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The Minimum Robust Domination Energy of a Connected Graph

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Abstract

Acharya et al. [1] introduced the notion of the *robust domination energy of a graph* G, denoted by $\varepsilon_{rd}(G)$ through the eigenvalues of an appropriate adjacency matrix and proved that $\varepsilon_{rd}(G)$ is the maximum number of edges between D and $V-D$, where D ranges over all the minimal dominating sets of G.

In this note, we prove that for any connected graph G of order n, $\varepsilon_{rd}(G) \geq n-1$, and this result is best possible for all $n \geq 2$. Further, a good characterization of all connected graphs of order n with $\varepsilon_{rd}(G) = n-1$ is given.

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1. Introduction

In this paper, unless mentioned otherwise, the terminology and notations in graph theory will be as in West [4].

Given a graph $G = (V, E)$, a subset $D \subseteq V$ is called a *dominating set* (or, simply a *domset*) of G if every vertex in G is either in D or is adjacent to a vertex in D. The set of all domsets in G is denoted by $\mathcal{D}(G)$. A domset is *minimal* if no proper subset of it is a domset of G and *minimum* if it has the least number $\gamma(G)$ of vertices amongst all the domsets in G; accordingly, the set of all minimal (minimum, respectively) domsets in G is denoted $\mathcal{D}^m(G)(\mathcal{D}^0(G))$ (for details, also see [2,3]).

Given a graph $G = (V, E)$ of order p and size q, and a nonempty proper subset $D \subset V$, say $D = \{u_1, u_2, \ldots, u_t\}$, we define the *boundary matrix* $B_D(G) = (b_{ij})_{t\times(p-t)}$ of D by letting b_{ij} to be the number of edges that join the i^{th} vertex u_i of D to the j^{th} vertex v_j of $V - D$. Clearly then, the i^{th} row sum of $B_D(G)$ yields the number of edges that join u_i to the vertices of $V - D$, so called *partial degree* $d_D(u_i)$ of u_i with respect to the given set D, and the sum $\sum_{i=1}^{t} d_D(u_i)$ gives precisely the number $\mathbf{m}(D, V - D)$ of *boundary edges* of D, viz., the edges that join the vertices of D with those of its *complement* $\overline{D} = V - D$. Then the D-*adjacency matrix* of G is defined in [1] as $A_D(G) = B_D(G) \cdot B_D(G)^T = (a_{ij})_{t\times t}$, where "$\cdot$" means usual matrix multiplication and $B_D(G)^T$ denotes the *transpose* of the boundary matrix $B_D(G)$.

Definition 1.1. *The energy of D, denoted $\varepsilon_G(D)$, is defined as the sum $\sum_{i=1}^{|D|} |\mu_i|$, where μ_i are the eigenvalues of $A_D(G)$ and $|\mu_i|$ is the usual modulus (called the magnitude) of the number μ_i. The robust domination energy (rd-energy, in short) of the graph G, denoted $\varepsilon_{rd}(G)$, is defined as the quantity $max\{\varepsilon_G(D) : D \in \mathcal{D}^m(G)\}$.*

The following lemma is basic and gives us an easy method to calculate the domination energies.

Lemma 1.2. [1] *Let D be any minimal domset of any graph G of order $n \geq 2$. Then*

(i) *$A_D(G)$ is a nonnegative semi-definite symmetric matrix of order t; and*

(ii) $\varepsilon_G(D) = \mathbf{m}(D, V - D)$.

Lemma 1.3. [1] *For any simple graph $G = (V, E)$ having size $q, \varepsilon_{rd}(G) = q$ if and only if G is bipartite.*

Definition 1.4. *The energy graph corresponding to a minimal domset D is the subgraph of G induced by the edges between D and $V - D$ and is denoted by $E_G(D)$. Note that, $E_G(D)$ is a bipartite graph.*

2. The main Result

All graphs considered here in this section are simple graphs. In this section, we prove that for any connected graph G of order n, $\varepsilon_{rd}(G) \geq n-1$ and obtain a characterization of all connected graphs of order n with $\varepsilon_{rd}(G) = n - 1$.

Lemma 2.1. *Let G_0 be an induced subgraph of a connected graph G of order n and D_0 be a minimal domset of G_0. Then there exists a minimal domset D of G such that*

(i) $D_0 \subseteq D$ *and* $D - D_0$ *is an independent set of vertices of* G ($D - D_0$ *may possibly be empty);*

(ii) *no vertex of* $D - D_0$ *is adjacent to a vertex of* D_0 *in* G *;*

(iii) *the energy subgraph of* D_0 *in* G_0 *denoted by* $E_{G_0}(D_0)$ *is a subgraph of the energy subgraph of* D *in* G, *namely* $E_G(D)$ *and*

(iv) *The number of connected components of the energy graph* D *in* G *is less than or equal to that of the energy subgraph* D_0 *in* G_0.

Proof. Let $V_1 = \bigcup_{u\in D_0} N[u]$, where $N[u]$ denotes the closed neighbourhood of u. If $V_1 = V(G)$, then D may be taken as D_0. If $V_1 \neq V(G)$, let u_1 be a vertex of $V(G) - V_1$ adjacent to a vertex of V_1 ; this is possible as G is connected, and let G_1 be the subgraph of G induced on $V_2 = V_1 \cup N[u_1]$; and $D_1 = D_0 \cup \{u_1\}$. If $V_2 = V(G)$ then D_1 is a minimal domset of G_1 and $\{u_1\}$ is an independent set of vertices and u_1 is nonadjacent to every vertex of D_0 as $u_1 \notin V_2$; and also (iii) is obvious. Since u_1 is adjacent to some vertex of V_1 and the number $C(E_G(D)) \leq C(E_{G_0}(D_0))$, where $C(H)$ is the number of connected components of the graph H. If $V_2 \neq V(G)$ then repeated use of the above selection finally results in a minimal domset D of G satisfying (i)-(iv). This completes the proof. □

In particular, if $G_0 = K_1$, then it follows that the energy graph of the selected minimal domset D of G is connected and hence, we have the following theorem:

Theorem 2.2. *For any connected simple graph* G *of order* n, $\varepsilon_{rd}(G) \geq n - 1$.

We now proceed to characterize connected graphs G of order n with $\varepsilon_{rd}(G) = n - 1$; such a graph is called a *robust energy maximal graph.*

Lemma 2.3. *Let* G *be a robust energy maximal graph of order* n. *Then*

(i) G *has no induced subgraph isomorphic to a cycle of even length and also an even cycle with exactly one chord;*

(ii) *If* G_0 *is a maximal complete subgraph of* G *then every vertex of* G *outside* G_0 *is joined to at most one vertex in* G_0 *;*

(iii) G *has no induced subgraph on* 6 *vertices* $v_1, v_2, v_3, v_4, v_5, v_6$ *such that* v_1, v_2, v_3, v_4 *induces a complete graph and* v_5, v_6 *are joined only to* v_1 *and* v_2, *respectively, in this induced subgraph.*

Proof. (i) Let $C = (v_1, v_2, \ldots, v_{2s})$ be an induced even cycle of G. As in the proof of Lemma 1.2, we may choose $u_1 = v_1, u_2 = v_3, \ldots, u_s = v_{2s-1}$ and extend it to an independent (minimal) domset D as described in the proof of Lemma 2.1. Then the connected spanning subgraph induced by the set of edges between D and $V-D$ contains the set of all edges of the cycle C, and hence this subgraph has at least n edges, which contradicts the extremality of G. If an even cycle C has exactly one chord v_2v_i, then arguing with the minimal domset D as above, it follows that $\varepsilon_{rd}(G) \geq n$ which contradicts the extremality of G.

(ii) Let G_0 be a maximal complete subgraph of G and suppose there is a vertex u outside of G_0 which is joined to two vertices v, w in G_0. As G_0 is maximal complete subgraph there is a vertex x in G_0 not joined to u, and hence the subgraph of G induced on these four vertices u, v, w, x is isomorphic to $K_4 - e$, which is an even cycle with exactly one chord which contradicts (i).

(iii) If G has such an induced subgraph, then v_1, v_2 can be extended to a minimal domset D in G (proof is similar to proof of (i)) such that the subgraph induced by the set of edges between D and $V-D$ is connected. However this subgraph contains the cycle v_1, v_3, v_2, v_4, implying that $\varepsilon_{rd}(G) \geq n$, which contradicts the extremality of G. □

Lemma 2.4. *If G is a block of order $n \geq 3$ and G is robust energy maximal, then G is isomorphic to a complete graph of order n or a cycle of length n, where in the latter case n is odd.*

Proof. Let G be a block of order $n \geq 3$ and G be robust energy maximal. Note that G has at least n edges. If every cycle of G is of even length then G is a bipartite graph, and by Lemma 1.2, the energy of G is $q(G) \geq n$. Therefore we may assume that G has an odd cycle. Let $\omega(G)$ be the maximum order of a complete subgraph in G. If $\omega(G) \geq 4$, let G_0 be a complete subgraph of G of order $c(G)$. If $c(G) \leq 3$, let G_0 be an induced odd cycle of maximum possible length in G. Then we consider two cases and in both cases we prove that $G \cong G_0$. Suppose $G \ncong G_0$. As G is a block there exist two distinct vertices u, v in G_0 and a path $P(uv)$ of length at least two in G such that all the internal vertices of $P(uv)$ are outside of G_0. Then we choose such a path of smallest possible length joining two distinct vertices of G_0.

Case 1. $\omega(G) \geq 4$.

If the path $P(uv)$ has exactly one internal vertex then $K_4 - e$ is an induced subgraph of G by the maximality of G_0. If the path $P(uv)$ has at least two internal vertices then G has an induced even cycle or an induced subgraph mentioned in (iii) of Lemma 2.3, contradicting the extremality of G.

Case 2. $\omega(G) \leq 3$.

The subgraph induced by $V(P_1) \cup V(P_2) \cup V(P_3)$ is either an even cycle with a chord or contains an induced even cycle, which is a contradiction. □

We now state and prove the main theorem of this paper.

Theorem 2.5. *A connected graph G is a robust energy maximal graph of order n if and only if* (*a*) *every non-end block of* G *is either* K_2 *or an odd cycle, and*

(*b*) *every end block of* G *is either a complete graph or an odd cycle.*

Proof. Let G be a connected robust energy maximal graph of order n. Then, by Lemma 2.1, every block B of G is a robust energy maximal graph of order $p(B)$, otherwise B will have a minimal domset D_0 such that the energy of D_0 is at least $p(B)$. Then by Lemma 2.1, this D_0 can be extended to a minimal domset D of G satisfying (i) through (iv) and $\varepsilon_G(D)$ is at least n, a contradiction. Therefore, every block of G, by Lemma 2.4, is a complete graph or an odd cycle. If now a non-end block of G is a complete graph K_t of order $t \geq 4$, then G has an induced subgraph on 6 vertices $v_1, v_2, v_3, v_4, v_5, v_6$ such v_1, v_2, v_3, v_4 induces a complete graph and v_5, v_6 are joined only to v_1 and v_2, is an induced subgraph of G, and by Lemma 2.3 (iii), we get a contradiction. This proves the necessity of the theorem.

To prove the sufficiency of the theorem, assume that G is a connected graph of order n, satisfying conditions (a) and (b) of the theorem. Then, it is easy to see that any minimal domset D of G contains exactly one vertex from each of the end-blocks of G which are complete graphs and the energy graph of D omits at least one edge from each of the odd cycles and hence the energy $\varepsilon_G(D)$ is at most $n-1$ and therefore, $\varepsilon_{rd}(G) = n-1$ by Theorem 2.2. □

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