. This shows, in particular, that the distribution of $X_n^{(m)}$ is log-concave.

3. HISTORY AND PREVIOUS APPLICATIONS

The earliest reference to the extended binomial coefficients can be found in the work of Abraham De Moivre [6]. A detailed "theoretical" discussion appeared in the third edition of [6], pp. 39-43, with many illustrative examples throughout the book. His main result appeared in the form of a lemma which stated: "To find how many chances there are upon any number of dice, each of them of the same number of faces, to throw any given number of points" [6, p. 39]. Without giving the reference, De Moivre stated in [6] that the lemma was published by him for the first time in 1711.

A look at [6] indicates that: (a) De Moivre dealt with a fair die with an arbitrary number of faces; (b) he calculated $C_m(n,r)$ numerically by explicit expansion of (7); (c) he was aware of the generating function for $C_m(n,r)$,

$$\left(\frac{1-t^m}{1-t}\right)^n = \sum_{r=0}^{(m-1)n} C_m(n,r)t^r,$$

which is given immediately after (5); (d) he was aware of the property of symmetry (9).

The distribution of $X_n^{(m)}+n$ for the case of a fair die appears as an exercise in [8, pp. 284-85]. Generating functions and limits for the cumulative probabilities of $X_n^{(m)}+n$ under this case are also presented as exercises by Feller [8, p. 285] who relates them to the work of Lagrange.

An important practical application of the extended binomial distribution was presented by Kalbfleisch and Sprott [14] in relation to the estimation of the "hit number," a parameter associated with an interesting dilution series model arising in virology. This model was originally proposed by Alling in [1]. The basics of the experiment, data, and assumptions are as follows: (a) a liquid medium containing a suspension of virus particles is successively diluted to form a geometric series of $k+1$ dilutions $a^0, a, a^2, \ldots, a^k$; (b) these dilutions are poured over replicate cell sheets; (c) after a period of growth, the number N_i of plaques occurring at dilution level a^i is observed $(0 \le i \le k)$; (d) the N_i's are independent with N_i having a Poisson distribution with mean $\eta\gamma^i$ $(0 \le i \le k)$. Here η is the expected number of plaques in the undiluted suspension ($i = 0$), and $\gamma = a^{-h}$, where a is the known dilution factor and h the "hit number," is the minimum number of virus particles that must attach themselves to a cell in order to form a plaque. The primary objective of the experiment was to estimate h.

In their statistical analysis, Kalbfleisch and Sprott [14] first show that the statistics $(S, T) = (\sum_{i=0}^k N_i, \sum_{i=0}^k iN_i)$ are jointly sufficient for (η, h). Then they derive the conditional distribution of T given S = s, which turns out to be the extended binomial distribution in the form (14) with $m = k+1, n = s$, and $\theta = \gamma$. They use this distribution to make inferences about $h = -\ln\gamma / \ln a$ that are unaffected by lack of knowledge on the remaining parameter η.

4. INFERENTIAL ISSUES

4.1 Sufficiency, Completeness, and Consequences

Since for given m and n the distribution of $X_n^{(m)}$ is a member of the family of power-series distributions, then $\{\Pr(X_n^{(m)} = \cdot; p)\colon 0 \le p \le 1\}$ is complete. Further, if $Y_1, Y_2, \ldots, Y_k$ are independent and identically distributed as $X_n^{(m)}$, then $S = \sum_{i=1}^k Y_i$ is sufficient for p or any one-to-one parametric function such as $\theta = p/q$. Due to the already noted reproductive property, it follows that $\{\Pr(S = \cdot; p)\colon 0 \le p \le 1\}$ is also complete.

These facts, in conjunction with the Rao-Blackwell theorem (e.g., see [12, pp. 349-52]), imply that the only parametric functions for which minimum variance unbiased estimators exist are the linear combinations of $\{p^r q^{(m-1)n-r}, 0 \le r \le (m-1)n\}$. In particular, the sample mean $\overline{Y} = \sum_{i=1}^k Y_i / k$ is the unique minimum variance unbiased estimator of the average value μ of $X_n^{(m)}$ given in (16).

4.2 Extended Fisher's Conditional Test and the Extended Hypergeometric Distribution

Consider two m-faced dice, labeled Die 1 and Die 2, with respective unknown parameter values p_1 and p_2. On the basis of the scores $Y_{11}, Y_{12}, \ldots, Y_{1n_1}$ in n_1 rolls of Die 1 and $Y_{21}, Y_{22}, \ldots, Y_{2n_2}$ in n_2 rolls of Die 2, we would like to test

$$H_0\colon p_1 = p_2 \text{ vs } H_1\colon p_1 \ne p_2.$$

In view of the sufficiency results of section 4.1, in developing a sensible test for H_0 vs H_1, one should focus on the total scores $Y_1 = \sum_{i=1}^{n_1} Y_{1i}$ and $Y_2 = \sum_{i=1}^{n_2} Y_{2i}$. Note that Y_1 and Y_2 are independent and have extended binomial distributions with parameters (m, n_1, p_1) and (m, n_2, p_2), respectively. Letting $\rho = p_1 q_2 / (q_1 p_2)$, one can show that

$$\Pr(Y_1 = a, Y_2 = b; p_1, p_2) = C_m(n_1, a) C_m(n_2, b) \rho^a \left(\frac{p_2}{q_2}\right)^{a+b} q_1^{(m-1)n_1} q_2^{(m-1)n_2},$$

from which it is readily seen that $T = Y_1 + Y_2$ is sufficient for p_2 when ρ is specified. Therefore, the conditional distribution of Y_1, given the observed value of T, depends on the parameters only through ρ. In fact,

$$\Pr(Y_1 = a; \rho \mid T = t) = \frac{C_m(n_1, a) C_m(n_2, t-a) \rho^a}{\Sigma_y C_m(n_1, y) C_m(n_2, t-y) \rho^y}, \quad 0 \le a \le t.$$

Since H_0 and H_1 are equivalent to H_0: $\rho = 1$ and H_1: $\rho \ne 1$, respectively, then a test for H_0 vs H_1 can be developed using Y_1 as a test statistic and its conditional null distribution

$$\Pr(Y_1 = a \mid T = t) = \frac{C_m(n_1, a) C_m(n_2, t-a)}{C_m(n_1 + n_2, t)}, \quad 0 \le a \le t. \tag{18}$$

P-values for testing H_0 vs H_1 can be calculated as tail probabilities from (18).

Note that the extended hypergeometric identity (8) has been used in deriving (18). Naturally, the test statistic reduces to Fisher's exact conditional test for homogeneity in 2×2 tables (see [9, pp. 89-92]) when $m = 2$ and (18) becomes the classical hypergeometric distribution. For these reasons, (18) will be called the *extended hypergeometric distribution of order m*.

Analogous to the well-known asymptotic relation between the classical hypergeometric and binomial distributions, it can be shown here that, for every m, (18) converges to $\binom{t}{a}\pi^a(1-\pi)^{t-a}$ as $n_1 \to \infty, n_2 \to \infty$ in such a way that $n_1 / (n_1 + n_2) \to \pi$.

5. NEGATIVE BINOMIAL EXTENSIONS

5.1 Total Score up to a Negative-Binomially-Stopped Roll

Consider consecutive rolls of the m-faced die with side probabilities (1)-(2). For a given positive integer k, define the random variable $Z_k^{(m)}$ as

$$Z_k^{(m)} = \text{total score until face marked 0 appears } k \text{ times.} \tag{19}$$

Clearly, $Z_k^{(m)}$ has the standard negative binomial distribution when $m = 2$.

In order to derive the PGF of $Z_k^{(m)}$, one can view the above experiment as a two-stage proccess as follows. First, generate a value n of T_k = (number of rolls until face marked 0 appears k times) $- k$. Then roll n times a "reduced" die with faces marked $\{1, 2, \ldots, m-1\}$ and corresponding side probabilities $pq^{m-2}/(1-q^{m-1}), p^2q^{m-3}/(1-q^{m-1}), \ldots, p^{m-1}/(1-q^{m-1})$. Then compute the total score among the n rolls to obtain $Z_k^{(m)}$ with the convention that $Z_k^{(m)} = 0$ whenever $n = 0$.

Note that T_k has the standard negative binomial distribution

$$\Pr(T_k = n; p) = \binom{n+k-1}{k-1} q^{(m-1)k} (1-q^{m-1})^n, \quad 0 \le n < \infty.$$

Thus, $Z_k^{(m)}$ can be seen as the total score from a negative binomial random number of rolls of the reduced die. From the basic theory on compounding of distributions, see [13, pp. 344-45], the PGF of $Z_k^{(m)}$ can be written

$$H(t) = E(t^{Z_k^{(m)}}) = G_{T_k}(G_R(t)),$$

where $G_{T_k}(t)$ is the PGF of T_k and $G_R(t)$ is the PGF of the score in one roll of the "reduced" die. Since

$$G_{T_k}(t) = \left(\frac{q^{m-1}}{1-(1-q^{m-1})t}\right)^k, \quad G_R(t) = \frac{pt}{1-q^{m-1}} \frac{q^{m-1} - p^{m-1}t^{m-1}}{q-pt},$$

then

$$H(t) = q^{(m-1)k}\left[1 - pt\frac{q^{m-1} - p^{m-1}t^{m-1}}{q-pt}\right]^{-k}. \tag{20}$$

Using the familiar negative binomial expansion in conjunction with the methods used to derive (4)-(5) yield

$$\Pr(Z_k^{(m)} = r; p) = q^{(m-1)k}\left(\frac{p}{q}\right)^r \sum_{i=0}^{r} \binom{k+i-1}{i} C_{m-1}(i, r-i) q^{(m-1)i}, \quad 0 \le r < \infty. \tag{21}$$

An alternative use of (20) is for moment calculations about $Z_k^{(m)}$. For instance, the average value of $Z_k^{(m)}$ is

$$E(Z_k^{(m)}) = H'(1) = \frac{kp}{q^{m-1}} \frac{1-mp^{m-1}}{q-p}. \tag{22}$$

When $m = 2$, (22) gives $E(Z_k^{(2)}) = kp/q$, which is the expected value of a standard negative binomial random variate.

5.2 An Extended Negative Binomial Distribution

Consider again the die with m faces and turn-up side probabilities given by (1)-(2). Perhaps a more natural negative binomial counterpart is the waiting time random variable

$$Y_N^{(m)} = \text{number of rolls until a total score of } N \text{ or more is observed for the first time.} \tag{23}$$

Clearly $Y_N^{(m)}$ is a standard negative binomial variate when $m = 2$. For this reason the distribution of $Y_N^{(m)}$ will be called the *extended negative binomial distribution of order m* and will be denoted as ENB(m, N, p).

It is readily seen that the fundamental identity

$$\Pr(Y_N^{(m)} \le n; p) = \Pr(X_n^{(m)} \ge N; p) \tag{24}$$

holds for every n. For the particular case $m = 2$, relationship (24) is well known from elementary probability courses. Using (24) in conjunction with (5) one can show that

$$\Pr(Y_N^{(m)} = n; p) = \sum_{r=0}^{N-1} C_m(n-1, r) p^r q^{(m-1)(n-1)-r} - \sum_{r=0}^{N-1} C_m(n, r) p^r q^{(m-1)n-r}, \tag{25}$$

for $n \ge$ the smallest integer not less than $N/(m-1)$.

Although (25) is adequate for numerical evaluations, further simplifications are possible in particular cases. For instance, when $1 \le N \le m$, with the help of (7) one can show that

$$\Pr(Y_N^{(m)} = n; p) = q^{(m-1)(n-1)} \sum_{r=0}^{N-1} \left(\frac{p}{q}\right)^r \left\{ \binom{n+r-2}{r} - q^{m-1} \binom{n+r-1}{r} \right\}, \tag{26}$$

for $1 \le n < \infty$. Note that $Y_1^{(m)}$ is a geometric random variable equivalent to the number of coin tosses until "heads" appears for the first time where the probability of "heads" is $1 - q^{m-1}$.

6. DISCUSSION

Richard C. Bollinger [4] concludes his article with the comment:

> In conclusion, we hope that the discussion has shown that the T_m arrays really are "extensions" of the Pascal triangle, with many similar properties that seem to be the natural generalizations of those of T_2, but perhaps with a few surprises also. T_2 has certainly been a rich source of interesting and useful mathematics. We suggest that its extended relatives potentially may serve as equally fruitful objects of study.

Here, T_m denotes the extended Pascal triangle formed by the extended binomial coefficients of order m, while T_2 is the familiar Pascal triangle of the classical binomial coefficients. Our article justifies to some extent the hopes of Bollinger; we too share his suggestions that these objects may serve as a source of many more discrete distributions.

A referee has pointed out the possibility of relating the extended binomial coefficients to the Fibonacci sequence of order m, $\{f_n^{(m)}\}_{n=0}^{\infty}$, for which an extensive literature is available. See, for example, the work of Philippou [15] and [16], Philippou and Muwafi [18], and Gabai [11]. Indeed, such a connection exists. Perhaps the simplest relationship is

$$f_{n+1}^{(m)} = \sum_{\ell=SIGE\left(\frac{n}{m}\right)}^{n} C_m(\ell, n-\ell), \tag{27}$$

where $SIGE(x)$ denotes the smallest integer that is greater than or equal to x, also called the "ceiling" of x. The validity of (27) can be established by expanding the generating function of $\{f_n^{(m)}\}_{n=0}^{\infty}$, given, e.g., in [15], in conjunction with the generating function for $\{C_m(n,r)\}_{r=0}^{(m-1)n}$ given in section 2, and then matching coefficients of identical powers in the generating variable. One immediate application of (27) is for numerical computation of $f_{n+1}^{(m)}$ by means of any of the methods for calculating $C_m(n,r)$ discussed in section 2. This approach is likely to be simpler than the formula for $f_{n+1}^{(m)}$ in terms of multinomial coefficients given by Theorem 1 in [15]. On the other hand, the interesting work on the Fibonacci sequence of order m done by Philippou and others has some bearing on the extended binomial coefficients in view of relationship (27). This avenue has not been explored in this article and merits further consideration.

An important application of the Fibonacci sequence of order m discussed by Philippou [16], Philippou, Georghiou, and Philippou [17], Philippou and Muwafi [18], and others, is in the calculation of the distribution of the discrete waiting time random variable

N_m = number of independent Bernoulli trials performed
until m consecutive successes are observed,

where each trial can result in "success" or "failure" with probabilities p and $q = 1 - p$, respectively. Working with the probability generating function of N_m, which was derived by Feller [8, p. 323], in a manner similar to the derivation of (27) one can show that

$$\Pr(N_m = m + j) = \sum_{\ell = SIGE\left(\frac{j}{m}\right)}^{j} C_m(\ell, j - \ell) p^{m+j-\ell} q^{\ell}, \tag{28}$$

for $j \geq 0$. Thus, (28) is an alternative to the formula for $\Pr(N_m = m + j)$ given by Theorem 3.1 in [18] in terms of multinomial coefficients.

It may also be of interest to look into possible continuous counterparts for the general discrete distributions presented in this article, just as beta and binomial, and exponential and geometric are naturally related. One interesting aspect is the study of the appropriate family of conjugate priors for the extended binomial distribution of order m. Work is currently being done in this direction.

ACKNOWLEDGMENTS

The authors gratefully acknowledge the helpful comments of D. A. Sprott and the referee who brought to our attention some relevant references. This research was supported by grants from the Natural Sciences and Engineering Research Council (NSERC) of Canada.

REFERENCES

1. D.W. Alling. "Estimation of Hit Number." *Biometrics* **27** (1971):605-13.
2. N. Balakrishnan, K. Balasubramanian, & R. Viveros. "On Sampling Inspection Plans Based on the Theory of Runs." *The Math. Scientist* **18** (1993):113-26.
3. R. C. Bollinger. "Fibonacci *k*-Sequences, Pascal-*T* Triangles, and *k*-in-a Row Problems." *The Fibonacci Quarterly* **17.1** (1984):23-28.
4. R. C. Bollinger. "Extended Pascal Triangles." *Math. Magazine* **66** (1993):87-94.
5. R. C. Bollinger & Ch. L. Burchard. "Lucas's Theorem and Some Related Results for Extended Pascal Triangles." *Amer. Math. Monthly* **97** (1990):198-204.
6. A. De Moivre. *The Doctrine of Chances: or, A Method of Calculating the Probabilities of Events in Play*. 3rd ed. London: Millar, 1756; rpt. New York: Chelsea, 1967.
7. S. Dharmadhikari & K. Joag-dev. *Unimodality, Convexity, and Applications*. San Diego: Academic Press, 1988.
8. W. Feller. *An Introduction to Probability Theory and Its Applications*. Vol. I. 3rd ed. New York: Wiley, 1968.
9. R. A. Fisher. *Statistical Methods and Scientific Inference*. New York: Hafner Press, 1973.
10. J. E. Freund. "Restricted Occupancy Theory—A Generalization of Pascal's Triangles." *Amer. Math. Monthly* **63** (1956):20-27.
11. H. Gabai. "Generalized Fibonacci *K*-Sequences." *The Fibonacci Quarterly* **8.1** (1970):31-38.
12. R. V. Hogg & A. T. Craig. *Introduction to Mathematical Statistics*. 4th ed. New York: Macmillan, 1978.
13. N. L. Johnson, S. Kotz, & A. W. Kemp. *Univariate Discrete Distributions*. 2nd ed. New York: Wiley, 1992.
14. J. G. Kalbfleisch & D. A. Sprott. "Inference about Hit Number in a Virological Model." *Biometrics* **30** (1974):199-208.
15. A. N. Philippou. "A Note on the Fibonacci Sequence of Order *K* and the Multinomial Coefficients." *The Fibonacci Quarterly* **21** (1983):82-86.

16. A. N. Philippou. "Recursive Theorems for Success Runs and Reliability of Consecutive-K-out of N: F Systems." In *Applications of Fibonacci Numbers* **1**:149-61. Ed. A. N. Philippou et al. Dordrecht: Kluwer, 1988.
17. A. N. Philippou, C. Georghiou, & G. N. Philippou. "A Generalized Geometric Distribution and Some of Its Properties." *Statistics & Probability Letters* **1** (1983):171-75.
18. A. N. Philippou & A. A. Muwafi. "Waiting for the Kth Consecutive Success and the Fibonacci Sequence of Order K." *The Fibonacci Quarterly* **20.1** (1982):28-32.

AMS Classification Numbers: 60C05, 62E15

❖❖❖