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SOME RESULTS ON THE RELATIVE AGEING OF TWO LIFE DISTRIBUTIONS

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Abstract

Kalashnikov and Rachev (1986) have proposed a partial ordering of life distributions which is equivalent to an increasing hazard ratio, when the ratio exists. This model can represent the phenomenon of crossing hazards, which has received considerable attention in recent years. In this paper we study this and two other models of relative ageing. Their connections with common partial orderings in the reliability literature are discussed. We examine the closure properties of the three orderings under several operations. Finally, we give reliability and moment bounds for a distribution when it is ordered with respect to a known distribution.

CROSSING HAZARD RATES; INCREASING HAZARDS RATIO; IFR; IFRA; NBU; RELIABILITY OF BOUNDS; STABILITY OF DISTRIBUTIONS

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1. Introduction and summary

Let X and Y be positive-valued random variables with cumulative distribution functions (c.d.f.) F and G, survival functions $\bar{F} = 1 - F$ and $\bar{G} = 1 - G$ and cumulative hazard functions $\Lambda_F = -\log \bar{F}$ and $\Lambda_G = -\log \bar{G}$, respectively. When the densities exist, we denote their probability density functions (p.d.f.) by f and g and hazard rates by $h_F = f/\bar{F}$ and $h_G = g/\bar{G}$, respectively. In the literature there exist many partial orderings between the random variables X and Y (or equivalently between the c.d.f.'s F and G). For example, F is said to be convex ordered with respect to G if $G^{-1} \circ F$ is a convex function (assuming G to be strictly increasing). This and many other partial orderings may be found in Barlow and Proschan (1975), Stoyan (1983), Ross (1983), Deshpande et al. (1990), etc. Here we study a relatively new partial ordering defined by Kalashnikov and Rachev (1986) which is found to be useful in reliability theory. The definition of this ordering is given below.

Definition 1. The random variable X is said to be *ageing faster than* Y (written '$X \underset{c}{\prec} Y$' or '$F \underset{c}{\prec} G$') if the random variable $Z = \Lambda_G(X)$ has an increasing failure rate (IFR) distribution.

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The importance of the above partial ordering stems from the fact that it provides a concept of relative ageing of two probability distributions. If the two failure rates are such that $h_F(x)/h_G(x)$ is a constant, then one may say that the two distributions age at the same rate. On the other hand if the ratio is an increasing (decreasing) function of the age x, then we say that X ages faster (slower) than Y. It is easily seen that this is equivalent to Definition 1 above, if the failure rates exist.

The phenomenon of crossing hazards also comes up in survival analysis. Pocock et al. (1982) have observed this phenomenon in connection with prognostic studies in the treatment of breast cancer. Champlin et al. (1983) and Begg et al. (1984) have also reported instances of the superiority of a treatment being short-lived. An increasing hazards ratio is a reasonable alternative to the proportional hazards model in these situations.

Returning to the problem of modelling, we can think of two simple generalizations of the '$\underset{c}{\prec}$' order by replacing 'IFR' in Definition 1 by 'IFRA' or 'NBU'. These orders can also be shown to have intuitive interpretations in terms of relative ageing. The formal definitions of these orders will be given in the next section.

The objective of this paper is to study the '$\underset{c}{\prec}$' ordering and the two related orderings. In Section 2 we note many characterizations of these orderings, some of which are originally due to Kalashnikov and Rachev (1986). The relative ageing interpretations of these orderings are highlighted. We also establish a connection among these partial orderings. In Section 3 we show that distributions within several parametric families are '$\underset{c}{\prec}$' ordered according to the values of some parameters. Some examples show that the three orderings discussed here do not imply or are not implied by the convex, star or superadditive orderings which are commonly known in reliability. The closure properties of the orderings are discussed in Section 4. In Section 5 we provide bounds for the Kolmogorov distance between the '$\underset{c}{\prec}$' ordered distributions. We also find upper and lower bounds for the survival function $\bar{F}$ if F is ordered with respect to a known distribution G in one of the three ways and shares a common moment with it. The cases of slower ageing with respect to a known distribution are also considered. Next we provide inequalities among the moments of the '$\underset{c}{\prec}$' ordered distributions. Finally the stability of F when it is ordered with respect to G is considered.

2. Properties and characterizations of the orders

At the outset we impose the constraint that the support of the distribution functions mentioned here includes the point 0. We begin with the following proposition which was noted (but not proved) by Kalashnikov and Rachev (1986).

Proposition 2.1

(i) If $X \underset{c}{\prec} Y$ and $Y \underset{c}{\prec} Z$ then $X \underset{c}{\prec} Z$.

(ii) The relations $X \underset{c}{\prec} Y$ and $Y \underset{c}{\prec} X$ hold simultaneously if and only if $\Lambda_F = c\Lambda_G$ for some $c > 0$.

(iii) Let E be an exponential random variable. Then X is IFR (DFR) if and only if $X \underset{c}{\prec} E$ ($E \underset{c}{\prec} X$).

The proof of the above, especially when the distributions are not strictly increasing, is non-trivial. We defer the proof until after the statement of the next proposition.

In view of (i) and (ii) above if one forms equivalence classes with random variables such that any two members in a class have proportional hazard rates, then the '$\underset{c}{\prec}$' order corresponds to a partial order of these equivalence classes. It is also clear that two distributions which age equally fast would belong to the same equivalence class and thereby to the same 'proportional hazards' or 'Lehman' family. Property (iii) brings out the fact that the IFR and DFR classes can be obtained by comparing probability distributions to the exponential distribution in the sense of the '$\underset{c}{\prec}$' order. One may recall that these classes may also be obtained as special cases of the convex ordering. It turns out that the '$\underset{c}{\prec}$' ordering can also be described by the notion of convexity. The next proposition gives this and two other characterizations of the '$\underset{c}{\prec}$' ordering. The proof follows from the definition of the ordering. In the following, we use the definition $\Lambda_G^{-1}(x) = \inf\{y : \Lambda_G(y) > x\}$. Consequently $\Lambda_G \circ \Lambda_G^{-1}(x) = x$ when Λ_G is continuous at x and $\Lambda_G^{-1} \circ \Lambda_G(x) = x$ when Λ_G is strictly increasing at x.

Proposition 2.2

(i) $X \underset{c}{\prec} Y$ if and only if $\Lambda_F \circ \Lambda_G^{-1}$ is convex on $[0, \infty)$.

(ii) $X \underset{c}{\prec} Y$ if and only if $\Lambda_F(Y)$ has a DFR distribution.

(iii) If h_F and h_G exist and $h_G \neq 0$, then $X \underset{c}{\prec} Y$ if and only if h_F/h_G is a non-decreasing function.

Proposition 2.2(i) should be contrasted with the definition of the convex ordering. The latter is equivalent to the convexity of $\bar{G}^{-1} \circ F$ (which is the same as $\Lambda_G^{-1} \circ \Lambda_F$). Later on we shall see that the '$\underset{c}{\prec}$' and convex orderings form quite different systems of equivalence classes, none implying the other. The characterizing property in Proposition 2.2(iii) appears to be the one which is easiest to interpret in terms of relative ageing as discussed in Section 1. When $h_F(0) \geqq h_G(0)$, the '$\underset{c}{\prec}$' ordering implies the 'hazard ratio' ordering (Ross (1983)), which means that h_F dominates h_G.

Proof of Proposition 2.1. Let X, Y and Z have cumulative hazard functions Λ_F, Λ_G and Λ_H, respectively. We use Proposition 2.2(i) repeatedly to prove the results. To prove part (i), let $\Lambda_F \circ \Lambda_G^{-1}$ and $\Lambda_G \circ \Lambda_H^{-1}$ be convex functions. Since these functions are also non-decreasing, the composition $\Lambda_F \circ \Lambda_G^{-1} \circ \Lambda_G \circ \Lambda_H^{-1}$ is a convex function. It is clear that $\Lambda_G^{-1} \circ \Lambda_G$ is the identity function over an interval where Λ_G is strictly increasing. Now suppose Λ_G is constant over the interval $[a, b)$. (The assumption that the support of G includes the point 0 implies that $a > 0$.) Then Λ_H must also be constant over $[a, b)$ so that the non-decreasing function $\Lambda_G \circ \Lambda_H^{-1}$ is convex. Thus the range of Λ_H^{-1} has no overlap with the interval $[a, b)$. Because of the right-continuity of the cumulative hazard functions, the domain of Λ_G must be a union of the two types of intervals described above. Therefore $\Lambda_F \circ \Lambda_H^{-1}$ is identical to $\Lambda_F \circ \Lambda_G^{-1} \circ \Lambda_G \circ \Lambda_H^{-1}$, which is convex. Using Proposition 2.2(i) we obtain 2.1(i). To prove 2.1(ii) note that $X \underset{c}{\prec} Y$ if and only if $\Lambda_F \circ \Lambda_G^{-1}$ is convex, which is equivalent to saying that $(\Lambda_F \circ \Lambda_G^{-1})^{-1}$ or $\Lambda_G \circ \Lambda_F^{-1}$ is concave. On the other hand $Y \underset{c}{\prec} X$ if and only if $\Lambda_G \circ \Lambda_F^{-1}$ is convex. These two hold simultaneously if and only if $\Lambda_G \circ \Lambda_F^{-1}$ is a linear function. The result follows. Part (iii) of 2.1 also follows from 2.2(i) by noticing that Λ_G is a linear function when G is exponential.

It is well known that successive generalizations of the property of convexity give rise to the star and superadditive orderings. In the same manner we now generalize the '$\underset{c}{<}$' ordering.

Definition 2. The random variable X is said to be *ageing faster than Y in average* (written '$X \underset{*}{<} Y$' or '$F \underset{*}{<} G$') if the random variable $Z = \Lambda_G(X)$ has an increasing failure rate average (IFRA) distribution.

Definition 3. The random variable X is said to be *ageing faster than Y in quantile* (written '$X \underset{su}{<} Y$' or '$F \underset{su}{<} G$') if the random variable $Z = \Lambda_G(X)$ has a new better than used (NBU) distribution.

We can immediately see that parts (i) and (ii) of Proposition 2.1 hold for these two orderings, while part (iii) holds with appropriate modifications. (For 2.1(i) to hold for the '$\underset{su}{<}$' ordering, we need the additional assumption that the distributions are strictly increasing.) Thus the IFRA, DFRA, NBU and NWU classes of life distributions can be viewed as special cases of these orders when either F or G is exponential. The following chain of implications is also obvious:

$$X \underset{c}{<} Y \Rightarrow X \underset{*}{<} Y \Rightarrow X \underset{su}{<} Y.$$

We now give two sets of characterizations which help understand the new partial orderings.

Proposition 2.3

(i) $X \underset{*}{<} Y$ if and only if $\Lambda_F \circ \Lambda_G^{-1}$ is star-shaped on $[0, \infty)$.

(ii) $X \underset{*}{<} Y$ if and only if $\Lambda_F(Y)$ has a DFRA distribution.

(iii) Suppose at least one of the two distributions F or G is continuous and strictly increasing. Then $X \underset{*}{<} Y$ if and only if Λ_F/Λ_G is an non-decreasing function.

Proof. Parts (i) and (ii) follow from the definition. Part (iii) is proved in two steps. Suppose G is continuous and strictly increasing. Then Λ_F/Λ_G is non-decreasing if and only if the function $\Lambda_F \circ \Lambda_G^{-1}(y)/y$ is non-decreasing. A similar argument involving $\Lambda_G \circ \Lambda_F^{-1}(y)/y$ can be made when F is continuous and strictly increasing. Putting these together and using Parts (i) and (ii) we have the desired result.

Proposition 2.4

(i) $X \underset{su}{<} Y$ if and only if $\Lambda_F \circ \Lambda_G^{-1}$ is superadditive on $[0, \infty)$.

(ii) $X \underset{su}{<} Y$ if and only if $\Lambda_F(Y)$ has a NWU distribution.

(iii) $X \underset{su}{<} Y$ if and only if

$$\bar{F}^{-1}\left(\frac{\bar{F}(x+t)}{\bar{F}(t)}\right) \geqq \bar{G}^{-1}\left(\frac{\bar{G}(x+t)}{\bar{G}(t)}\right) \qquad \text{for each } x, t > 0,$$

assuming $\bar{F}$ and $\bar{G}$ to be continuous and strictly decreasing functions.

Proof. Parts (i) and (ii) are easy to prove. In order to prove part (iii), let $a, b > 0$, $t = \Lambda_G^{-1}(a)$ and $x + t = \Lambda_G^{-1}(a + b)$, and note the following sequence of equivalent statements:

$$\Lambda_F \circ \Lambda_G^{-1}(a+b) \geqq \Lambda_F \circ \Lambda_G^{-1}(a) + \Lambda_F \circ \Lambda_G^{-1}(b) \qquad \text{for each } a, b > 0$$

$$\Leftrightarrow -\log\left(\frac{\bar{F}(\Lambda_G^{-1}(a+b))}{\bar{F}(\Lambda_G^{-1}(a))}\right) \geqq -\log \bar{F}(\Lambda_G^{-1}(b)) \qquad \text{for each } a, b > 0$$

$$\Leftrightarrow \bar{F}^{-1}\left(\frac{\bar{F}(\Lambda_G^{-1}(a+b))}{\bar{F}(\Lambda_G^{-1}(a))}\right) \geqq \Lambda_G^{-1}(b) \qquad \text{for each } a, b > 0$$

$$\Leftrightarrow \bar{F}^{-1}\left(\frac{\bar{F}(x+t)}{\bar{F}(t)}\right) \geqq \Lambda_G^{-1}\left(-\log\frac{\bar{G}(x+t)}{\bar{G}(t)}\right) = \bar{G}^{-1}\left(\frac{\bar{G}(x+t)}{\bar{G}(t)}\right) \qquad \text{for each } x, t > 0.$$

The quantity $\bar{F}^{-1}(\bar{F}(x+t)/\bar{F}(t))$ may be interpreted as the *rescaled quantile of the life distribution of a component which is t units old*, that is,

$$\boldsymbol{P}\left[X > \bar{F}^{-1}\left(\frac{\bar{F}(x+t)}{\bar{F}(t)}\right)\right] = \boldsymbol{P}[X_t > x],$$

X_t being the remaining life of a unit that is t units old, and $X = X_0$. Part (iii) of Proposition 2.4 indicates that the rescaled quantile of one distribution is smaller than that of the other distribution at all ages. Thus the last parts of Propositions 2.3 and 2.4 provide natural ageing interpretations to the new partial orderings and justify the names given in Definitions 2 and 3.

Since $\bar{G}^{-1} \circ \bar{F}$ is the same as $\Lambda_G^{-1} \circ \Lambda_F$, star-shapedness or superadditivity of this function represents the star and superadditive ordering, respectively. The first parts of Propositions 2.3 and 2.4 indicate that the functions Λ_G^{-1} and Λ_F are composed in a different way in the case of '$\underset{*}{\lesssim}$' and '$\underset{su}{\lesssim}$' orderings.

The '$\underset{c}{\lesssim}$', '$\underset{*}{\lesssim}$' and '$\underset{su}{\lesssim}$' orders share several properties which will be discussed below. We shall use the generic symbol '$\prec$' for simplicity, while describing any result that holds for each of these orderings.

Theorem 2.1. $X \prec Y$ if and only if $u(X) \prec u(Y)$ for every strictly increasing positive function u passing through (0, 0).

Proof. Let Λ_F^u and Λ_G^u be the cumulative hazard functions of $u(X)$ and $u(Y)$, respectively. It is easy to observe that $\Lambda_F^u \circ (\Lambda_G^u)^{-1} = \Lambda_F \circ \Lambda_G^{-1}$.

Corollary 2.1. $X \prec Y$ if and only if $aX \prec aY$ for all $a > 0$.

Corollary 2.2. $X \prec Y$ if and only if $X^r \prec Y^r$ for all $r > 0$.

It is seen that the result in Corollary 2.1 is weaker than the result available for the convex, star and superadditive orderings, where the ordering of X with respect to Y

implies the corresponding ordering of aX with respect to bY for all $a, b > 0$. We shall come back to Corollary 2.2 in Section 5.

We conclude this section with the statement of a property of the '$\underset{*}{<}$' ordering which is analogous to a similar property of the star ordering.

Proposition 2.5. If $F \underset{*}{<} G$ and F and G have no common point of discontinuity, then $\bar{F}$ crosses $\bar{G}$ at most once and from above.

Proof. The condition $F \underset{*}{<} G$ implies by Proposition 2.3(i) that the graph of $\Lambda_F \circ \Lambda_G^{-1}(x)$ crosses the x at most once and from below. This ensures that over an interval where Λ_G is continuous, Λ_F crosses Λ_G at most once and from below. The same argument holds over an interval where Λ_F is continuous, using the fact that $\Lambda_G \circ \Lambda_F$ is inverse star-shaped. The result follows.

3. Parametric classes

When the '$\underset{c}{<}$' ordering (or the other two orderings defined in the previous section) is restricted to certain parametric classes, it reduces to a natural ordering among the parameters as shown in the following examples.

Example 3.1 (linear failure rate). Let $\bar{F}_i(x) = \exp\{-\alpha_i(x + \frac{1}{2}\theta_i x^2)\}$ for $i = 1, 2$. In this case $F_1 \underset{c}{<} F_2$ if and only if $\theta_1 \geqq \theta_2$, irrespective of α_1 and α_2.

Example 3.2 (Weibull). Let $\bar{F}_i(x) = \exp\{-\alpha_i x^{\theta_i}\}$ for $i = 1, 2$. Then $F_1 \underset{c}{<} F_2$ if and only if $\theta_1 \geqq \theta_2$, irrespective of α_1 and α_2.

Example 3.3 (Pareto). Let $\bar{F}_i(x) = (1 + x/\theta_i)^{\alpha_i}$ for $i = 1, 2$. Then $F_1 \underset{c}{<} F_2$ if and only if $\theta_1 \geqq \theta_2$, irrespective of α_1 and α_2.

Example 3.4 (Makeham). Let $\bar{F}_i(x) = \exp[-\{\alpha_i xx + \theta_i(\exp(-\alpha_i x) - 1 + \alpha_i x)\}]$ for $i = 1, 2$. Now, if $\alpha_1 = \alpha_2$, then $F_1 \underset{c}{<} F_2$ if and only if $\theta_1 \geqq \theta_2$.

Example 3.5 (gamma). Let $\bar{F}_i(x) = \int_x^\infty \{\alpha_i^{\theta_i} y^{\theta_i - 1}/\Gamma(\theta_i)\}\exp(-\alpha_i y)dy$ for $i = 1, 2$. Now, if $\alpha_1 = \alpha_2$, then $F_1 \underset{c}{<} F_2$ if and only if $\theta_1 \geqq \theta_2$.

The last case is proved by writing $\bar{F}_1$ as

$$\bar{F}_1(x) = \alpha_1^{\theta_1 - \theta_2} \frac{\Gamma(\theta_2)}{\Gamma(\theta_1)} \left[x^{\theta_1 - \theta_2}\bar{F}_2(x) + \int_x^\infty (\theta_1 - \theta_2) y^{\theta_1 - \theta_2 - 1} \bar{F}_2(y)dy\right]$$

after adjustment of power and integration by parts. Then the ratio of the hazard rates can be written as

$$\frac{h_1(x)}{h_2(x)} = \frac{1}{1 + \int_1^\infty (\theta_1 - \theta_2) z^{\theta_1 - \theta_2 - 1}\{\bar{F}_2(zx)/\bar{F}_2(x)\}dz}.$$

Using arguments similar to those of Barlow and Proschan (1975), p. 74, and making use of the fact that $\bar{F}_2(zx)/\bar{F}_2(x)$ decreases with x whenever $z > 1$, we reach the stated conclusion.

The above examples illustrate how the members of various parametric families are ordered through the '$\underset{c}{<}$', '$\underset{*}{<}$' and '$\underset{su}{<}$' orderings. In general the results are not unexpected. In the Weibull and gamma families they are ordered according to the values of the shape parameter. In the Pareto family, however, it is seen that by fixing the values of α_1, α_2, θ_1 and θ_2 appropriately, the '$\underset{c}{<}$' order may be made to agree with the convex ordering or go against the convex ordering. (It may be recalled that the values of α_1 and α_2 determine the convex order of the Pareto distributions.) This brings out the fact that there is no implication relationship between these two orderings which are based on the convexity of $\Lambda_F \circ \Lambda_G^{-1}$ and $\Lambda_G^{-1} \circ \Lambda_F$, respectively.

4. Closure properties

In reliability we often deal with complex systems of independent components, with convolutions, with mixture distributions, etc. Hence it is of interest to know whether the orderings '$\underset{c}{<}$' '$\underset{*}{<}$' and '$\underset{su}{<}$' are preserved under such operations. The following theorem states two closure properties.

Theorem 4.1. Let $X_i \sim F_i$ and $Y_i \sim G_i$ for $i = 1, \cdots, n$ and all the random variables are independent. Further, let $F_i < G_i$ for each i.

(i) If $F_1 = F_2 = \cdots = F_n = F$ and $G_1 = G_2 = \cdots = G_n = G$, then

$$\min(X_1, X_2, \cdots, X_n) < \min(Y_1, Y_2, \cdots, Y_n).$$

(ii) If $\lim_{n\to\infty} F_n = F$, $\lim_{n\to\infty} G_n = G$ and F and G have no common point of discontinuity, then $F < G$.

Proof. Part (i) is easy to prove. To prove part (ii) it is enough to show that the convexity (or star-shapedness or superadditivity) of $-\log P[\Lambda_{G_n}(X_n) > x]$ on $[0, \infty)$ implies that of $-\log \boldsymbol{P}[\Lambda_G(X) > x]$. We prove that the latter is the limit of the former, or equivalently,

$$\lim_{n\to\infty} \boldsymbol{P}[G_n(X_n) \leqq y] = \boldsymbol{P}[G(X) \leqq y] \qquad \text{for each } y \in [0, 1).$$

The sequence of functions $\{G_n\}$ is uniformly bounded. In such a case a simplification of Theorem 1 of Topsøe (1967) gives the following necessary and sufficient conditions for the above:

(a) $\lim_{n\to\infty} \boldsymbol{E}[G_n(X)] = \boldsymbol{E}[G(X)]$;

(b) For every $\varepsilon > 0$ the identity

$$\boldsymbol{P}\left[X \in \bigcap_{k=1}^{\infty} \{x : G_{n_k}(x + \delta_k) - G_{n_k}(z - \delta_k) > \varepsilon\}\right] = 0$$

holds for each sequence $\{\delta_k\}$ of positive numbers converging to 0 and each subsequence $\{G_{n_k}\}$ of $\{G_n\}$.

Since $G_n \rightarrow G$ pointwise, the condition (a) follows by the bounded convergence theorem. In order to check condition (b) let x_0 be a continuity point of G. For a given $\varepsilon > 0$ pick $\delta > 0$ such that $(x_0 - \delta)$ and $(x_0 + \delta)$ are also continuity points of G and $G(x_0 + \delta) - G(x_0 - \delta) \leqq \varepsilon/3$. If $\{\delta_k\}$ is a sequence converging to 0, then for large enough k we have $\delta_k < \delta$, $|G_{n_k}(x_0 + \delta) - G(x_0 + \delta)| \leqq \varepsilon/3$ and $|G_{n_k}(x_0 - \delta) - G(x_0 - \delta)| \leqq \varepsilon/3$. It follows that x_0 does not belong to the set

$$\bigcap_{k+1}^{\infty} \{x : G_{n_k}(z + \delta_k) - G_{n_k}(x - \delta_k) > \varepsilon\}.$$

Thus the above set consists of at most the discontinuity points of G. In fact it can be shown that only finitely many of them, having jump size greater than ε, are included in it. Consequently condition (b) is satisfied if and only if F and G have no common point of discontinuity.

The result in (i) above says that the '$<$' ordering is preserved under formation of series systems of i.i.d. components. However, that it is not preserved under formation of general coherent systems is brought out if one considers the parallel system of i.i.d. components. Similarly, the ordering is not preserved under formation of convolutions or mixtures either. Simple counterexamples can be constructed out of exponential random variables in order to prove these non-closures. Whether the closure properties hold for the above operations after imposing additional relevant conditions on the distributions is an open problem.

5. Reliability bounds and other inequalities

5.1. *Reliability bounds with one known moment*. Barlow and Proschan (1975), p. 111 give a lower bound on $\bar{F}(t)$ when F is convex ordered with respect to G and shares a common moment with it. Similar upper and lower bounds for star-shaped and superadditive orderings have been provided by Sengupta (1994). These results are useful in understanding the nature of an unknown distribution F which is ordered with respect to a known distribution G. For example, if F is convex ordered with respect to a Weibull distribution G with a known shape parameter greater than 1, then one can provide a bound on F which is sharper than the IFR bound. Following the technique of Sengupta (1994), and assuming that G is strictly increasing, the corresponding bounds for the '$\underset{c}{<}$', '$\underset{*}{<}$' and '$\underset{su}{<}$' orderings can be obtained. Another set of bounds exist for the reverse order between F and G. All these bounds, which can be shown to be sharp, are summarized in Table 1. The bounds are obtained by writing $\Lambda_F(x)$ as $\Lambda \circ \Lambda_G$, where Λ is a convex, star-shaped or superadditive function, and then optimizing $\Lambda_F(x)$ over the class of these special functions subject to the constraint

$$\int_0^{\infty} \exp(-\Lambda \circ \Lambda_G(y)) r z^{r-1} dz = \mu_r,$$

where μ_r is the known rth moment of F $(r > 0)$. The derivations are omitted in order to save space.

TABLE 1

Bounds on $\bar{F}(x)$ when $\int_0^\infty rx^{r-1}\bar{F}(x)dx = \mu_r$

Case	Upper bound	Lower bound
$F \underset{c}{\prec} G$	$\bar{F}(x) \leqq \begin{cases} 1 & \text{if } x < \mu_r^{1/r} \\ \bar{G}^\alpha(x) & \text{if } x \geqq \mu_r^{1/r} \end{cases}$ where α is such that $\int_0^x rz^{r-1}\bar{G}^\alpha(z)dz = \mu_r$	$\bar{F}(x) \geqq \begin{cases} \inf_{0 \leq \beta \leq x} [\bar{G}(x)/\bar{G}(\beta)]^\alpha & \text{if } x < \mu_r^{1/r} \\ 0 & \text{if } x \geqq \mu_r^{1/r} \end{cases}$ where α is such that $\beta^r + \int_\beta^\infty rz^{r-1}[\bar{G}(z)/\bar{G}(\beta)]^\alpha dz = \mu_r$
$F \underset{*}{\prec} G$	$\bar{F}(x) \leqq \begin{cases} 1 & \text{if } x < \mu_r^{1/r} \\ \bar{G}^\alpha(x) & \text{if } x \geqq \mu_r^{1/r} \end{cases}$ where α is such that $\int_0^x rz^{r-1}\bar{G}^\alpha(z)dz = \mu_r$	$\bar{F}(x) \geqq \begin{cases} \bar{G}^\alpha(x) & \text{if } x < \mu_r^{1/r} \\ 0 & \text{if } x \geqq \mu_r^{1/r} \end{cases}$ where α is such that $x^r + \int_x^\infty rz^{r-1}\bar{G}^\alpha(z)dz = \mu_r$
$F \underset{su}{\prec} G$	$\bar{F}(x) \leqq \begin{cases} 1 & \text{if } x < \mu_r^{1/r} \\ \bar{G}^\alpha(x) & \text{if } x \geqq \mu_r^{1/r} \end{cases}$ where α is such that $\int_0^x rz^{r-1}\bar{G}^\alpha(z)dz = \mu_r$	$\bar{F}(x) \geqq \begin{cases} e^{-\alpha} & \text{if } x < \mu_r^{1/r} \\ 0 & \text{if } x \geqq \mu_r^{1/r} \end{cases}$ where α is such that $(e^\alpha - 1) \sum_{n=1}^\infty e^{-n\alpha}[\bar{G}^{-1}(\bar{G}^n(x))]^r = \mu_r$
$G \underset{c}{\prec} F$	$\bar{F}(x) \leqq \sup_{0 \leq \alpha \leq \alpha_0} \dfrac{\mu_r \bar{G}^\alpha(x)}{\int_x^\infty rz^{r-1}\bar{G}^\alpha(z)dz}$ where α_0 is such that $\int_0^\infty rz^{r-1}\bar{G}^{\alpha_0}(z)dz = \mu_r$	$\bar{F}(x) \geqq 0$
$G \underset{*}{\prec} F$	$\bar{F}(x) \leqq \bar{G}^\alpha(x)$ where α is such that $\bar{G}^\alpha(x)x^r + \int_x^\infty rz^{r-1}\bar{G}^\alpha(z)dz = \mu_r$	$\bar{F}(x) \geqq 0$
$G \underset{su}{\prec} F$	$\bar{F}(x) \leqq e^{-\alpha}$ where α is such that $(1 - e^{-\alpha}) \sum_{n=1}^\infty e^{-n\alpha}[\bar{G}^{-1}(\bar{G}^n(x))]^r = \mu_r$	$\bar{F}(x) \geqq 0$

As an example, suppose we know that $F \underset{c}{\prec} G$ where $\bar{G}(x) = \exp(-x^2)$ and that the first moment of F is μ. Then it follows from parts (i) and (iii) of Proposition 2.1 that F is IFR and hence the IFR lower bound (see Barlow and Proschan (1975)) is applicable here. However, Table 1 gives a sharper lower bound, as shown in Figure 1.

5.2. *Reliability bounds with one known quantile.* Barlow and Proschan (1975), p. 110 and 188, give a set of bounds for IFRA, NBU and NWU distributions when a quantile of the distribution is given. Similarly one can find upper and lower bounds on

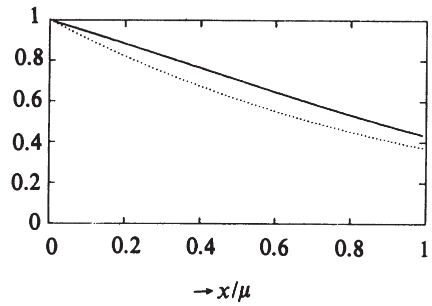

Figure 1. Lower bounds on $\bar{F}(x)$ as a function of x/μ, where μ is the mean of F: dotted line shows IFR lower bound, bold line shows lower bound when $F \underset{c}{\preceq} G$ and $\bar{G}(x) = \exp(-x^2)$.

TABLE 2
Bounds on $\bar{F}(x)$ when $\bar{F}(x_p) = p$

Case	Upper bound	Lower bound
$F \underset{c}{\preceq} G$	$\bar{F}(x) \leqq \begin{cases} 1 & \text{if } x < x_p \\ \bar{G}^{\alpha}(x) & \text{if } x \geqq x_p \end{cases}$ where $\alpha = \log p / \log \bar{G}(x_p)$	$\bar{F}(x) \geqq \begin{cases} \bar{G}^{\alpha}(x) & \text{if } x < x_p \\ 0 & \text{if } x \geqq x_p \end{cases}$ where $\alpha = \log p / \log \bar{G}(x_p)$
$F \underset{*}{\preceq} G$	$\bar{F}(x) \leqq \begin{cases} 1 & \text{if } x < x_p \\ \bar{G}^{\alpha}(x) & \text{if } x \geqq x_p \end{cases}$ where $\alpha = \log p / \log \bar{G}(x_p)$	$\bar{F}(x) \geqq \begin{cases} \bar{G}^{\alpha}(x) & \text{if } x < x_p \\ 0 & \text{if } x \geqq x_p \end{cases}$ where $\alpha = \log p / \log \bar{G}(x_p)$
$F \underset{su}{\preceq} G$	$\bar{F}(x) \leqq \begin{cases} 1 & \text{if } x < x_p \\ p^n & \text{if } \bar{G}^{n+1}(x_p) < \bar{G}(x) \leqq \bar{G}^{n}(x_p) \end{cases}$	$\bar{F}(x) \geqq \begin{cases} p^{1/n} & \text{if } \bar{G}^{1/n}(x_p) < \bar{G}(x) \leqq \bar{G}^{1/(n+1)}(x_p) \\ 1 & \text{if } x \geqq x_p \end{cases}$
$G \underset{c}{\preceq} F$	$\bar{F}(x) \leqq \begin{cases} \bar{G}^{\alpha}(x) & \text{if } x < x_p \\ p & \text{if } x \geqq x_p \end{cases}$ where $\alpha = \log p / \log \bar{G}(x_p)$	$\bar{F}(x) \geqq \begin{cases} p & \text{if } x < x_p \\ \bar{G}^{\alpha}(x) & \text{if } x \geqq x_p \end{cases}$ where $\alpha = \log p / \log \bar{G}(x_p)$
$G \underset{*}{\preceq} F$	$\bar{F}(x) \leqq \begin{cases} \bar{G}^{\alpha}(x) & \text{if } x < x_p \\ p & \text{if } x \geqq x_p \end{cases}$ where $\alpha = \log p / \log \bar{G}(x_p)$	$\bar{F}(x) \geqq \begin{cases} p & \text{if } x < x_p \\ \bar{G}^{\alpha}(x) & \text{if } x \geqq x_p \end{cases}$ where $\alpha = \log p / \log \bar{G}(x_p)$
$G \underset{su}{\preceq} F$	$\bar{F}(x) \leqq \begin{cases} p^{1/(n+1)} & \text{if } \bar{G}^{1/n}(x_p) < \bar{G}(x) \leqq \bar{G}^{1/(n+1)}(x_p) \\ 1 & \text{if } x \geqq x_p \end{cases}$	$\bar{F}(x) \geqq p^n \quad \text{if } \bar{G}^{n+1}(x_p) < \bar{G}(x) \leqq \bar{G}^{n}(x_p)$

$\bar{F}(x)$ when it is known that $\bar{F}(x_p) = p$ and that F is ordered with respect to a known distribution G. Assuming that $\bar{G}$ is continuous and strictly increasing, we summarize the results in Table 2 and omit the proofs. These bounds are also sharp.

5.3. *Moment inequalities*. Given the first moment of an unknown distribution, sometimes one is interested in knowing the range of its other moments. If the distribution is IFRA, then one such result can be found in Barlow and Proschan (1975), p. 116. We consider the general problem of finding a bound on the sth moment of F, given the rth moment and the fact that it is ordered with respect to G. The following theorem gives the required generalization.

Theorem 5.1. Suppose $X \underset{*}{\prec} Y$ and $E(X^r) = E(Y^r)$ for some $r > 0$. Then

$$E(X^s) \begin{cases} \geqq E(Y^s) & \text{for all } s \in (0, r), \\ \leqq E(Y^s) & \text{for all } s > r, \end{cases}$$

assuming the moments exist. Further, the inequalities are strict when the distributions of X and Y are not identical.

Proof. Lemma 4.6.4 of Barlow and Proschan (1975), p. 112 goes through for the $\underset{*}{\prec}$ ordering. Moreover, the inequality given there is strict whenever ψ is strictly monotone, assuming the distributions to be non-identical. The stated result follows by choosing ψ to be an appropriate power function.

Note that $F \underset{*}{\prec} G$ implies $F \underset{*}{\prec} (1 - \bar{G}^{\alpha})$ for all $\alpha > 0$. Therefore the equality of rth moments in the above theorem is not a very restrictive requirement. It is enough to know the rth moment of X; then a suitable modification of Y can be found to match this moment.

It is also important to note that the above theorem holds for $\underset{c}{\prec}$, convex and star orderings.

Corollary 5.1. If $X \underset{*}{\prec} Y$ and X and Y share two common moments, then they have identical distributions.

5.4. *Stability*. As a special case of Corollary 5.1, if $F \underset{*}{\prec} G$ and the distributions have common first and second moments, then $F = G$. One may wish to find out how close F and G are when they only share the first moment but the second moments are known. Rachev (1991), p. 260 has shown that

$$\sup_{x \geqq 0} |F(x) - G(x)| \leqq 3b^{2/3}[\zeta(X, Y)]^{1/3},$$

where b is the supremum of the density of either F or G and $\zeta(X, Y)$ is the average metric (Zolotarev (1976)) $\zeta(X, Y)$ given by

$$\zeta(X, Y) = \int_0^\infty \left| \int_x^\infty [F(t) - G(t)]dt \right| dx.$$

If both the distributions have densities with known supremum, one can choose b to be the smaller of the two.

We begin with a generalization of Theorem 4.1 of Kalashnikov and Rachev (1986) in order to obtain an expression for the average metric.

Theorem 5.2. Suppose $X \sim F$, $Y \sim G$, $F \underset{*}{<} G$, $\boldsymbol{E}(X) \leqq \boldsymbol{E}(Y)$ and the second moments of F and G are finite. Let the additional condition $F(x) < G(x)$ hold for some x whenever $\boldsymbol{E}(X) < \boldsymbol{E}(Y)$. Then the average metric $\zeta(X, Y)$ is equal to $\frac{1}{2}[\boldsymbol{E}(Y^2) - \boldsymbol{E}(X^2)]$.

Proof. By Proposition 2.5, F crosses G at most once and from below. The conditions involving the first moments ensure that such a crossing does occur. Now we follow Kalashnikov and Rachev to define

$$D(x) = \int_x^\infty [\bar{F}(t) - \bar{G}(t)]dt$$

and argue that $D'(x)$ changes sign (from negative to positive) only once between 0 and ∞, while $D(0) \leqq D(\infty) = 0$. Thus $D \leqq 0$ and hence $\zeta(X, Y)$ is the integral of $-D(x)$ from 0 to ∞. The result follows.

Corollary 5.2. Under the conditions of the above theorem, $F = G$ if and only if $\boldsymbol{E}(X^2) = \boldsymbol{E}(Y^2)$. (This is a corollary to Theorem 5.1 as well.)

Corollary 5.3. If either F or G has a density uniformly bounded by b, then under the conditions of Theorem 5.2,

$$\sup_{x \geqq 0} |F(x) - G(x)| \leqq 3\left[\frac{b^2}{2}\{\boldsymbol{E}(Y^2) - \boldsymbol{E}(X^2)\}\right]^{1/3}.$$

We actually get a family of bounds from Corollary 5.3, since $F \underset{*}{<} G$ is equivalent to $F \underset{*}{<} (1 - \bar{G}^\alpha)$ for each $\alpha > 0$. To be more specific, let α_1 be the largest value of α such that $\bar{F}$ is completely dominated by $\bar{G}^\alpha$ and α_2 be such that the first moment of $(1 - \bar{G}^{\alpha_2})$ is the same as that of F. Then we have for all $\alpha \in (\alpha_1, \alpha_2]$

$$\sup_{x \geqq 0} |F(x) - G(x)| \leqq 3\left[\frac{b^2(\alpha)}{2}\{v(\alpha) - \boldsymbol{E}(X^2)\}\right]^{1/3},$$

where

$$b(\alpha) = \sup_{x \geqq 0} \alpha g(x)\bar{G}^{\alpha-1}(x),$$

$$v(\alpha) = \int_0^\infty 2x\bar{G}^\alpha(x)dx.$$

Evidently the right-hand side of the above inequality may be minimized with respect to α over $(\alpha_1, \alpha_2]$ in order to obtain the 'best possible approximation'. For example, for any IFRA distribution with the first two moments μ_1 and μ_2 (where G is the exponential distribution) we have $\alpha_1 = 0$ and $\alpha_2 = 1/\mu_1$. Corollary 5.3 says that

$$\sup_{x \geqq 0} |\bar{F}(x) - \exp(-\alpha x)| \leqq 3[1 - \mu_2\alpha^2/2]^{1/3}$$

for all $\alpha \in (0, 1/\mu_1]$. The minimum value of the right-hand side is $3[1 - \mu_2/(2\mu_1^2)]^{1/3}$, which corresponds to $\alpha = 1/\mu_1$. This bound actually holds for any HNBUE distribution (Rachev (1991), p. 260). In any case, this simple illustration shows how useful Corollary 5.3 can be.

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