Sankhyā: The Indian Journal of Statistics
1983, Volume 45, Series A, Pt. 2, pp. 168-180.

NEYMAN FACTORISATION THEOREMS FOR EXPERIMENTS ADMITTING DENSITIES

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SUMMARY. $(X, \mathcal{A}, \mathcal{P})$ be an experiment and m a measure such that $\frac{dP}{dm}$ exists for each P. It is shown that a subfield $\mathcal{B}$ is pairwise sufficient and contains carriers iff $\frac{dP}{dm}$ can be factored as $g_P(x)\, h(x, A)$ on sets A of m σ-finite measure. The theorem is proved using pivotal measures and two constructions of the pivotal measure is also given. Finally, an example is given to show that the theorem cannot be improved.

1. Introduction

An experiment or a statistical structure is a triplet $(X, \mathcal{A}, \mathcal{P})$ consisting of a set X, a σ-field $\mathcal{A}$ of subsets of X and a family $\mathcal{P}$ of probability measures on $(X, \mathcal{A})$. A measure m on $(X, \mathcal{A})$ is said to dominate $\mathcal{P}$, if m dominates each P in $\mathcal{P}$ and further if each P in $\mathcal{P}$ has a density $\frac{dP}{dm}$ with respect to m.

Let $\mathcal{B}$ be a subfield of $\mathcal{A}$. We would say that $\left\{\frac{dP}{dm} : P \epsilon \mathcal{P}\right\}$ has a Neyman factorisation with respect to $\mathcal{B}$ or that there is a $\mathcal{B}$-measurable factorisation if there exist $\mathcal{B}$-measurable functions g_P and an $\mathcal{A}$ measurable function h such that $\frac{dP}{dm}(x) = g_P(x)h(x)[m]$. In this paper we are concerned with the relationship between the notion of sufficiency and subfields admitting factorisation.

The earliest Neyman factorisation theorem is in the case when m is σ-finite. It was proved by Halmos and Savage (1949), that when m is σ-finite "A subfield $\mathcal{B}$ of $\mathcal{A}$ is sufficient iff there is a $\mathcal{B}$-measurable factorisation". In this case, to obtain the factorisation, Halmos and Savage, first show the existence of a measure λ satisfying

(a) $\lambda \equiv \mathcal{P}$, i.e., $\lambda(A) = 0$ iff $P(A) = 0$ for all P in $\mathcal{P}$.

(b) $\mathcal{B}$ is sufficient iff there is a version $g_P(x)$ of $\frac{dP}{d\lambda}$ which is $\mathcal{B}$-measurable. Having obtained λ, the required factorisation is obtained by writing

$$\frac{dP}{dm} = \frac{dP}{d\lambda} \cdot \frac{d\lambda}{dm} = g_P(x) . \frac{d\lambda}{dm} .$$

The situation is somewhat different when m is not σ-finite. On the one hand simple examples can be constructed where (i) there is a $\mathcal{B}$-measurable factorisation but $\mathcal{B}$ is only pairwise sufficient, (ii) $\mathcal{B}$ is pairwise sufficient but there is no $\mathcal{B}$-measurable factorisation. On the other hand, it is known that the existence of a $\mathcal{B}$-measurable factorisation implies that $\mathcal{B}$ is pairwise sufficient. It is thus clear that factorisation involves a notion somewhere in between pairwise sufficiency and sufficiency. It is shown in Ghosh, Morimoto and Yamada (1981) that the appropriate notion is "pairwise sufficient containing carriers".

When m is localizable it is proved in Ghosh, Morimoto and Yamada (1981) and Yamada (1981) that "$\mathcal{B}$ is pairwise sufficient and contains carriers iff $\frac{dP}{dm} = g_P.h$ where g_P is $\mathcal{B}$ measurable and h is $\mathcal{A}$-measurable". In a further generalisation, when m is only locally localizable (see Ghosh et al., 1981, Yamada, 1981) a similar theorem holds excpet that h is only locally $\mathcal{A}$-measurable with respect to m. In both these situations, motivated by the σ-finite case, one proceeds to construct a factorization as follows. First get a measure λ, satisfying

(a′) $\mathcal{P} \equiv \lambda$ i.e., $\lambda(A) = 0$ iff $P(A) = 0 \ \forall \ P \in \mathcal{P}$.

(b′) $\mathcal{B}$ is pairwise sufficient and contains carriers iff there is a version $g_P(x)$ of $\frac{dP}{d\lambda}$ which is $\mathcal{B}$-measurable.

Such measures will be called pivotal measures. Having got a pivotal measure λ, to obtain a factorisation, $\frac{d\lambda}{dm} |_A$ is defined for sets A of m σ-finite measure. These $\frac{d\lambda}{dm} |_A$ are then put together using localizability or local localizability, to get a $\mathcal{A}$-measurable or locally $\mathcal{A}$-measurable version of $\frac{d\lambda}{dm}$.

In this paper we give two constructions of a pivotal measure, without any assumptions on m and prove a weaker form of the Neyman factorisation theorem. From this theorem, the Neyman factorisation theorem in the localizable cases follow as corollaries. We give in the last section an example to show that, without additional assumptions on m, our form of the Neyman factorisation theorem cannot be improved.

2. Notations and preliminaries

Let $(X, \mathcal{A}, \mathcal{P})$ be a statistical structure. Then for any P in $\mathcal{P}$ we denote by $N_P = \{A \in \mathcal{A} : P(A) = 0\}$ and by $N_{\mathcal{P}}, \bigcap_{P \in \mathcal{P}} N_P$. We write $\mathcal{P} \equiv m$ if $N_m = N_{\mathcal{P}}$ for a measure m.

We now define the notion of a carrier (Yamada, 1981).

Definition : Let $(X, \mathcal{A}, \mathcal{P})$ be a statistical structure and $P \in \mathcal{P}$. A set C_P in $\mathcal{A}$ is said to be a carrier of P with respect to $(X, \mathcal{A}, \mathcal{P})$ if $P(C_P) = 1$ and further if $A \in \mathcal{A}$, $A \subset C_P$ and $P(A) = 0$ then $A \in N_{\mathcal{P}}$.

Since the underlying statistical structure would generally be clear from the context, we shall write carrier of P rather than carrier of P with respect to $(X, \mathcal{A}, \mathcal{P})$.

Definition : $(X, \mathcal{A}, \mathcal{P})$ is said to contain carriers if every P in $\mathcal{P}$ has a carrier C_P.

It is easy to construct statistical structures not containing carriers. The next proposition states that the condition of containing carriers is equivalent to the existence of a dominating measure. This proposition already appears in (Diepenbrock, 1971). Also see Ghosh *et al.*, (1981). As Diepenbrock's paper is unpublished we will here sketch his proof for convenience.

Proposition 1 : *$(X, \mathcal{A}, \mathcal{P})$ contains carriers iff there is a measure μ on $(X, \mathcal{A})$ such that*

(i) $\mathcal{P} \equiv \mu$

(ii) $\dfrac{dP}{d\mu}$ *exists for every P in $\mathcal{P}$.*

Proof : Suppose there is a μ-satisfying (i) and (ii). Then for C_P choose

$$C_P = \left\{x : \frac{dP}{d\mu} > 0\right\}.$$

To see the converse, consider

$\underline{\overline{F}} = \{\underline{F} : $ (i) $\underline{F} \subset \mathcal{A}$

(ii) $F_1, F_2 \in \underline{F}$ and $F_1 \neq F_2 \Longrightarrow F_1 \cap F_2 \in N(\mathcal{P})$

(iii) $F \in \underline{F}$ implies there exists a P in $\mathcal{P}$ such that $P(F) > 0$ and $F - C_P \in N_{\mathcal{P}}\}$.

$\bar{F}$ is partially ordered under inclusion. An application of Zorn's lemma yields a maximal element $\mathcal{F} = \{F_\gamma : \gamma \in \Gamma\}$ in $\bar{F}$. Then the measure μ defined by $(X, \mathcal{A})$ by

$\mu(A) = \sum_{\gamma \in \Gamma} P^\gamma(A \cap F_\gamma)$ satisfies (i) and (ii) of the proposition.

3. Construction of pivotal measure-(1)

Let $(X, \mathcal{A}, \mathcal{P})$ be a statistical structure containing carriers. Let $\mathcal{B}$ be the smallest subfield which is pairwise sufficient and contains carriers. The existence of such a subfield is shown in Theorem 5 of (Ghosh *et al.*, 1981). Let $\{C_P : P \in \mathcal{P}\}$ be the carriers of $\mathcal{P}$ which belong to $\mathcal{B}$. Now consider the statistical structure $(X, \mathcal{B}, \mathcal{P})$. This statistical structure then contains carriers and imitating the proof of proposition 1, we can get a maximal decomposition $\{F_\gamma : \gamma \in \Gamma\}$, $F_\gamma \in \mathcal{B}$ and then construct a measure λ on $(X, \mathcal{A})$ by

$$\lambda(A) = \sum_{\gamma \in \Gamma} P^\gamma(A \cap F_\gamma).$$

It is easy to see that, on $\mathcal{B}$

(i) $\mathcal{P} \equiv \lambda$

(ii) $\dfrac{dP_{\mathcal{B}}}{d\lambda_{\mathcal{B}}}$ exist for every P in $\mathcal{P}$.

We shall next show that (i) and (ii) above hold on $\mathcal{A}$ itself and then show that λ is indeed a Pivotal measure.

Proposition 2 : *On* $(X, \mathcal{A})$

(i) $\mathcal{P} \equiv \lambda$

(ii) $\dfrac{dP}{d\lambda}$ *exists for each* P *in* $\mathcal{P}$.

Proof : $\mathcal{P} \gg \lambda$ is trivial. To see that $\mathcal{P} \ll \lambda$ consider A in $\mathcal{A}$ with $\lambda(A) = 0$. Fix P in $\mathcal{P}$. There is a countable set of F_γ's, say, $F_{\gamma_1}, F_{\gamma_2}, \ldots$ such that $P\left(\bigcup_1^\infty F_{\gamma_i}\right) = 1$. Now since $\lambda(A) = 0$, for all i $P^{\gamma_i}(A \cap F_{\gamma_i}) = 0$. Further $F_{\gamma_i} \subseteq C_{P^{\gamma_i}}[\mathcal{P}]$ so $P(A \cap F_{\gamma_i}) = 0$. Hence, $P(A) = \sum_1^\infty P(A \cap F_{\gamma_i}) = 0$. The above argument shows also that P is concentrated on a set of positive σ-finite measure. Hence $\dfrac{dP}{d\lambda}$ exists.

Proposition 3 : *A subfield $\mathcal{B}_0$ of $\mathcal{A}$ is pairwise sufficient and contains carriers iff for each P in $\mathcal{P}$, there exists a $\mathcal{B}_0$-measurable version of* $\frac{dP}{d\lambda}$.

Proof : Recall that $\mathcal{B}$ is the smallest pairwise sufficient σ-field containing carriers of $\mathcal{P}$. It can be easily seen from the construction of λ that if $F_{\gamma_1}, F_{\gamma_2}, \ldots, \epsilon\, \mathcal{B}$ and if $P\left(\bigcup_1^\infty F_{\gamma_i}\right) = 1$. Then $\frac{dP}{d\sum_1^\infty P^{\gamma_i}} I_{\left(\bigcup_1^\infty F_{\gamma_i}\right)}(x)$ is a version of $\frac{dP}{d\lambda}$. Since $\mathcal{B}$ is pairwise sufficient $\frac{dP}{d\sum_1^\infty P^{\gamma_i}} I_{\left(\bigcup_1^\infty F_{\gamma_i}\right)}$ can be taken to be $\mathcal{B}$-measurable and hence is a $\mathcal{B}$-measurable version of $\frac{dP}{d\lambda}$. Now suppose $\mathcal{B}_0$ is pairwise sufficient and contains carriers then $\mathcal{B} \subseteq \mathcal{B}_0 \vee N_{\mathcal{P}}$. By the above remark, there is a version of $\frac{dP}{d\lambda}$ which is $\mathcal{B}_0 \vee N_{\mathcal{P}}$ measurable and consequently there is also a $\mathcal{B}_0$ measurable version.

Suppose P, $\mathcal{Q}$ belong to $\mathcal{P}$ and admit $\mathcal{B}_0$-measurable densities $\frac{dP}{d\lambda}$ and $\frac{dQ}{d\lambda}$. Then $\frac{dP}{dP+Q} \dfrac{\frac{dP}{d\lambda}}{\frac{dP+Q}{d\lambda}}$ is a $\mathcal{B}_0$-measurable version of $\frac{dP}{dP+Q}$. This shows that $\mathcal{B}_0$ is sufficient for (P, Q). Support of P can be obtained by setting $C_P = \left\{x : \frac{dP}{d\lambda} > 0.\right\}$

4. Construction of pivotal measure–(2)

In this section we obtain a Pivotal measure for a structure $(X, \mathcal{A}, \mathcal{P})$ containing carriers, out of a pivotal measure for statistical sturctures dominated by a locally localizable measure.

Let $\{F_\gamma : \gamma \,\epsilon\, \Gamma\}$ be the maximal decomposition of Proposition 1, and μ the measure constructed therein. Firstly we assume that the measure μ is complete. Let us define a new space X' by

$$X' = \biguplus_{\gamma \epsilon \Gamma} (F_\gamma, \gamma),$$

$A' = \{B \subseteq X'$:

(i) $B \bigcap (F_\gamma, \gamma) = (B_\gamma, \gamma)$ implies $B_\gamma \,\epsilon\, \mathcal{A}$

(ii) There exists a set A in $\mathcal{A}$ such that for all γ. $(B_\gamma \Delta A) \bigcap F_\gamma \,\epsilon\, N_\mu\}$.

We now define a measure μ' on $(X', \mathcal{A}')$ by $\mu'(B) = \mu(A)$ where A is the set satisfying $\forall\ \gamma$, $(B_\gamma \Delta A) \cap F_\gamma \in N_\mu$. The set A will be called as the set approximating B. Then we have the following results most of which are easy to prove.

(1) $(X', \mathcal{A}', \mu')$ is a complete measure space such that for all γ $\mu'(F_\gamma, \gamma) = \mu(F_\gamma)$.

(2) $\{(F_\gamma; \gamma) : \gamma \in \Gamma\}$ is a disjoint maximal decomposition of μ'.

(3) μ' is locally localizable.

This follows from a theorem of Dipenbrock (1971, Theorem 4.5.2) that a complete measure having a disjoint maximal decomposition is locally localizable.

(4) For any A in $\mathcal{A}$, we define $T'(A)$ by

$$T'(A) = \bigcup_{\gamma \in \Gamma} (A \cap F_\gamma, \gamma)$$

Then $T'(A) \in \mathcal{A}'$.

(5) For any $B \in \mathcal{A}'$, there exists a set A such that $\mu'(B \Delta T'(A)) = 0$.

(6) For any $P \in \mathcal{P}$ let us define a probability measure P' on $(X', \mathcal{A}')$ by $P'(B) = P(A)$ where A is a set approximating B and put $\mathcal{P}' = \{P' : P \in \mathcal{P}\}$. Then $(X', \mathcal{A}', \mathcal{P}')$ is dominated the locally localizable measure $\mu' (\equiv \mathcal{P}')$.

This follows from the fact that $T'(T_P) \in \mathcal{A}'_\sigma(\mu')$ and $P'(T'(T_P)) = P(T_P) = 1$ where $T_P = \left\{\frac{dP}{d\mu} > 0\right\}$ and $\mathcal{A}'_\sigma(\mu')$ is all sets in $\mathcal{A}'$ of σ-finite, μ' measure.

From (6), there exists a pivotal measure λ for $(X', \mathcal{A}, \mathcal{P}')$ (Yamada, 1981, Theorem 3). Let us define a measure λ on $(X', \mathcal{A})$ by

$$\lambda(A) = \lambda'(T'(A)),\ A \in \mathcal{A}.$$

We will prove that this measure λ is a pivotal measure for $(X, \mathcal{A}, \mathcal{P})$. For this purpose we will need some more results.

(7) For any subfield $\mathcal{B}$ of $\mathcal{A}$, we can define a subfield $\mathcal{B}$ of $\mathcal{A}'$ by

$\mathcal{B}' = \{B \subseteq X'$:

(i) $B \cap (F_\gamma, \gamma) = (B_\gamma, \gamma)$ implies $B_\gamma \in \mathcal{A}$ for all $\gamma \in \Gamma$.}

(ii) There exists a set $A \in \mathcal{B}$ such that $(B_\gamma \Delta A) \cap F_\gamma \in N_\mu$ for all $\gamma \in \Gamma$.

(8) For any $\mathscr{A}'$-measurable non-negative function f', there exists a non-negative $\mathscr{A}$-measurable function f such that

$$\{f \neq f_\gamma\} \cap F_\gamma \in N_\mu \text{ for all } \gamma.$$

where f_γ is an $\mathscr{A}$-measurable function defined by

$$f_\gamma(x) = \begin{cases} f'(x, \gamma) & \text{if } x \in F_\gamma \\ 0 & \text{otherwise.} \end{cases}$$

(8′) For any non-negative $\mathscr{B}$ measurable function g' there exists a non-negative $\mathscr{B}$-measurable function g such that

$$\{g \neq g_\gamma\} \cap F_\gamma \in N_\mu \text{ for all } \gamma \in \Gamma.$$

(9) For any $\mathscr{A}$-measurable non-negative function f there exists a non-negative $\mathscr{A}'$-measurable function f' such that

$$\{f = f'_\gamma\} \cap F_\gamma \in N_\mu \text{ for all } \gamma \in \Gamma.$$

These, (8), (8′) and (9) can be proved step by step starting from an indicator function.

(10) If $\mathscr{B}$ is pairwise sufficient and contains carriers then $\mathscr{B}'$ is pairwise sufficient for $(X, \mathscr{A}, \mathscr{P}')$ and contains carriers.

Let C_P be a carrier of P belonging to $\mathscr{B}$. Then $T'(C_P)$ is a carrier of P' and belongs to $\mathscr{B}'$. Pairwise sufficiency will follow from (9).

We now turn to the measure λ.

(11) $\mathscr{P} \equiv \lambda$ and each P in $\mathscr{P}$ has a density with respect to λ.

Proof: Let $p(x', P')$ be any version of the density of P' with respect to λ'. Then from (8), there exists a non-negative $\mathscr{A}$-measurable function $q(x, P)$ such that

$$\{q(x, P) \neq p_\gamma(x, P)\} \cap F_\gamma \in N(\mu) \text{ for all } \gamma \text{ in } \Gamma.$$

Let us denote by $T_{P'} = \{x' : p(x, P') > 0\} \in \mathscr{A}'_\sigma(\lambda')$. Hence $T_{P'} = \bigcup_1^\infty E'_i$; $E'_i \in \mathscr{A}'$ and $\lambda'_*(E'_i) < \infty$. Let E_i be a set approximating E'_i and put $F_P = \bigcup_1^\infty E_i$. Then $F_P \in \mathscr{A}_\sigma(\lambda)$ and $P(F_P) = 1$. Define a function $p(x, P)$ by

$$p(x, P) = \begin{cases} q(x, P) & \text{if } x \in F_P \\ 0 & \text{otherwise.} \end{cases}$$

Then $p(x, P)$ is a version of $\dfrac{dP}{d\lambda}$.

(12) λ is a pivotal measure for $(X, \mathcal{A}, \mathcal{P})$.

Proof: Let $\mathcal{B}$ be any subfield which is pairwise sufficient and contains carriers. Then $\mathcal{B}'$ is pairwise sufficient and contains carriers with respect to $(X', \mathcal{A}', \mathcal{P}')$ by (10). Hence there exists for each P', a $\mathcal{B}$-measurable version $p(x', P')$ of $\frac{dP'}{d\lambda'}$. For this $p(x', P')$ there is from (11), a version of the density $p(x, P)$ of P with respect to λ. In the proof of (11) the $q(x, P)$ can be taken to be $\mathcal{B}$-measurable by (8′). Since $F_P \in \mathcal{B}$, $p(x, P)$ is $\mathcal{B}$-measurable. Therefore λ is a pivotal measure.

We shall next construct a pivotal measure when the measure μ is not necessarily complete. For this purpose, let $\mathcal{A}$ be the completion of $\mathcal{A}$ with respect to μ and μ be the extension. Also let P denote the natural extension of P to $\mathcal{A}$. Then $(X, \mathcal{A}, \mathcal{P})$; where $\mathcal{P} = P, P \in \mathcal{P}$ and μ play the same role as $(X, \mathcal{A}, \mathcal{P})$ and μ in the preceding case. We have thus a pivotal measure λ for $(X, \mathcal{A}, \mathcal{P})$ and it can be readily verified that $\lambda|_{\mathcal{A}}$ is a pivotal measure for $(X, \mathcal{A}, \mathcal{P})$.

5. The Neyman factorisation theorem

In this section we will prove a factorisation theorem under the only assumption that there exist a dominating measure for the experiment.

Theorem: *Assume that there exists a measure m on $(X, \mathcal{A})$ such that each P in $\mathcal{P}$ has a density $\frac{dP}{dm}$ with respect to m. Then a subfield $\mathcal{B}$ is pairwise sufficient and contains carriers iff for each P in $\mathcal{P}$ and $A \in \mathcal{A}_\sigma$ (m)*

$$\frac{dP}{dm} = g_P(x) h(x, A)[m] \text{ on } A$$

where $g_P(x)$ is a non-negative $\mathcal{B}$-measurable function and $h(x, A)$ is a non-negative $\mathcal{A}$-measurable function and vanishes outside A.

Proof: "Only if" part.

Let us assume that $\mathcal{B}$ is pairwise sufficient and contains carreirs. Also let λ be a pivotal measure for $(X, \mathcal{A}, \mathcal{P})$. For any A in $\mathcal{A}_\sigma(m)$, there exists a non-negative Radon-Nikodym derivative on A say $k(x, A)$ of λ with respect to m. Define an $\mathcal{A}$-measurable non-negative function on $h(x, A)$ by

$$h(x, A) = \begin{cases} k(x, A), & x \in A \\ 0 & , x \in X - A. \end{cases}$$

Let $g_P(x)$ be a $\mathcal{B}$-measurable version of $\frac{dP}{d\lambda}$. Then it is clear that

$$\frac{dP}{dm} = g_P(x)h(x, A)[m] \text{ on } A.$$

"If part".

Let $B_P = \{x : g_P(x) > 0\} \in \mathcal{B}$. Then we have

$$P((X-B_P)\cap A) = \int_{(X-B_P)\cap A} g_P(x)h(x, A)dm = 0 \text{ for all } A \text{ in } \mathcal{A}_0(m).$$

Hence $P(B_P) = 1$. Next take any A in $\mathcal{A}$ such that $A \subseteq B_P$ and $P(A) = 0$. Then for any $B \in \mathcal{A}_0(m)$ we have

$$0 = P(A \cap B) = \int_{A\cap B} g_P(x)h(x, B)dm = \int_{A\cap B\cap(h)(x,B)>0)} (g_Px)h(x, B)dm.$$

Hence $m(A \cap B \cap \{h(x, B) > 0\}) = 0$. It then follows that for any Q in $\mathcal{P}$, letting $B = \left\{\frac{dQ}{dm} > 0\right\}$

$$Q\left(A \cap \left\{\frac{dQ}{dm} > 0\right\} \cap h\left(x, \left\{\frac{dQ}{dm} > 0\right\}\right)\right) = 0.$$

Therefore

$$\int_{A\cap\left(\frac{dQ}{dm}>0\right)} g_Q(x)h\left(x, \left\{\frac{dQ}{dm} > 0\right\}\right) dm = 0$$

or

$$Q(A) = Q\left(A \cap \frac{dQ}{dm} > 0\right) = 0.$$

Therefore B_P is a carrier of P. Next we will prove that $\mathcal{B}$ is pairwise sufficient.

Let P_1 and P_2 be two measures from $\mathcal{P}$. Denote by B_{P_1} and B_{P_2} the suppor's of P_1 and P_2 respectively, given by $\left(\frac{dP}{dm} > 0\right)$ and let $B = B_{P_1} \cup B_{P_2}$. Define a function h on X by

$$h(x) = \begin{cases} h(x, B_{P_1}) & \text{on } B_{P_1} \\ h(x, B_{P_2}) & \text{on } B_{P_2} - B_{P_1} \\ 0 & \text{otherwise.} \end{cases}$$

Noting that P_1 and P_2 are equivalent to m on $B_{P_1} \cap B_{P_2}$ it follows that

$$\frac{dP_1}{dm} = g_{P_1}(x)h(x)[m]$$

and

$$\frac{dP_2}{dm} = g_{P_2}(x)h(x)[m].$$

Since B_{P_1} and B_{P_2} are in $\mathcal{A}_\sigma$ (m) so is B. Hence m_B, m restricted to B is σ-finite and further

$$\frac{dP_1}{dm_B} = g_{P_1}(x)h(x)[m_B]$$

$$\frac{dP_2}{dm_B} = g_{P_2}(x)h(x)[m_B].$$

The Halmos-Savage theorem now yields the sufficiency of $\mathcal{B}$ for $P_1 P_2$.

Corollary 1 : *It can be easily seen that* $\{h(x, A) : A \in \mathcal{A}_\sigma(m)\}$ *gives an m-cross section. Thus if m is localizable or locally localizable, we can replace* $\{h(x, A) : A \in \mathcal{A}_\sigma(m)\}$ *by an* $\mathcal{A}$*-measurable or locally* $\mathcal{A}$*-measurable function* $h(x)$. *This yield the factorisation theorem in Ghosh et al.* (1981).

Corollary 2 : *For each P let* $A(P)$ *denote the completion of* $\mathcal{A}$ *by subsets of P null sets. It can then be easily seen that if there is a function* $h(x)$ *such that* $h(x) = h(x, A)[m]$ *for all A in* $\mathcal{A}_\sigma(m)$; *then* $h(x)$ *is* $A(P)$ *measurable for each P and hence* $\mathcal{A} = \bigcap_{P \in \mathcal{P}} \mathcal{A}_P$, *measurable.*

6. An example

Apropos to the corollaries in Section 4, if m is not locally localizable then it may not be possible to get a single function $h(x)$ to represent $\{h(x, A) : A \in \mathcal{A}\}$. In this section we give example of such a situation.

Proposition : *Suppose* $\mathcal{P} \equiv m$, *then* $\mathcal{A} = \mathcal{A}(m)$ *where*

$$\mathcal{A}(m) = \{A \subset X : \forall\, F \in \mathcal{A}_\sigma(m) \ \exists\, B_F \in \mathcal{A} \ \text{ such that } (A \cap F) \,\Delta\, B_F \subset N_F \text{ and } m(N_F) = 0\}.$$

Proof : That $\mathcal{A}(m) \subset \mathcal{A}(P)$ for all P in $\mathcal{P}$ is easy. Therefore $\mathcal{A}(m) \subset \mathcal{A}$.

Let $A \in \mathcal{A} = \bigcap_P \mathcal{A}(P)$ and let $F \in \mathcal{A}_\sigma(m)$.

Since $\mathscr{P} \equiv m$, and $F \in \mathscr{A}_\sigma(m)$, there is (see Halmos and Savage, 1949) a countable subset of $\mathscr{P}$, say $P_1, P_2, \ldots$ such that $F \subseteq \| B_{P_i}$ where B_{P_i} is a support of P_i for $i = 1, 2, \ldots$. For each P_i there is a set $B_i \in \mathscr{A}$ such that $(A \Delta B_i) \subseteq N_i$ and $P_i(N_i) = 0$. Then $B = \bigcup_i (B_i \cap B_{P_i})$ is in $\mathscr{A}$ and satisfies $A \cap F \Delta B \subseteq N$, where $m(N) = 0$.

Remark: It follows from the proceding proposition that if there is a function $h(x)$ representing $\{h(x, A), A \in \mathscr{A}_\sigma(m)\}$ and if $\mathscr{P} \equiv m$, then $h(x)$ is necessarily $\mathscr{A}(m)$-measurable.

We will now give an example where $\{h(x, A) : \mathscr{A}_\sigma(m)\}$ cannot be represented by a single function and consequently a situation where Neyman factorisation of the form discussed in the introduction does not hold. Our example is based on some results of Fremlin (1978). Fremlin (1978) gives example of a localizable space $(Z, \mathscr{C}, \mu)$ for which $\mathscr{C} = \mathscr{C}(\mu)$ but which is not decomposable. Consequently, it follows from proposition of Fremlin (1978) that there exist a probability space (Ω, a, γ) such that the product measure $\mu \times \gamma$ on $(Z \times \Omega, \mathscr{C} \times a)$ is not localizable even if we extend $\mu \times \gamma$ to $\widetilde{\mathscr{C} \times a}(\mu \times \gamma)$.

Now let $X = Z \times \Omega$

$$\mathscr{A} = \mathscr{C} \times a(\mu \times \gamma)$$
$$m' = \mu \times \gamma$$
$$\mathscr{B} = \mathscr{C} \times \Omega$$

then $(X, \mathscr{A}, m')$ satisfies the following properties

(i) $(X, \mathscr{A}, m')$ is not localizable, in fact not even locally localizable;

(ii) $(X, \mathscr{B}, m')$ is localizable;

(iii) If $\{F_\gamma : \gamma \in \Gamma\}$ is a maximal decomposition of $(X, \mathscr{B}, m')$ then it is also a decomposition of $(X, \mathscr{A}, m')$.

Since $(X, \mathscr{A}, m')$ is not localizable there is a maximal decomposition $\{E_\gamma : \gamma \in \Gamma\}$ of m' and a set $\Gamma_1 \subset \Gamma$ such that $\{E_\gamma : \gamma \in \Gamma_1\}$ does not have an m' essential supremum.

Define a measure m on $(X, \mathscr{A})$ by

$$m(A) = m'(A) + \sum_{\gamma \in \Gamma_1} m'(A \cap E_\gamma)$$

(•) Then m is equivalent to m', and a local density of m' with respect to m' does not exist. That is, there does not exist a $\mathscr{A}$-measurable function $f(x)$ such that

$$\int_A f\, dm = m'(A) \text{ whenever } m(A) < \infty.$$

To see this, note that if such a f exists then $E = \{x : f(x) < 1\}$ is an essential supremum for $\{E_\gamma : \gamma \in \Gamma_1\}$.

Now let $\mathcal{F} = \{F \in \mathcal{B} : 0 < m'(F) < \infty\}$ and define $\mathcal{P} = \{P_F : F \in \mathcal{F}\}$ where

$$P_F(A) = \frac{m'(F \cap A)}{m'(F)}.$$

Then

(a) $\mathcal{P} \equiv m$ on $\mathcal{A}$, since a $\mathcal{B}$ decomposition is also a $\mathcal{A}$ decomposition.

(b) $\mathcal{B}$ is pairwise sufficient for $(X, \mathcal{A}, \mathcal{P})$ and contains carriers of $\mathcal{P}$.

We will now show that $\frac{dP_F}{dm}$ cannot be factorised. For suppose

$$\frac{dP_F}{dm} = g_F(x)h(x)$$

where g_F is $\mathcal{B}$-measurable.

Since $\mathcal{A}(m) = \mathcal{A}$ and $\mathcal{P} \equiv m$, h is $\mathcal{A}$-measurable.

Now

$$\frac{dP_F}{dm} = \frac{1}{m'(F)} \left. \frac{dm'}{dm} \right|_F = g_F(x)h(x)$$

or

$$\left. \frac{dm'}{dm} \right|_F = m'(F)g_F(x)h(x).$$

Hence $\{m'(F)g_F(x) : F \in \mathcal{F}$ is a $\mathcal{B}$-measurable cross section. Therefore there is a $g(x)$ such that $I_F(x)g(x) = g_F(x).m'(F)[m']$. and so $g(x).h(x)I_F(x) = \frac{dm'}{dm}$ F so that $g(x).h(x)$ is a local density of m' with respect to m. And this contradicts (•).

Acknowledgement. The authors would like to thank Professors J. K. Ghosh, H. Morimoto and K. K. Roy for many useful discussions.

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Paper received: February, 1981.